LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 15: Rising sea

Seminar

15.1. Alexander Grothendieck was a special mathematician. Both extremely creative, charismatic as well as eccentric, he not only revolutionized huge parts of mathematics, he also chose in the later part of his life to shut himself off from the world and live alone as a hermit. Today we learn about a picture which Grothendieck drew about methods to solve mathematical problems. It is the **hammer and chisel principle** versus the **rising sea approach** to solve mathematical problems. We look at this here in the context of determinants.

15.2. The **Hammer and Chisel principle** is to put the cutting edge of the chisel against the shell and to strike hard. If needed, begin again at many different points until the shell cracks-and you are satisfied. Grothendieck puts this poetically by comparing the theorem to a nut to be opened. The mathematician uses the Hammer and Chisel to reach "the nourishing flesh protected by the shell". It can also be called the Sledgehammer method. You just hit the problem with all you have and grind it through until the problem is solved.

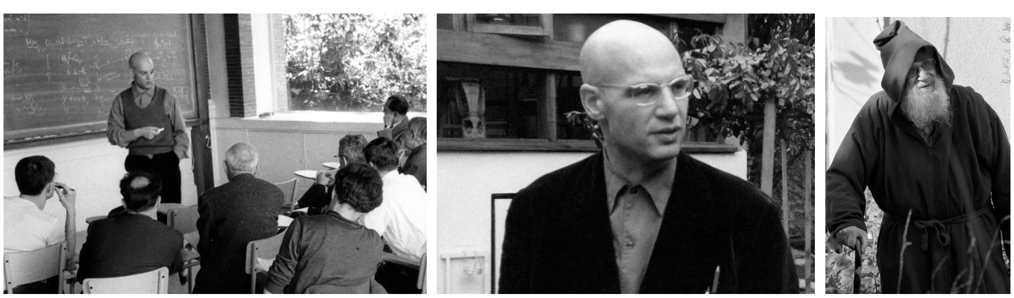

Figure 1. Three pictures of Alexander Grothendieck.

15.3. The **rising sea approach** submerges the problem first into a theory, going well beyond the results originally to be established. Grothendieck imagined to immerse the nut in some softening liquid until the shell becomes flexible. After weeks and months, by mere hand pressure, the shell opens like a perfectly ripened avocado.

15.4. We will illustrate the two principles when looking at some properties of determinants and permutation matrices. One can learn a lot about proofs in this context. Here is a statement you have proven in a homework already.

Theorem: $\det(A^T) = \det(A)$.

15.5. The proof goes by noticing that every permutation π has a dual permutation π^T which has the same sign. We just noticed that the number of up-crossings of π and π^T are the same. This might be a bit hard to see.

15.6. The "rising sea" approach is to put the problem into the larger context of **permutation groups**. The set of all permutations become a group if one realizes the elements as permutation matrices as matrices can be multiplied.

Problem A: Given a permutation matrix A belonging to a permutation π. What is the permutation matrix of the inverse permutation?

15.7. A permutation or pattern can be written as a 0-1 matrix where $P_{ij} = 1$ if $\pi(j) = i$. Now, building a theory takes time and effort. In the case of permutations, one first shows that every permutation can be written as a product of **transpositions**. A transposition is a matrix obtained from the identity matrix by swapping two rows. As we are not doing an abstract algebra course, let us assume the following fact to be for granted. The number of up-crossings of π is even if and only if one can generate π with an even number of transpositions.

15.8.

Problem B: Is it true that the number of up-crossings is equal to the minimal number of transpositions needed to realize the permutation? If yes, prove it. If not, give a counter example.

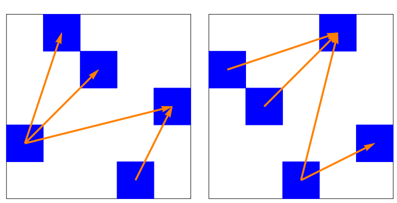

FIGURE 2. A pattern π and the inverse (transpose) pattern π^T We also draw the up-crossings. There are the same number of up-crossings for π and π^T.

15.9. This rising sea approach is cool, but one needs time to build a theory. Let us therefore play a bit more with the hammer and chisel. It is actually quite a nice task if one has the help of a computer.

15.10. Permutation matrices are matrices in which each of the columns is a basis vector e_k and no columns are the same. They solve the 8-**rook problem**:

Problem C: Here is a combinatorial problem in chess. Assume you have a chess game with 8 rooks (rooks can move freely along rows or columns). How many chess rook configurations are there on a 8×8 board in which no rook threatens any other rook?

Problem D: Find a chess game configuration with 4 queens on a 4×4 board (queens can move also diagonally) such that none hits the other. How many are there with 4 queens?

15.11. The Mathematica code to find the number of non-attacking Queen positions for a general n is in the homework part. In the case $n = 4$ you can do it by hand.

15.12. Here is another identity we have mentioned. You have checked it for 2×2 and 3×3 cases, but we did not prove it yet in full generality. We do that now.

Theorem: $\det(AB) = \det(A)\det(B)$.

15.13. Proof: Let us see whether we can understand the proof. Let $C = AB$. Now write

$$A|B = C$$

Now, if we scale a row in A, this corresponds to scaling a row in C. Switching two rows of A corresponds to switching two rows in C. Subtracting a multiple of a row to another row also produces a subtraction of a multiple of a row to another row. While row reducing A, we do the same operations also on C. The sign changes on the left and right hand side are the same. Every division by λ on the left also reduces the determinant of C by λ. After finishing up, we have on the left hand side the determinant of $1 \cdot B$ which is the determinant of B and on the right hand side a matrix with determinant $\det(C)/\det(A)$. We see that $\det(B) = \det(C)/\det(A)$ meaning $\det(A)\det(B) = \det(C)$. QED.

Problem E: The above proof does not quite work if A is not invertible. What happens in that case? Why is it that if A is not invertible, still, the identity holds?

15.14. In the proof that A^T and A have the same determinant, we need the fact that for a pattern, the transposed pattern has the same number of up-crossings. A pattern P defines an orthogonal matrix P. Now, the transpose pattern is the inverse matrix. If we can show that P and P^T have the same determinant, then we are done.

Homework

Problem 15.1 Find the characteristic polynomial of the permutation matrix where e_{k-1} is in the k'th column and e_n is in the first column. What are the roots of this polynomial in the case $n = 4$?

Problem 15.2 The Matrix Tree theorem assures that if L is the Laplacian of a graph with n nodes, then the coefficient a_{n-1} of the characteristic polynomial counts the number of rooted spanning trees in the graph. For the complete graph with 5 elements, the Laplacian is

$$L = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}.$$

What are the eigenvalues of L. What is the characteristic polynomial of L? What is a_4, the number of rooted spanning trees?

Problem 15.3 Here is the Laplacian of the circular graph with 5 nodes. Write this matrix in the form $L = 2 - Q - Q^{-1}$.

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Find the eigenvalues of Q and express the eigenvalues of L using the eigenvalues of Q.

Problem 15.4 Hammer and chisel: Experiment with the following code to get the number of n queen problems for $n = 3, ..., 8$. Alternatively, outline the life of Alexander Grothendieck.

```
PermutationMatrix[p_]:=Module[{n=Length[p],A},A=Table[0,{n},{n}];
   Do[A[[p[[k]],k]]=1,{k,Length[p]}]; A];
QueenConflicts[p_]:=Sum[Sum[If[Abs[p[[i]]-p[[j]]]==Abs[i-j],1,0],
   {i,j+1,Length[p]}],{j,Length[p]}];
F[n_]:=Module[{P=Permutations[Range[n]]},
    U=Flatten[Position[Map[QueenConflicts,P],0]];
    Table[PermutationMatrix[P[[U[[k]]]]],{k,Length[U]}]];
MatrixForm[F[6]]
```

Problem 15.5 Hammer and chisel: Experiment with the following code to get the number of n super queen solutions for $n = 10$. If you have the energy while waiting, write a mathematical essay of the length SGA (a document written by Alexander Grothendieck).

```
PermutationMatrix[p_]:=Module[{n=Length[p],A},A=Table[0,{n},{n}];
   Do[A[[p[[k]],k]]=1,{k,Length[p]}]; A];
SuperQueenConflicts[p_]:=Sum[Sum[If[Abs[p[[i]]-p[[j]]]==Abs[i-j]  ||
   (Abs[i-j]==2 && Abs[p[[i]]-p[[j]]]==1)  ||
   (Abs[i-j]==1 && Abs[p[[i]]-p[[j]]]==2),1,0],
   {i,j+1,Length[p]}],{j,Length[p]}];
F[n_]:=Module[{P=Permutations[Range[n]]},
    U=Flatten[Position[Map[SuperQueenConflicts,P],0]];
    Table[PermutationMatrix[P[[U[[k]]]]],{k,Length[U]}]];
MatrixForm[F[10]]   (* PATIENCE, SOAKING FOR AN HOUR!!!! *)
```