Unit 19: Discrete dynamical systems

Lecture

19.1. A $n \times n$ matrix A defines a linear transformation $T(x) = Ax$. Iterating gives a sequence of vectors $x, Ax, A^2x, \dots A^nx, \dots$. It is called the **orbit** of x of the **discrete dynamical system** defined by A. We write $x(t) = A^tx$ so that $x(0)$ is the **initial condition**. We think of t as **time** and $x(t)$ as the situation at time t.

19.2. For example, for $A = \begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix}$ we have $A^{10} = \begin{bmatrix} 40330467 & 20135709 \\ 40271418 & 20194758 \end{bmatrix}$. For an initial condition like $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, we get $A^{10}x = \begin{bmatrix} 201534237 \\ 201593286 \end{bmatrix}$. Of course, we want a better way to compute the orbit than just multiplying the matrix again and again.

19.3. Eigenvectors provide relief: if v is an eigenvector, then $Av = \lambda v$ and $A^tv = \lambda^tv$. Using linearity of $Tx = Ax$, we see that if x is a sum of eigenvectors like $x = c_1v_1 + \cdots + c_nv_n$, then $Ax = A(c_1v_1 + \cdots + c_nv_n) = c_1Av_1 + \cdots + c_nAv_n = c_1\lambda_1v_1 + \cdots + c_n\lambda_nv_n$.

> If $\mathcal{B} = \{v_1, \cdots, v_n\}$ is an eigenbasis of A and $x(0) = c_1v_1 + \cdots + c_nv_n$ then $x(t) = c_1\lambda_1^tv_1 + \cdots + c_n\lambda_n^tv_n$ solves $x(t+1) = Ax(t)$.

This solution is called the **closed-form solution**.

19.4. In the above example, we have the eigenvectors $v_1 = [1, 1]^T, v_2 = [-1, 2]^T$ to the eigenvalues $\lambda_1 = 6, \lambda_2 = 3$ and $x = [3, 4]^T$ is $c_1v_1 + c_2v_2$ with $c_1 = 10/3$ and $c_2 = 1/3$. The closed-form solution is

$$x(t) = \frac{10}{3}6^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{3}3^t \begin{bmatrix} -1 \\ 2 \end{bmatrix} .$$

It is no problem to evaluate this at any t, like $t = 10$ which gives the above result for $A^{10}x$. We can even evaluate this for $t = 10^{100}$ but can not produce A^t.

Examples

19.5. The sequence of numbers $u(0), u(1), u(2), \dots, u(t), \dots$ defined by the **recursion**

$$u(t+1) = u(t) + u(t-1)$$

with initial condition $u(0) = 0$, $u(1) = 1$ is called the **Fibonacci sequence**. We will solve this system in class. It leads to a recursion

$$\begin{bmatrix} u(t+1) \\ u(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ u(t-1) \end{bmatrix}$$

so that we want to compute the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. This leads to the formula of Binet

$$u(t) = \frac{\left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^t - \left(\frac{1}{2}\left(1-\sqrt{5}\right)\right)^t}{\sqrt{5}} .$$

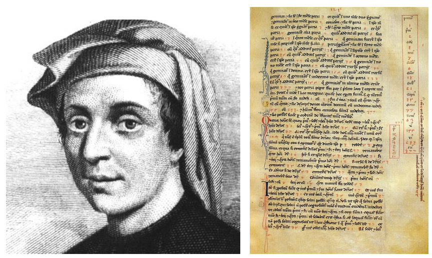

FIGURE 1. Leonardo Pisano 1170-1250 was later called Fibonacci by historians. He published his book "Liber Abaci" in 1202, in which the Hindo-Arabic numeral system and place value was introduced to the West. Also discussed was the Fibonacci sequence. The picture of Fibonacci shown is of unknown origin and almost certainly a work of fiction. The original edition of the text is lost, but there is a version from 1228. Mathematics related to the Fibonacci numbers already appeared in a combinatorial context between 450 and 200 BC in "Chanda-Sutra" = "The art of prosody" written by the Indian mathematician Acharya Pingala.

[1]

19.6. For a **discrete recursion** equation like $u(t+1) = 2u(t) + u(t-1)$ and initial conditions like $u(0) = 1$ and $u(1) = 1$ and get all the other values fixed. We have $u(2) = 3, u(3) = 10$, etc. A discrete recursion can always be written as a discrete dynamical system. Just use the vector $x(t) = [u(t), u(t-1)]^T$ and write

$$x(t+1) = \begin{bmatrix} u(t+1) \\ u(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ u(t-1) \end{bmatrix} = AX(t) .$$

Now we can compute the closed-form solution. The eigenvalues are $1+\sqrt{2}$ and $1-\sqrt{2}$ to the eigenvectors $v_1 = [1+\sqrt{2}, 1]^T$, $v_2 = [1-\sqrt{2}, 1]^T$. The initial condition $[1, 1]$ is $x(0) = (1/2)v_1 + (1/2)v_2$. The closed form solution is $x(t) = (1/2)(1+\sqrt{2})^t v_1 + (1/2)(1-\sqrt{2})^t v_2$.

[1]K. Plofker, Mathematics in India, Princeton University press, 2009

19.7. If $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is a rotation dilation then we can compute A^n quickly. As it is a rotation dilation matrix with angle $\theta = \arg(a+ib)$ and scaling factor $r = \sqrt{a^2+b^2}$, we have $A^n = r^n \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}$.

19.8. The recursion $u(t+1) = u(t) - u(t-1)$ with $u(0) = 0, u(1) = 1$ leads to the discrete dynamical system with the matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$. We see that A^6 is the identity. Every initial vector is mapped after 6 iterations back to its original starting point. The eigenvalues are complex.

19.9. A matrix with non-negative entries for which the sum of the columns entries add up to 1 is called **stochastic matrix** or **Markov matrix**.

> **Theorem:** Markov Matrices have an eigenvalue 1

Proof. The matrix A^T has rows which add up to 1. The vector $[1, 1, \ldots, 1]^T$ therefore is an eigenvector of A^T to the eigenvalue 1. As A has the same eigenvalues as A^T, also A has an eigenvalue 1. □

19.10. The next result is part of the **Perron-Frobenius theorem**.

> **Theorem:** If all entries of the Markov matrix A are positive, then the eigenvalue 1 is maximal and has algebraic multiplicity 1.

Proof. Given $Av = v$, we show that all v_k are the same. We first prove by contradiction that $|v_k|$ is constant. Otherwise, let m be the index where $|v_k|$ is largest and let i be an index where $|v_k|$ is smallest. $|v_i| < |v_m|$ and

$$|v_m| = |\sum_{k=1}^n A_{mk} v_k| \leq \sum_{k=1}^n A_{mk}|v_k| \leq \sum_{k=1, k\neq i}^n A_{mk}|v_m| + A_{mi}|v_i| < \sum_{k=1}^n A_{mk}|v_m| = |v_m| \, .$$

Contradiction. Now, we show that v_k everywhere has the same sign. By scaling, we can assume $v_k = \pm 1$. If there should be different signs let m be an index where $v_m = 1$ and i be an index where $v_i = -1$. Repeat the same computation:

$$1 = v_m = \sum_{k=1}^n A_{mk} v_k \leq \sum_{k=1}^n A_{mk}|v_k| \leq \sum_{k=1, k\neq i}^n A_{mk} 1 + A_{mi}(-1) < \sum_{k=1}^n A_{mk} 1 = 1$$

which is again a contradiction. □

19.11. Intuitively, applying the matrix A averages the coordinate values of A and smooths out the distribution making it constant. In the homework you prove

> **Theorem:** The product of Markov matrices is a Markov matrix.

Homework

This homework is due on Tuesday, 3/26/2019. [2]

Problem 19.1: The vector $A^t x$ gives pollution levels in the Silvaplana, Sils and St Moritz lake t weeks after an oil spill. The matrix is $A = \begin{bmatrix} 0.7 & 0 & 0 \\ 0.1 & 0.6 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix}$ and $x(0) = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$ is the initial pollution level. Find a closed-form solution for $x(t)$.

Problem 19.2: A **Lilac bush** has $n(t)$ new branches and $o(t)$ old branches at the beginning of each year t. During the year, each old branch will grow two new branches and remain old and every new branch will becomes an old branch. Write down the matrix A such that $\begin{bmatrix} n(t+1) \\ o(t+1) \end{bmatrix} = A \begin{bmatrix} n(t) \\ o(t) \end{bmatrix}$ and find closed-formulas for $n(t), o(t)$ if $\begin{bmatrix} n(0) \\ o(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

Problem 19.3: Given a polyhedron with vertex,edge and face data $x = [v, e, f]^T$, one can do a **Barycentric refinement**. This produces a new polyhedron with data $x(1) = Ax(0)$ written out as $\begin{bmatrix} v(1) \\ e(1) \\ f(1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} v(0) \\ e(0) \\ f(0) \end{bmatrix}$. How many vertices, edges and faces does the icosahedron with $[v(0), e(0), f(0)]^T = [12, 30, 20]^T$ have after 10 Barycentric refinements?

Problem 19.4: The **Lucas numbers** are defined by the same recursion as the Fibonacci numbers. It is only that we don't start with $(u(0), u(1)) = (0, 1)$ but with $(u(0), u(1)) = (2, 1)$. Find a close formula for the t'th Lucas numbers $2, 1, 3, 4, 7, 11, 18, ...$

Problem 19.5: a) Prove that the product of two Markov matrices A, B is a Markov matrix.
b) What happens if we would define Markov as the row sum adding up to 1. Why does the result still hold? Hint for a) You have to show that if $\sum_{j=1}^n A_{jk} = 1$ for all k and $\sum_{j=1}^n B_{jk} = 1$ for all k then $\sum_{j=1}^n \sum_{l=1}^n A_{jl} B_{lk} = 1$ for all k.

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[2] 19.1 and 19.2 are slightly modified problems from the book of Otto Bretscher.