# Unit 20: Differential equations

### Lecture

**20.1.** A **differential equation** is an equation for an **unknown function** or vector valued function $x(t)$ which involves at least one derivative $\frac{dx}{dt} = x'$ for $x$. An example is $x'(t) = x(t)^2 + t^3$ or $x''(t) + tx'(t)x(t) = \sin(x(t))$. The **order** of a differential equation is the highest derivative which appears. The first example was a **first order differential equation**, the later was a **second order differential equation**.

**20.2.** Many differential equations can be solved using **separation of variables**. Let us look at the equation
$$\frac{dx}{dt} = 3x \ .$$
To solve this, put all $x$ on one side and all $t$ on the other side:
$$\frac{dx}{x} = 3dt, \qquad \Rightarrow \qquad \int \frac{dx}{x} = \int 3dt + c$$
which gives $\log(x) = 3t + c$ and $x(t) = e^{3t+c} = Ce^{3t}$ for some constant $C$.

**20.3.** It is custom to write the constant $C$ as $x(0)$ because this is the value we get for $t = 0$. Also used is the notation $x'(t)$ or $\dot{x}(t)$ for the derivative. The former has been used by Leibniz, the later by Newton.

> $x' = \lambda x$ has the solution $x(0)e^{\lambda t}$.

**20.4.** For positive $\lambda$ we get **exponential growth**. This is the most primitive model for population growth. For negative $\lambda$ we get **exponential decay**. This is used to model **radioactive decay** and is used in **carbon-14 dating**. In both cases the rate of change is proportional to the sample size. The $\lambda$ is the fertility in population dynamics or **reciprocal mean life** which is $1/8.267$ years for $C^{14}$. This means that in 8.2 years $1/e=1/2.718$. Using $\log_2$ this means that in 5.72 years half of the $C^{14}$ isotopes have decayed to $C^{12}$.

**20.5.** Given a $n \times n$ matrix $A$, we can look at the **system of linear differential equations** $x'(t) = Ax(t)$. If $x(0)$ is given, one can write $x(t) = e^{At}x(0)$. But as before in the discrete case, where the power $A^t$ was hard to compute, also the exponential $e^{At}$ is not convenient. We get more insight if we use an eigenbasis.
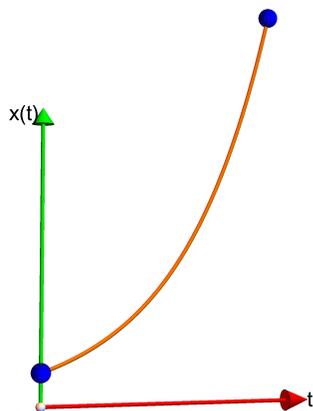
FIGURE 1. The exponential function.

**20.6.** For example, for $A = \begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix}$ and an initial condition like $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, we proceed exactly in the same way. We have already the eigenvectors $v_1 = [1,1]^T, v_2 = [-1,2]^T$ to the eigenvalues $\lambda_1 = 6, \lambda_2 = 3$ and $x = [3,4]^T$ is $c_1 v_1 + c_2 v_2$ with $c_1 = 10/3$ and $c_2 = 1/3$. The closed-form solution is

$$x(t) = \frac{10}{3} e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{3} e^{3t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Also here, we have no problem evaluating this at any time $t$.

> If $\mathcal{B} = \{v_1, \cdots, v_n\}$ is an eigenbasis of $A$ and $x(0) = c_1 v_1 + \cdots + c_n v_n$ then $x(t) = c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n$ solves $x'(t) = Ax(t)$.

**20.7.** Both for differential equations $\dot{x}(t) = Ax(t)$ in two dimensions as well as for discrete systems $x(t+1) = Ax(t)$, one can see the right hand side as a vector field and the solution curve $x(t)$ as a **flow line** of $x' = F(x(t))$. These **phase portraits** together with some solution curves can give more qualitative information.
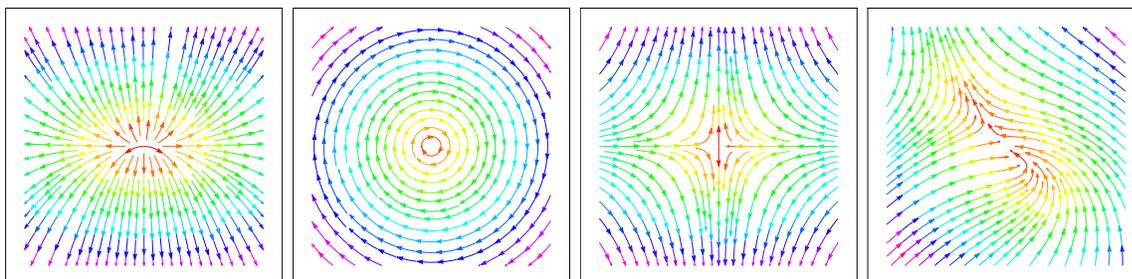


FIGURE 2. Stream plots of $F(x,y) = (x,2y)$, $F(x,y) = (y,-x)$, $F(x,y) = (-x,2y)$, $F(x,y) = (-y-x)$, $x^2 + y^2)$. The first three are linear systems, the last one is not linear.

**20.8.** The linear equation $x' = ax$ produces exponential growth $x(t) = x(0)e^{at}$. This is not sustainable in the long term. For a population, the food might run out for example. The **logistic equation** $x' = ax - x^2$ takes care of this. It is a model, where the growth is stopped when $x = a$. Let us solve a concrete problem. Assume $x' = x(1 - x)$ and $x(0) = 10$. What is $x(t)$? Using **separation of variables** we get $dx/(x(1 - x)) = dt$ so that $\log(x) - \log(1 - x) = t + c$. This Leads to $x(t) = 10e^t/(10e^t - 9)$.

**20.9.** The equation $x' = x^2$ with $x(0) = 1$ is solved by $1/(1 - t)$. The solution blows up in finite time. This example shows the need for some conditions so that solutions exist for all times. We will not go into this here. For linear differential equations, we always have solutions for all times.

**20.10.** The equation $x' = 2\sqrt{x}$ with $x(0) = 0$ is solved by $x(t) = t^2$ as well as with $x(t) = 0$. We see that there is no unique solution. Also this is a phenomenon which only happens for non-linear systems.

**20.11.** Find a closed-formula for the solution of the system

$$
\begin{aligned}
\dot{x}_1 &= x_1 + 2x_2 \\
\dot{x}_2 &= 4x_1 + 3x_2
\end{aligned}
$$

with $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The system can be written as $\dot{x} = Ax$ with $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. The matrix $A$ has the eigenvector $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ to the eigenvalue $-1$ and the eigenvector $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ to the eigenvalue 5.

Because $Av_1 = -v_1$, we have $v_1(t) = e^{-t}v$. Because $Av_2 = 5v_1$, we have $v_2(t) = e^{5t}v$. The vector $v$ can be written as a linear-combination of $v_1$ and $v_2$: $v = \frac{1}{3}v_2 + \frac{2}{3}v_1$. Therefore, $x(t) = \frac{1}{3}e^{5t}v_2 + \frac{2}{3}e^{-t}v_1$.

**20.12.** Finding solutions of nonlinear differential equations can be hard. Sometimes, we can not find closed-form solutions. Like for $x'(t) = e^{-t^2}$, where $x(t)$ as an anti-derivative of $e^{-t^2}$ has no elementary solution. But if you write down a random equation, you will probably not be able to give a concrete solution. Try Mathematica's "Dsolve" procedure on $x' = x + \sin(x)$ for example.

The closed-form solution like $x(t) = e^{At}x(0)$ for $\dot{x} = Ax$ does not give us much insight what happens. One wants to understand the solution quantitatively and answer questions like: what happens in the long term? Is the origin stable? Are there periodic solutions? Can one decompose the system into simpler subsystems?

HOMEWORK

This homework is due on Tuesday, 3/27/2019.

**Problem 20.1:** Solve the differential equations.
a) $x' = \sin(t)x$ with $x(0) = 2$,
b) $x' = t/(5x^4)$ ,with $x(0) = 2$.
c) $x' = 1 + x^2$ ,with $x(0) = 0$,
d) $x' = 1/\cos(x)$ ,with $x(0) = 9$.

**Problem 20.2:** Solve the differential equation $x' = 6x^{3/2}$. This equation does not have a unique solution with $x(0) = 0$. Find two.

**Problem 20.3:** Solve the system
$$\frac{dx}{dt} = Ax, \quad A = \begin{bmatrix} 4 & 9 \\ 7 & 6 \end{bmatrix}$$
with initial condition $x(0) = \begin{bmatrix} 10 \\ -6 \end{bmatrix}$. Draw the phase portrait.

**Problem 20.4:** A population model is given by
$$\frac{dx}{dt} = 15x - 13y$$
$$\frac{dy}{dt} = 8x - 11y$$
First decide whether it is a symbiosis, competition or predator-prey model. then sketch the phase portrait in the first quadrant and decide for which initial conditions the populations die out.

**Problem 20.5:** A door opens on one side only. A spring mechanism closes the door which forms an angle $\theta(t)$ with the frame. The angular velocity is $\omega(t) = \frac{d\theta}{dt}(t)$. The differential equations are
$$\frac{d\theta}{dt} = \omega$$
$$\frac{d\omega}{dt} = -2\theta - 3\omega$$
The first equation is the definition, the second incorporates the force $-2\theta$ of the spring and the friction $-3\omega$.
Sketch a phase portrait for the system and use this to answer the question, for which initial conditions, the door reaches $\theta = 0$ with negative $\omega$.

1

---

[1]20.5 is the slamming door problem by Otto Bretscher.