

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 21: The golden mean

SEMINAR

21.1. The **Fibonacci recursion** $F_{n+1} = F_n + F_{n-1}$ leads to the **Fibonacci sequence**

$$1, 1, 2, 3, 5, 8, \dots$$

There is an explicit formula of Binet involving the **golden mean**

$$\phi = \frac{(1 + \sqrt{5})}{2}.$$

The golden mean has also been called the “**divine proportion.**” The number appeared first in Euclid’s elements around 350 BC.

Problem A: Verify from the **formula of Binet** that $F_{n+1}/F_n \rightarrow \phi$.

21.2. Both the Fibonacci numbers and the golden ratio are popular subjects. A few books dedicated to the subject are listed at the end. If we look at the number ϕ , except maybe for π , there is no other number about which so much has been written about. Fibonacci numbers connect with nature: Devlin writes in “Fibonacci’s arithmetic revolution” *“an iris has 3 petals; primroses, buttercups, wild roses, larkspur, and columbine have 5; delphiniums have 8; ragwort, corn marigold, and cineria 13; asters, black-eyed Susan, and chicory 21; daisies 13, 21, or 34; and Michaelmas daisies 55 or 89. Sunflower heads, and the bases of pine cones, exhibit spirals going in opposite directions. The sunflower has 21, 34, 55, 89, or 144 clockwise, paired respectively with 34, 55, 89, 144, or 233 counterclockwise; a pine-cone has 8 clockwise spirals and 13 counterclockwise. All Fibonacci numbers.*

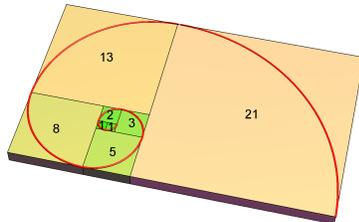


FIGURE 1. The Fibonacci spiral.

21.3.

Problem B: Which number x has the property that if you subtract one of it, then it is its reciprocal?

Problem C: Which rectangle has the property that if you cut away a square with the length of the smaller side, you get a similar rectangle.

Problem D: Which number is a fixed point of the map $T(x) = 1 + 1/x$?

Problem E: Which number is given by the limit $1/(1+1/(1+1/1+\dots))$?

Problem F: Which number is given by $x = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}$

21.4. A regular **pentagon** can be obtained conveniently in the complex plane as the solution set of the equation $z^5 = 1$. The solutions are $e^{2\pi ik/5}$, with $k = 1, 2, 3, 4, 5$.

Problem G: Use this to find the ratio between the diagonal of the pentagon and the side length.

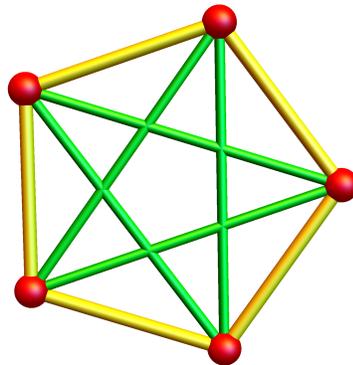


FIGURE 2. The regular Pentagon, when equipped with diagonals becomes the Pentagram. Don't draw it, the figure is known to be magic.

21.5. The **Fibonacci spiral** is obtained by drawing quarter-circle arcs into squares of size F_n . Putting these arcs together produces an approximation of the **golden spiral**.

Problem H: What is $1 + 2 \cos(2\pi/5)$? Compute $1 + 2 \sin(\pi/10)$.

Problem I: The **triacontahedron** is a thirty-faced convex Catalan solid. Each of the 30 congruent rhombic faces is a **golden rhombus**. The coordinates of one of these rhombi is $A = (-\phi, 0, 0), B = (0, -1, 0), C = (\phi, 0, 0), D = (0, 1, 0)$. What is the surface area of the triacontahedron?

21.6. Some literature:

M. Livio, The golden ratio, Broadway books, 2002

G.B. Meisner, The golden ratio, Race Point, 2018.

N.N. Vorobev, Fibonacci numbers

R.A. Dunlap The Golden Ratio and Fibonacci Numbers

A.S. Posamentier, I. Lehmann The Fabulous Fibonacci Numbers

R. Knot, Fibonacci Numbers and the Golden Section, 2001

L.E. Sigler, Fibonacci's Liber Abaci, 2003

T. Koshy, Fibonacci and Lucas Numbers with Applications, Wiley, 2001

K. Devlin, The Man of Numbers, Fibonacci's Arithmetic Revolution, Bloomsbury, 2011

G.E.Bergum, A.N.Philippou, A.F.Horadam, Applications of Fibonacci Numbers, Springer, 1998

R.C. Johnson, Fibonacci Numbers and Matrices, 2016

R.Herz-Fischler, A mathematical History of the golden number, 1998

H. Walser, J. Pedersen, P. Hilton, The golden section, 2001

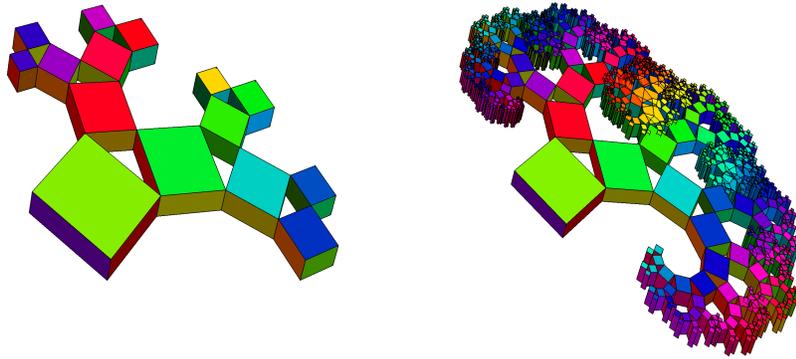


FIGURE 3. Stage 2 and 8 of the tree of Pythagoras with golden ratio.

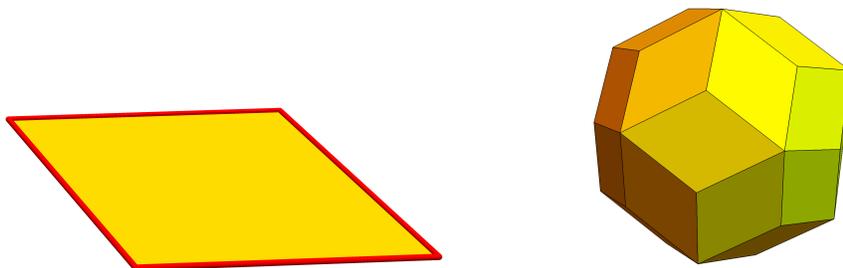


FIGURE 4. The golden rhombus and the triacontahedron.

HOMEWORK

Problem 21.1 a) The **sublime triangle** is an isosceles triangles with 36,72,72 degrees. Prove that the ratio between the long and short side is the golden mean. It is the reason why the triangle is also called the **golden triangle**.

b) The **golden Gnomon** is an isosceles triangle with angles 36,36,108 degrees. What is the ratio between the long and short side?

Problem 21.2 Let us experiment with determinants again and define $L(n)$ as the $n \times n$ matrix with -1 in the upper side diagonal and 1 in the diagonal and 1 in the lower side diagonal. For example,

$$L(7) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} .$$

Compute $\det(L(n))$ using the Laplace expansion and verify that $\det(L(n))/\det(L(n-1)) \rightarrow \phi$.

Problem 21.3 Verify that the regular octagon has width which compares to the base as the **silver ratio** $1 + \sqrt{2}$. Why is this number called the silver ratio? What is the third **metallic mean** satisfying $x^2 - 3x = 1$ called?

Problem 21.4 Find a picture of a plant in which some Fibonacci numbers are present. Either sketch it or print out that picture and write in the Fibonacci numbers.

Problem 21.5

a) How is the **lute of Pythagoras** related to the golden mean?

b) How is the **tree of Pythagoras** related to the golden mean?

Look up and draw both objects.