Unit 22: Stability

Lecture

22.1. A **linear dynamical system** is either a **discrete time dynamical system** $x(t+1) = Ax(t)$ or a **continuous time dynamical systems** $x'(t) = Ax(t)$. It is called **asymptotically stable** if for all initial conditions $x(0)$, the orbit $x(t)$ converges to the origin 0 as $t \to \infty$. The one-dimensional case is clear: the discrete time system $x(t+1) = \lambda x(t)$ has the solution $x(t) = \lambda^t x(0)$ and is asymptotically stable if and only if $|\lambda| < 1$. The continuous time system $x'(t) = \lambda x(t)$ has the solution $x(t) = e^{\lambda t} x(0)$. This is asymptotically stable if and only if $\lambda < 0$. If $\lambda = a + ib$, then $e^{(a+ib)t} = e^{at} e^{ibt}$ shows that asymptotic stability happens if $\mathrm{Re}(\lambda) = a < 0$.

22.2. Let us first discuss the stability for discrete dynamical systems $x(t+1) = Ax(t)$, where A is a $n \times n$ matrix.

22.3.

> **Theorem:** A discrete dynamical system $x(t+1) = Ax(t)$ is asymptotically stable if and only if all eigenvalues of A satisfy $|\lambda_j| < 1$.

Proof. (i) If A has an eigenbasis, this follows from the closed-form solution: assume $||\lambda_k| \leq \lambda < 1$. From $|x(t)| \leq |c_1||\lambda_1|^t + \cdots + |c_n||\lambda_n|^t \leq (\sum_i |c_i|)|\lambda|^t$, we see that the solution approaches 0 exponentially fast.
(ii) The general case needs the **Jordan normal form theorem** proven below which tells that every matrix A can be conjugated to $B + N$, where B is the diagonal matrix containing the eigenvalues and $N^n = 0$. We have now $(B+N)^t = B^t + B(n,1)B^{t-1}N + \cdots + B(n,n)B^{t-n}N^{n-1}$, where $B(n,k)$ are the Binomial coefficients. The eigenvalues of A are the same as the eigenvalues of B. By (i), we have $B^t \to 0$. So, also $A^t \to 0$. $\square$

22.4. In the case of **continuous time dynamical system** $x'(t) = Ax(t)$. the complex eigenvalues will later play an important role but they are also important for discrete dynamical systems.

22.5.

> **Theorem:** A continuous dynamical system is asymptotically stable if and only if all eigenvalues satisfy $\mathrm{Re}(\lambda_j) < 0$.

Proof. We can see this as a discrete time dynamical system with time step $U = e^A$ because the solution e^{At} can be written as U^t. We need therefore that $e^{\lambda_j(U)} < 1$. Which is equivalent to $\text{Re}(\lambda_j) < 0$. $\square$

22.6. A $m \times m$ matrix J is a **Jordan block**, if $Je_1 = \lambda e_1$, and $Je_k = \lambda e_k + e_{k+1}$ for $k = 2, \dots, m$. A matrix is A in **Jordan normal form** if it is block diagonal, where each block is a **Jordan block**. The shear matrix $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is an example of a 2×2 Jordan block.

Theorem: Every $A \in M(n,n)$ is similar to $B \in M(n,n)$ in Jordan normal form.

Proof. A vector v is called a **generalized eigenvector** of the eigenvalue λ of A if $(A-\lambda)^m v = 0$ for some $m > 0$. The smallest integer m for which the relation holds is called the **eigenvector rank** of v. If the eigenvector rank is 1, then v is an actual eigenvector.

22.7. Take an eigenvalue λ of A and pick the maximal m for which there is a generalized eigenvector of rank m. This means that there is a vector v_m such that $(A-\lambda)^m v_m = 0$ but $(A-\lambda)^{m-1} v_m \neq 0$. Now define $v_k = (A-\lambda)^{m-k} v_m$. The vector $v_1 = (A-\lambda)^{m-1} v_m$ is an eigenvector of A because $(A-\lambda)v_1 = 0$. Because $(A-\lambda)v_{k+1} = v_k$ and $(A-\lambda)v_1 = 0$, on the space V spanned by $\mathcal{B} = \{v_1, v_2, \cdots, v_m\}$, the matrix is given by a Jordan block with 0 in the diagonal. This space V is called a **generalized eigenspace** of A. It is left invariant by A.

22.8. Take a generalized eigenspace V defined by an eigenvector v_1 and a generalized eigenspace W defined by another eigenvector w_1 not parallel to v_1. Claim: V and W have no common vector. Proof: Assume $x = \sum_{i=1}^m a_i v_i = \sum_{j=1}^l b_j w_j$. Then $(A-\lambda)x = \sum_{i=1}^m a_i v_{i-1} = \sum_{j=1}^l b_j w_{j-1}$. If $m \neq l$, and say $m < l$ then applying $(A-\lambda)$ $m-1$ times gives $a_m v_1 = b_{1+l-m} w_{1+l-m}$ but since v_1 is an eigenvector and v_{1-l-m} is not, this does not work. If $m = l$, then we end up with $a_m v_1 = b_m w_1$ which is not possible as v_1 and w_1 are not parallel unless $a_m = b_m = 0$. Now repeat the same argument but only apply $m-2$ times. This gives $a_{m-1} v_{m-1} = b_{m-1} w_{m-1}$ but this implies v_{m-1}, w_{m-1} are parallel showing again v_m is parallel to w_m We get $a_{m-1} = b_{m-1} = 0$ and eventually see that all $a_k = b_k = 0$.

22.9. The proof of the theorem for $n \times n$ matrices uses induction with respect to n. The case $n = 1$ is clear as a 1×1 matrix is already in Jordan normal form. Now assume that for every $k < n$, every $k \times k$ matrix is similar to a matrix in Jordan normal form. Take a generalized eigenvector v and build the Jordan normal block acting on the generalized eigenspace V. By the previous paragraph, we can find a basis such that $V^\perp$ is invariant. Using induction, there is a Jordan decomposition for A acting on the $V^\perp$. The matrix A has in this bases now a Jordan decomposition with an additional block. $\square$

22.10. This implies the **Cayley-Hamilton theorem**:

> **Theorem:** If p is the characteristic polynomial of A, then $p(A) = 0$.

Proof. It is enough to show this for a matrix in Jordan normal form for which the characteristic polynomial is λ^m. But $A^m = 0$. $\square$

Examples

22.11. You have a **checking account** of one thousand dollars and a **savings account** with one thousand dollars. Every day, you pay 0.001 percent to the bank for both accounts. But the deal is sweet: your checking account will get every day 1000 times the amount in the savings account. Will you get rich?

$$\begin{aligned} c(t+1) &= 0.999c_n + 1000s_n \\ s(t+1) &= 0.999s_n \ . \end{aligned}$$

We will discuss this in class.

22.12. For which constants a is the system $x(t+1) = Ax(t)$ stable?
a) $A = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$, b) $A = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}$, c) $A = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$.
Solution.
a) The trace is zero, the determinant is $-a^2$. We have stability if $|a| < 1$. You can also see this from the eigenvalues, $a, -a$.
b) Look at the trace-determinant plane. The trace is a, the determinant -1. This is nowhere inside the stability triangle so that the system is always unstable.
c) The eigenvalues are $0, 2a$. The system is stable if and only if $|2a| < 1$ which means $|a| < 1/2$.

22.13. In two dimensions, we can see asymptotic stability from the trace and determinant. The reason is that the characteristic polynomial and so the eigenvalues only need the trace and determinant.

> A two dimensional discrete dynamical system has asymptotic stability if and only if $(\mathrm{tr}(A), \det(A))$ is contained in the interior of the **stability triangle** bounded by the lines $\det(A) = 1, \det(A) = \mathrm{tr}(A) - 1$ and $\det(A) = -\mathrm{tr}(A) - 1$.

Proof: Write $T = \mathrm{tr}(A)/2, D = \det(A)$. If $|D| \geq 1$, there is no asymptotic stability. If $\lambda = T + \sqrt{T^2 - D} = \pm 1$, then $T^2 - D = (\pm 1 - T)^2$ and $D = 1 \pm 2T$. For $D \leq -1 + |2T|$ we have a real eigenvalue ≥ 1. The conditions for stability is therefore $D > |2T| - 1$. It implies automatically $D > -1$ so that the triangle can be described shortly as $\boxed{|\mathrm{tr}(A)| - 1 < \det(A) < 1}$.

> For a two-dimensional continuous dynamical system we have asymptotic stability if and only if $\mathrm{tr}(A) < 0$ and $\det(A) > 0$.

Homework

This homework is due on Tuesday, 4/02/2019.

Problem 22.1: Determine whether the matrix is stable for the discrete dynamical system or for the continuous dynamical system or for both: a) $A = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix}$, b) $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, c) $C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Problem 22.2: True or false? We say A is stable if the origin $\vec{0}$ is asymptotically stable for $x(t+1) = A(x(t))$. Give short explanations:
a) 1 is stable. b) 0 matrix is stable.
c) a horizontal shear is stable d) a reflection matrix is stable.
e) A is stable if and only if A^T is stable.
f) A is stable if and only if A^{-1} is stable.
g) A is stable if and only if $A+1$ is stable.
h) A is stable if and only if A^2 is stable.
i) A is stable if $A^2 = 0$. j) A is unstable if $A^2 = A$.
k) A is stable if A is diagonalizable.

Problem 22.3: a) Check the Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$. b) Ditto for a rotation dilation matrix defined by a, b.

Problem 22.4: For which real values k does the drawing rule

$$\begin{aligned} x(t+1) &= x(t) - ky(t) \\ y(t+1) &= y(t) + kx(t+1) \end{aligned}$$

produce trajectories which are ellipses? Write first the system as a discrete dynamical system $v(t+1) = Av(t)$, then look for k values for which the eigenvalues satisfy $|\lambda_k| = 1$.

Problem 22.5: Find the eigenvalues of

$$A = \begin{bmatrix} 0 & a & b & c & 0 & 0 \\ 0 & 0 & a & b & c & 0 \\ 0 & 0 & 0 & a & b & c \\ c & 0 & 0 & 0 & a & b \\ b & c & 0 & 0 & 0 & a \\ a & b & c & 0 & 0 & 0 \end{bmatrix}$$

Where a, b and c are arbitrary constants. Verify that the discrete dynamical system is stable for $|a| + |b| + |c| < 1$.