LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 24: Chaos

Seminar

24.1. Simple transformations can produce **chaotic orbits**. Let us experiment! Make sure your calculator is in the "Rad" mode, in which 2π radians mean 360 degrees. You can check the mode by computing $\cos(\pi)$. You should see the result -1.

FIGURE 1. Rotate the phone by 90 degrees to get the scientific calculator. Be in Rad mode!

Problem A: Push repetitively $\boxed{\cos}$. What do you observe?

Problem B: Don't clear and repeat pushing $\boxed{x^2}$. Observe.

Problem C: Repeat pushing $\boxed{\sqrt{x}}$. What limit do you get?

Problem D: Repeat pushing $\boxed{\sin}$ then $\boxed{1/x}$ button. Observe.

Problem E: Repeat pushing $\boxed{\tan}$ then $\boxed{1/x}$ button. Observe.

24.2. Experiments like what you just did have led **Mitchell Feigenbaum** to the discovery of **universality**. It is now part of **chaos theory**. Feigenbaum played with a calculator and looked for bifurcation points in the sequence of systems like $T(x) = c\sin(\pi x)$ on $[0,1]$. For small c, all orbits converge to a fixed point, then there will be an interval in which there is a periodic point of periodic 2, then 4 etc. This period doubling bifurcations will accumulate at a parameter point in a way which is independent of the map. It would be the same for the map $T(x) = cx(1-x)$ on $[0,1]$.

24.3. Given a **discrete time dynamical system** $x(t+1) = F(x(t))$ or a **continuous time dynamical system** $x'(t) = F(x(t))$, we say that the orbit $x(t)$ shows **sensitive dependence of initial conditions** if the map $U(t)$ which maps $x(0)$ to $x(t)$ has the property that $|dU(t)|$ grows exponentially in the sense that the **Lyapunov exponent** $\gamma(x) = \liminf_{t\to\infty}(1/t)\log|dU(t)|$ is positive.

Remember that for a matrix A, we defined $|A| = \mathrm{tr}(A^TA)$ which is the length of A if we look at the matrix as a vector. The liminf γ means that $(1/t)\log|dU(t)| \geq \gamma$ for arbitrarily large t and that we have taken the largest γ with that property. For example, $\liminf \sin(t) = -1$. It is a basic fact that if a function $f(t)$ is bounded, then $\liminf_t f(t)$ exists. It follows that if F is differentiable and $|dF|$ is globally bounded, then the Lyaponuov exponent exists. This does not mean that the limit exists. It is a liminf only.

24.4. Look at the system on the square $X = \mathbb{T}^2 = [0,1) \times [0,1)$ with $T(x,y) = (2x+y, x+y) \bmod 1$. The notation $u \bmod 1$ is a number in $[0,1)$ defined by removing the integer part. for example $1.73 \bmod 1 = 0.73$. Mathematically X is the **two-dimensional torus** which is the square on which the left and right side are identified and where the bottom side and top side are identified too. The map is defined by a linear relation but we also wrap things around the torus. We have $dT = A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. The map T is called the **Arnold cat map**.

24.5. We say that a system shows **chaos** on a bounded subset X of the phase space if X is left invariant and the **metric entropy** $\mu(T) = \int_X \gamma(x)\, dV(x) > 0$. The subset X can be a bounded region of Euclidean space or then a bounded surface $S = r(R)$ which is left invariant by the system and where R is a parameter domain. In that case, $dV(x) = |dr(u)|du$ is the usual measure we take for example to compute length, area or volume.

24.6. A popular choice is the **torus** $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. It can be visualized as the unit cube $[0,1]\times[0,1]\times\cdots\times[0,1]$, where in each interval the left and right side are identified. The cat map $(x,y) \to (2x+y, x+y)$ is a transformation on the 2-torus $\mathbb{T}^2$. When iterating the map, we just discard any integer parts. For example $T(0.3, 0.5) = (0.1, 0.8)$.

Theorem: If $T(x) = Ax$ is given by a $n \times n$ matrix A with integer coefficients and the eigenvalues of A are all positive, then the corresponding map on the torus $\mathbb{T}^n$ shows chaos. The Lyapunov exponent is $\log|\lambda_1|$ where λ_1 is the largest eigenvalue of A.

24.7. In the case of the cat map, the eigenvalues of A are $(3 \pm \sqrt{5})/2$. The Lyapunov exponent is constant $\log((3+\sqrt{5})/2) = 0.962424\ldots$

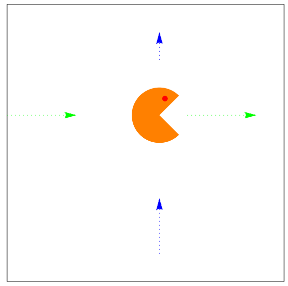

FIGURE 2. Pac-Man is played on a torus. When reaching the right wall, it comes in on the other side. When identifying the left and right side we get a finite piece of cylinder. Now identifying the two ends produces a torus.

FIGURE 3. The cat map $T(x, y) = (2x + y, x + y)$ on $\mathbb{T}^2$. Vladimir Arnold visualized the map with a cat subjected to the dynamics. In the middle we see a set evolve under the Standard map $T(x, y) = (2x - y + 4\sin(x), x)$. To the right, some orbits of the standard map.

24.8. There are many open problems in chaos theory. Also, when dealing with differential equations. For example, nobody has shown that a system like the ABC system shows chaos in the above sense. It is a differential equation on the three dimensional torus.

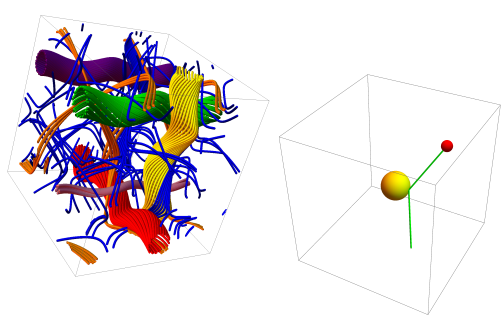

FIGURE 4. The ABC system $x' = A\sin(z) + C\cos(y)$, $y' = B\sin(x) + A\cos(z)$, $z' = C\sin(y) + B\cos(x)$, with $A = 1, B = 1.5, C = 1.5$ is a differential equation on the three dimensional torus. The Sinai billiard to the right too. A ball is a fixed obstacle at which a gas particle reflects with the billiard law. This system is known to be chaotic. It is a model for chaos of a Boltzman gas in the kinetic theory of gases.

Homework

Problem 24.1 a) If $T(x) = 10x \bmod 1$, find $T^5(\pi)$.
b) What is the entropy of the map $T(x) = 10x \bmod 1$?
c) What is the entropy of the Fibonacci map $T(x, y) = (x + y, x) \text{mod } 1$?

Problem 24.2 Experiment with the map $f(x) = c\cos(e^x)$). There is a threshold c_0 such that for $c > c_0$ is chaotic. Find it approximately.

```
T[x_] := 1.9 Cos[Exp[x]];
ListPlot[NestList[T, 0.3, 100000],PlotRange -> {-1, 1}]
```

Problem 24.3 Find your own two key combination on the calculator which produces chaos. We have seen that the combination $\boxed{1/x}$ and $\boxed{\tan}$ works. There are more. If you have found a good one, plot the orbit using the above mathematica code and submit that.

Problem 24.4 Use the page http://www.dynamical-systems.org/twist/Entropy/Entropy.html from 20 years ago, to compute numerically the entropy of the standard map $T(x, y) = (2x-y+ksin(x), x)$ on the torus with $k = 5.5$. Or use the following Mathematica code to compute an estimate for $k = 5.5$.

```
(* Open Standard map entropy conjecture Entropy(g) >= log(g/2)      *)
pi=N[Pi]; R:=N[2*pi*Random[]];
T[{x_,y_},g_]:=Mod[{2*x-y+g*Sin[x],x},2*pi];   (* Chirikov map       *)
A[{x_,y_},g_]:={{2+g*Cos[x],-1},{1,0}};        (* Jacobian matrix    *)
Lya[{x_,y_},g_,n_]:=Module[{B=B={{1,0},{0,1}},p={x,y},t=0,a},
 Do[B=A[p,g].B;p=T[p,g];a=Abs[B[[1,1]]];If[a>1,B=B/a;t+=Log[a]],{n}];
  (t+Log[Sqrt[Tr[Transpose[B].B]]])/n];         (* Lyapunov exponent *)
entropy[g_,n_,m_]:={Sum[Lya[{R,R},g,n],{m}]/m,N[Log[g/2]]};
entropy[4.0,1000,100] (* g=4, got 100 orbits of length 1000          *)
```

Problem 24.5 A piecewise smooth convex closed curve in the plane defines a **billiard dynamical system** in which the billiard ball reflects at the boundary. An example is a rectangle or a circle. These are examples of billiards which do not show chaos. Look whether you find something about chaotic billiards. There are tables which produce chaos. Find one, then use ruler and compass (or a good eye) to plot some orbits in that table.