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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total:

Unit 28: Second Hourly Practice

Welcome to the second hourly. It will take place on April 9, 2019 at 9:00 AM sharp in Hall D. Please fill out your name in the box above.

- You only need this booklet and something to write. Please stow away any other material and electronic devices. Remember the honor code.
- Please write neatly and give details. We want to see details, even if the answer should be obvious to you.
- Try to answer the question on the same page. There is also space on the back of each page.
- If you finish a problem somewhere else, please indicate on the problem page so that we find it.
- You have 75 minutes for this hourly.

PROBLEMS

Problem 28P.1 (10 points):

- a) Prove, using one of the theorems we have seen in the course that any matrix of the form $A - A^T$ is diagonalizable if A is an arbitrary real $n \times n$ matrix.
- b) Prove, using one of the theorems we have seen in the course that any matrix of the form AA^T is diagonalizable if A is an arbitrary real $n \times n$ matrix.
- c) The matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

has the characteristic polynomial $-\lambda^5 + 1 = 0$. What does the theorem of Cayley-Hamilton state?

Solution:

- a) $A - A^T$ is normal. By the spectral theorem it is diagonalizable.
- b) AA^T is symmetric. By the spectral theorem, it is diagonalizable.
- c) When plugging in the matrix in the characteristic polynomial we get zero: $A^5 - 1 = 0$.

Problem 28P.2 (10 points):

Find the characteristic polynomial and the eigenvalues of the following matrices:

- a) $\begin{bmatrix} -1 & 3 \\ 4 & 0 \end{bmatrix}$
 b) $\begin{bmatrix} 1 & 4 & 3 \\ 1 & 4 & 3 \\ 3 & 4 & 1 \end{bmatrix}$
 c) $\begin{bmatrix} 2 & 3 & 1 & 4 \\ 0 & 5 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 6 \end{bmatrix}$.

Solution:

a) The eigenvalues are $-4, 3$.

b) The eigenvalues are $8, 0, -2$ (sum of row entries, 0 because not invertible, trace)

c) $2, 5, 2, 6$ are the eigenvalues (diagonal entries)

The characteristic polynomial in each case is given as a product $(\lambda_1 - \lambda)\dots(\lambda_n - \lambda)$

Problem 28P.3 (10 points):

Check the boxes which apply for all matrices of the type:

	invertible	diagonalizable	symmetric	real eigenvalues
Projection matrix				
Shear matrix				
Rotation matrix				
Reflection matrix				
$A^2 = 0, A \neq 0$				
Diagonal matrix				

Solution:

	invertible	diagonalizable	symmetric	real eigenvalues
Projection matrix		x	x	x
Shear matrix	x			x
Rotation matrix	x			
Reflection matrix	x	x	x	x
$A^2 = 0, A \neq 0$				x
Diagonal matrix		x	x	x

Problem 28P.4 (10 points, each sub problem is 2 points):

a) (5 points) Find the determinant of the following matrix. You have to give reasoning!

$$A = \begin{bmatrix} 22 & 2 & 2 & 2 & 2 \\ 2 & 22 & 2 & 2 & 2 \\ 2 & 2 & 22 & 2 & 2 \\ 2 & 2 & 2 & 22 & 2 \\ 2 & 2 & 2 & 2 & 22 \end{bmatrix}$$

b) (5 points) Find an eigenbasis of A . It does not have to be orthonormal.

Solution:

a) $B = A - 20$ has the eigenvalues $10, 0, 0, 0, 0$ so that A has eigenvalues $30, 20, 20, 20, 20$. The determinant is the product of the eigenvalues which is 4800000 .

b) One is $[1, 1, 1, 1, 1]$ as the sum of the row entries is constant. The others span the kernel of B which gives $[-1, 1, 0, 0, 0], [-1, 0, 1, 0, 0], [-1, 0, 0, 1, 0], [-1, 0, 0, 0, 1]$. The eigenbasis is $\mathcal{B} = \{[1, 1, 1, 1, 1], [-1, 1, 0, 0, 0], [-1, 0, 1, 0, 0], [-1, 0, 0, 1, 0], [-1, 0, 0, 0, 1]\}$.

Problem 28P.5 (10 points):

Find the possibly complex eigenvalues for the following matrices:

a) (2 points)

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

b) (2 points)

$$B = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

c) (2 points)

$$C = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

d) (2 points)

$$D = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$$

e) (2 points)

$$E = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \end{bmatrix}.$$

Solution:

- a) 3 with eigenvector e_1 and 3 with eigenvector e_2 . Note that any non-zero vector is an eigenvector as the matrix is just 3 times the identity matrix.
 b) 5 with eigenvector $[-1, 2]^T$ and -5 with eigenvector $[2, 1]^T$. This is a reflection dilation.
 c) $3 \pm i4$ with eigenvectors $[\pm i, 1]$. d) 7 with eigenvector $[1, 1]$ and -1 with eigenvector $[-1, 1]^T$.
 e) $\lambda_k = 2e^{2\pi ik/4}$ with $k = 1, 2, 3, 4$ and $v_k = [1, \lambda_k, \lambda_k^2, \lambda_k^3]^T$.

Problem 28P.6 (10 points):

Find a closed-form solution of the recursion

$$x_{n+1} = 3x_n - 2x_{n-1}$$

with $x(1) = 1, x(0) = 4$ and determine with the system is stable. Make sure to write this first as a discrete dynamical system $(x(t+1), x(t)) = A(x(t), x(t-1))$ then use the initial condition $[1, 4]$.

Solution:

The matrix is $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$. It has eigenvalues $\lambda_1 = 1$ with eigenvector $[1, 1]$ and $\lambda_2 = 2$ with eigenvector $[2, 1]$. Now write $[1, 4] = c_1[1, 1] + c_2[2, 1]$. The solution is $c_1 = 7, c_2 = -3$. The closed-form solution is

$$[x(t), y(t)] = 1^t 7[1, 1] + 2^t (-3)[2, 1].$$

Problem 28P.7 (10 points):

When we try to find a closed-form solution of the system

$$\begin{aligned} x' &= x - 2y \\ y' &= 2x - 3y \end{aligned}$$

with $x(0) = 2, y(0) = 1$ we run into trouble. Outline why, then tell how we still can find a closed-form solution. You don't have to do that explicitly. Just show to which matrix A of the system is similar to. What we want to know is whether the system is stable.

Solution:

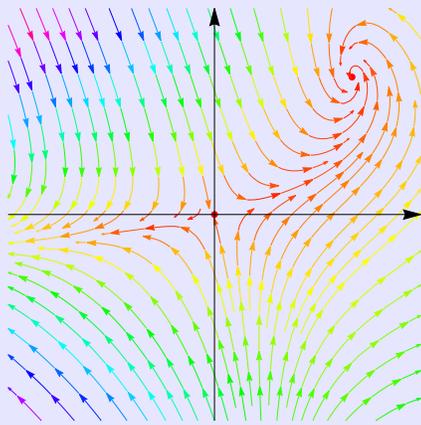
The eigenvalues of the matrix are -1 and -1 . There is no eigenbasis! The Jordan normal form theorem shows that A is conjugated to $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$. Since the real part of the eigenvalues are negative, the system is still stable.

Problem 28P.8 (10 points):

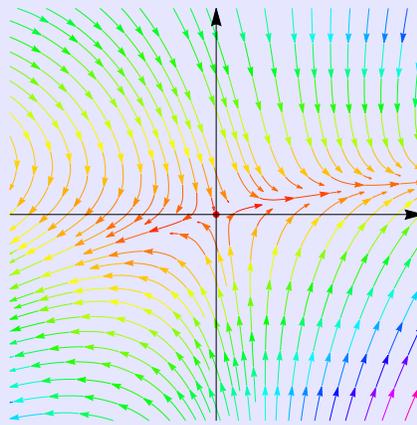
We consider the nonlinear system of differential equations

$$\begin{aligned}\frac{d}{dt}x &= x + y - xy \\ \frac{d}{dt}y &= x - 3y + xy.\end{aligned}$$

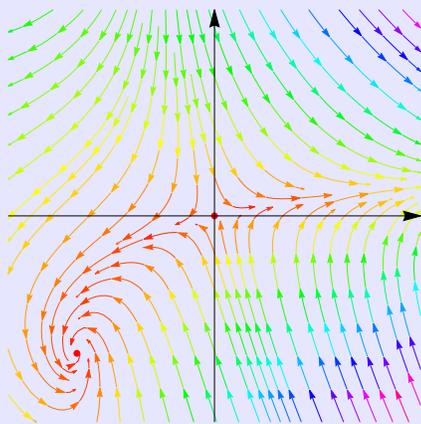
- (2 points) Find the equilibrium points.
- (3 points) Find the Jacobian matrix at each equilibrium point.
- (3 points) Use the Jacobean matrix at an equilibrium to determine for each equilibrium point whether it is stable or not.
- (2 points) Which of the diagrams A-D is the phase portrait of the system above?



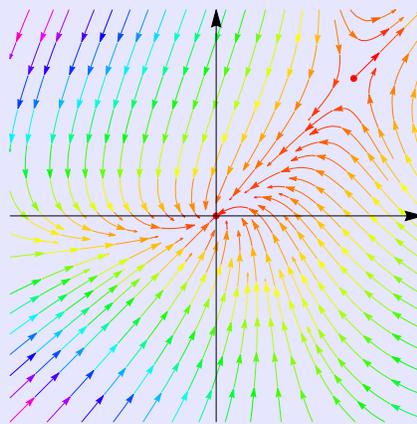
A



B



C



D

Solution:

a) The null clines are $y=x/(x-1)$ and $y=x/(x-3)$. They intersect at the equilibrium points which are $(0, 0)$ and $(2, 2)$.

b) The Jacobian matrices are

$$\begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 3 & -1 \end{bmatrix}.$$

c) The first equilibrium point is unstable with one positive $-1 + \sqrt{5}$ and one negative eigenvalue $-1 - \sqrt{5}$, the second is stable with spirals as the eigenvalues are $-1 + \sqrt{3}i$ and $-1 - \sqrt{3}i$.

d) We have phase portrait A.

Problem 28P.9 (10 points):

(4 points) a) (5 points) Assume T is a transformation on $C^\infty(\mathbb{T})$, the linear space of 2π -periodic functions on the real line. Which transformations are linear?

Transformation	Check if linear
$Tf(x) = f(x + 1)$	
$Tf(x) = f(\cos(x))$	
$Tf(x) = f'(x + \cos(x))$	
$Tf(x) = f(f(x) \cos(x))$	
$Tf(x) = \cos(f(x))$	
$Tf(x) = \cos(x) + f(x)$	

(5 points) The rest are knowledge questions which do not need any reasoning.

a) Which mathematician was a hermite in the later part of his life?

b) What is the entropy of the map $T(x) = 22x$ on $\mathbb{R}^1/\mathbb{Z}^1$.

c) Assume a 5×5 matrix has 2 Jordan blocks. Is it diagonalizable?

d) Find the eigenvector of the eigenvalue problem $Df = 3f$.

e) How big is the dimension of the solution space $(D^5 + D^3 + D)f = 0$?

Solution:

- the first 3 are linear, the last 3 not. a) Grothendieck
 b) $\ln(22)$
 c) no, it would need 5 Jordan blocks of size 1
 d) $\exp(3x)$
 e) 5 dimensional

Problem 28P.10 (10 points):

Find the general solution to the following differential equations:

- a) (1 point)

$$f'(t) = 1/(t + 1)$$

- b) (1 point)

$$f''(t) = e^t + t$$

- c) (2 points)

$$f''(t) + f(t) = t + 2$$

- d) (2 points)

$$f''(t) - 2f'(t) + f(t) = e^t$$

- e) (2 points)

$$f''(t) - f(t) = e^t + \sin(t)$$

- f) (2 points)

$$f''(t) - f(t) = e^{-3t}$$

Solution:

- a) $\log(t + 1) + C$
 b) $e^t + t^3/6 + C_1t + C_2$.
 c) $C_1 \cos(t) + C_2 \sin(t) + t + 2$
 d) $C_1e^t + C_2te^t + t^2e^t/2$
 e) $C_1e^t + C_2e^{-t} + te^t/2 - 1/2 \sin(t)$
 f) $C_1e^t + C_2e^{-t} + e^{-3t}/8$