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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total :

Unit 28: Second Hourly

Welcome to the second hourly. Please don't get started yet. We start all together at 9:00 AM. You can already fill out your name in the box above. Then grab some cereal in the beautiful kitchen (a Povray scene using code of Jaime Vives Piqueres from 2004).

- You only need this booklet and something to write. Please stow away any other material and electronic devices. Remember the honor code.
- Please write neatly and give details. Except when stated otherwise, we want to see details, even if the answer should be obvious to you.
- Try to answer the question on the same page. There is also space on the back of each page and at the end.
- If you finish a problem somewhere else, please indicate on the problem page so that we find it. Make sure we find additional work.
- You have 75 minutes for this hourly.

Problems

Problem 28.1 (10 points):
a) (3 points) What basic fundamental theorem in mathematics is involved to prove that the sum of the algebraic multiplicities of a $n \times n$ matrix is equal to n?
Name of the theorem: (1 point)

State the theorem (1 point) and tell why it implies the statement (1 point) .

b) (3 points) What theorem in linear algebra implies that the sum of the geometric multiplicities of an **orthogonal** $n \times n$ matrix is n so that A is diagonalizable over the complex numbers?
Name of the theorem: (1 point)

State the theorem (1 point) and why does the theorem imply the statement? (1 point)

c) (4 points) What theorem mentioned in this course assures that **any matrix** (not only diagonalizable ones) with eigenvalues 0 or 1 is similar to a matrix in which every entry is 0 or 1.
Name of the theorem: (1 point)

State the theorem (2 points) and why does the theorem imply the statement? (1 point)

Problem 28.2 (10 points):
Match the following matrices with the sets of eigenvalues. You are told that there is a unique match. It is not always necessary to compute all the eigenvalues to do so. You have to give a reason although for each choice (one reason could be that it is the last possible match as you are told there is an exact match). Two points for each sub problem.

Enter 1-5	The matrix
	$A = \begin{bmatrix} -1 & -2 & 8 \\ -7 & -3 & 19 \\ -3 & -2 & 10 \end{bmatrix}$
	$A = \begin{bmatrix} 5 & -9 & -7 \\ 0 & 5 & 2 \\ 0 & 0 & 6 \end{bmatrix}$
	$A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
	$A = \begin{bmatrix} 5 & -6 & 0 \\ 6 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$
	$A = \begin{bmatrix} 13 & 11 & 13 \\ -2 & -1 & -2 \\ -8 & -7 & -8 \end{bmatrix}$

1) $\{3, 2, 1\}$.

2) $\{1, 0, 3\}$.

3) $\{6, 5, 5\}$.

4) $\{1, i, -i\}$.

5) $\{5 + 6i, 5 - 6i, 5\}$.

Problem 28.3 (10 points):
Which of the following matrices are diagonalizable?
If it is not diagonalizable, tell why. If it is, write down the diagonal matrix B it is conjugated to.

a) (2 points) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

b) (2 points) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

c) (2 points) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

d) (2 points) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

e) (2 points) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$.

Problem 28.4 (10 points, each sub problem is 2 points):
a) (4 points) Fill in $\leq, =, \geq$ so that the statement is true for an arbitrary real $n \times n$ matrix A. The "number of eigenvalues" is the sum of all algebraic multiplicities of all eigenvalues. No justifications are needed in this problem.

The algebraic multiplicity of an eigenvalue of A is		its geometric multiplicity.
The number of $\mathbb{C}$ eigenvalues of A is		n.
The number of $\mathbb{R}$ eigenvalues of A is		n.
The rank of A is		the number of nonzero eigenvalues of A

b) (2 points) Give an example of a normal 3×3 matrix A, which is not symmetric.

A =

c) (2 points) Give an example of a real 2×2 matrix B which has eigenvalues $6 + 7i$ and $6 - 7i$.

B =

d) (2 points) Give an example of a real 3×3 matrix C which is not diagonalizable.

C =

Problem 28.5 (10 points):
The following 6 matrices can be grouped into 3 pairs of similar transformations. Find these three pairs and justify in each case why the matrices are similar.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Group 1:

Why are they similar?

Group 2:

Why are they similar?

Group 3:

Why are they similar?

Problem 28.6 (10 points):
a) (4 points) Find the eigenvalues λ_1, λ_2 and eigenvectors v_1, v_2 of the matrix

$$A = \begin{bmatrix} 9 & 1 \\ 2 & 8 \end{bmatrix} .$$

b) (6 points) Write down a closed-form solution for the discrete dynamical system

$$\begin{aligned} x(t+1) &= 9x(t) + y(t) \\ y(t+1) &= 2x(t) + 8y(t) \end{aligned}$$

for which $x(0) = 2$, $y(0) = -1$.

Problem 28.7 (10 points):
a) (6 points) Find a closed-form solution to the system

$$\begin{aligned} x'(t) &= 9x(t) + y(t) \\ y'(t) &= 2x(t) + 8y(t) , \end{aligned}$$

for which $x(0) = 2$, $y(0) = -1$.

b) (4 points) Determine the stability of the linear system of differential equations:

$$\begin{aligned} x'(t) &= x(t) - 7y(t) - 9z(t) \\ y'(t) &= x(t) - 2y(t) + z(t) \\ z'(t) &= x(t) + 5y(t) + z(t) . \end{aligned}$$

Problem 28.8 (10 points):
We consider the nonlinear system of differential equations

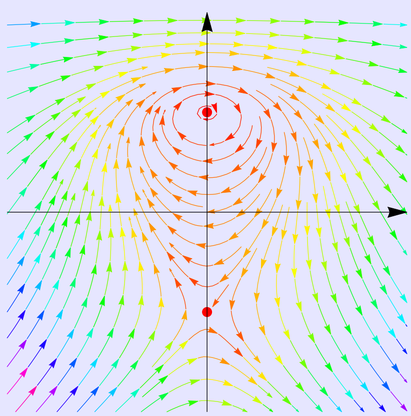
$$\begin{aligned} \frac{d}{dt}x &= x^2 + y^2 - 1 \\ \frac{d}{dt}y &= xy - 2x \ . \end{aligned}$$

a) (2 points) Find the nullclines and equilibrium points.

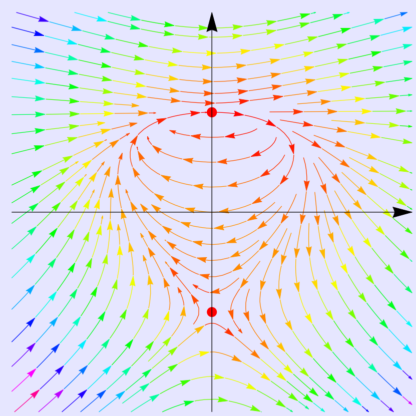
b) (3 points) Find the Jacobian matrix at each equilibrium point.

c) (3 points) Use the Jacobean matrix at an equilibrium to determine for each equilibrium point whether it is stable or not.

d) (2 points) Which of the diagrams A-D is the phase portrait of the system above? **Draw the nullclines and equilibrium points into the portrait!**



A



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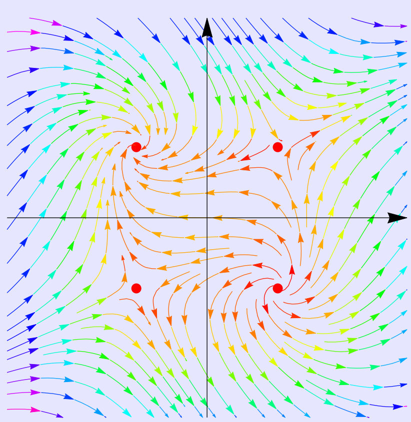

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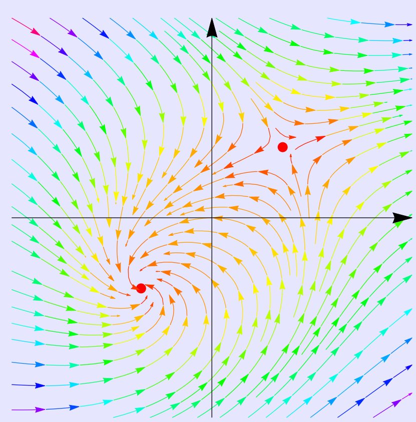

D

Problem 28.9 (10 points):
Which of the following are linear spaces? If the space is a linear space we need to see you checked its properties. If it is not, we want to see why it is not linear.

a) (2 points) The space of smooth functions f satisfying $f(x) = x + \sin(f(x))$.

b) (2 points) The space of smooth functions f satisfying $f(x) \geq -100$.

c) (2 points) The space of smooth functions f satisfying $f(x) + f(x)\sin(x) = 0$.

Which of the following are linear operators? If it is a linear operator, we need to see you have checked its properties. If it is not, we want to see a reason why it fails to be linear.

d) (2 points) The operator $T(f)(x) = f'(\cos(x))\sin(x)$.

e) (2 points) The operator $T(f)(x) = e^{f(x)} - 1$.

Problem 28.10 (10 points):
Cookbook or operator method. It is your choice! But we want to see protocol and steps, not just the answer!

a) (2 points) Find the general solution of the system $f''' = 24t$.

b) (2 points) Find the general solution of the system $f'' + 9f = 1$.

c) (3 points) Find the general solution of the system $f'' - 4f = 2t$.

d) (3 points) Find the general solution of the system $f'' + 10f' + 16f = 2t$.