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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total :

Unit 28: Second Hourly

Welcome to the second hourly. Please don't get started yet. We start all together at 9:00 AM. You can already fill out your name in the box above. Then grab some cereal in the beautiful kitchen (a Povray scene using code of Jaime Vives Piqueres from 2004).

- You only need this booklet and something to write. Please stow away any other material and electronic devices. Remember the honor code.
- Please write neatly and give details. Except when stated otherwise, we want to see details, even if the answer should be obvious to you.
- Try to answer the question on the same page. There is also space on the back of each page and at the end.
- If you finish a problem somewhere else, please indicate on the problem page so that we find it. Make sure we find additional work.
- You have 75 minutes for this hourly.

Problems

Problem 28.1 (10 points):

a) (3 points) What basic fundamental theorem in mathematics is involved to prove that the sum of the algebraic multiplicities of a $n \times n$ matrix is equal to n?

Name of the theorem: (1 point)

State the theorem (1 point) and tell why it implies the statement (1 point) .

b) (3 points) What theorem in linear algebra implies that the sum of the geometric multiplicities of an **orthogonal** $n \times n$ matrix is n so that A is diagonalizable over the complex numbers?

Name of the theorem: (1 point)

State the theorem (1 point) and why does the theorem imply the statement? (1 point)

c) (4 points) What theorem mentioned in this course assures that **any matrix** (not only diagonalizable ones) with eigenvalues 0 or 1 is similar to a matrix in which every entry is 0 or 1.

Name of the theorem: (1 point)

State the theorem (2 points) and why does the theorem imply the statement? (1 point)

Solution:
a) The fundamental theorem of algebra. Since the characteristic polynomial of a $n \times n$ matrix is a polynomial of degree n and every root is an eigenvalue, the theorem implies that there are n eigenvalues.
b) The spectral theorem for normal matrices. Every orthogonal matrix is normal because $A^T A = AA^T = 1$. The theorem assures that there is an eigenbasis.
c) The Jordan Normal form theorem. As every jordan block now has 1 or 0 in the diagonal and 1 in the super diagonal, the entire Jordan normal form B of A is a $0 - 1$ matrix.

Problem 28.2 (10 points):
Match the following matrices with the sets of eigenvalues. You are told that there is a unique match. It is not always necessary to compute all the eigenvalues to do so. You have to give a reason although for each choice (one reason could be that it is the last possible match as you are told there is an exact match). Two points for each sub problem.

Enter 1-5	The matrix
	$A = \begin{bmatrix} -1 & -2 & 8 \\ -7 & -3 & 19 \\ -3 & -2 & 10 \end{bmatrix}$
	$A = \begin{bmatrix} 5 & -9 & -7 \\ 0 & 5 & 2 \\ 0 & 0 & 6 \end{bmatrix}$
	$A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
	$A = \begin{bmatrix} 5 & -6 & 0 \\ 6 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$
	$A = \begin{bmatrix} 13 & 11 & 13 \\ -2 & -1 & -2 \\ -8 & -7 & -8 \end{bmatrix}$

1) $\{3, 2, 1\}$.

2) $\{1, 0, 3\}$.

3) $\{6, 5, 5\}$.

4) $\{1, i, -i\}$.

5) $\{5 + 6i, 5 - 6i, 5\}$.

Solution:
The trace determines the match already. 1,3,4,5,2.

Problem 28.3 (10 points):
Which of the following matrices are diagonalizable?
If it is not diagonalizable, tell why. If it is, write down the diagonal matrix B it is conjugated to.

a) (2 points) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

b) (2 points) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

c) (2 points) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

d) (2 points) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

e) (2 points) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$.

Solution:
a) no, geometric multiplicity of the eigenvalue 1 is only 1.
b) yes, this is a symmetric matrix with eigenvalues 4,0,0,0. B is the diagonal matrix with diagonal entries 4,0,0,0.
c) yes, this is a matrix with simple spectrum. B is the diagonal matrix with entries 1,2,3,4.
d) no, the geometric multiplicity of the eigenvalue 0 is only 3 and not 4 which would be needed to have an eigenbasis.
e) yes, this is a symmetric matrix. There are two eigenvalues 0. Since the remaining eigenvalues solve a quadratic equation they are (the trace is 1) of the form $1/2 + a/2$ and $1/2 - a/2$, where a is some square root. (it actually is $a = \sqrt{117}$).

Problem 28.4 (10 points, each sub problem is 2 points):
a) (4 points) Fill in $\leq, =, \geq$ so that the statement is true for an arbitrary real $n \times n$ matrix A. The "number of eigenvalues" is the sum of all algebraic multiplicities of all eigenvalues. No justifications are needed in this problem.

The algebraic multiplicity of an eigenvalue of A is		its geometric multiplicity.
The number of $\mathbb{C}$ eigenvalues of A is		n.
The number of $\mathbb{R}$ eigenvalues of A is		n.
The rank of A is		the number of nonzero eigenvalues of A

b) (2 points) Give an example of a normal 3×3 matrix A, which is not symmetric.

A =

c) (2 points) Give an example of a real 2×2 matrix B which has eigenvalues $6 + 7i$ and $6 - 7i$.

B =

d) (2 points) Give an example of a real 3×3 matrix C which is not diagonalizable.

C =

Solution:
a) $\geq$ because in general, geometric multiplicities are smaller or equal than algebraic multiplicities.
$=$ by the fundamental theorem of calculus.
$\leq$ because some eigenvalues could be complex
$\geq$ by the rank-nullity theorem. The magic matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is and example, where the rank is larger than the number of nonzero eigenvalues of A.
b) Take a rotation-dilation matrix with $a = 6$ and $b = 7$.
c) Take an Jordan matrix with some eigenvalue.

Problem 28.5 (10 points):
The following 6 matrices can be grouped into 3 pairs of similar transformations. Find these three pairs and justify in each case why the matrices are similar.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$
$$D = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Group 1:

Why are they similar?

Group 2:

Why are they similar?

Group 3:

Why are they similar?

Solution:
A and C both have eigenvalues $1, 0, 3$ so that they both are similar to the same diagonal matrix. B and D are transpose of each other. They are similar. E and F both have geometric multiplicity 2 for the eigenvalues $\lambda_1 = 1, \lambda_2 = 1$ and another common eigenvalue $\lambda_3 = 2$. Both matrices therefore are similar to a common diagonal matrix. So, they are similar.

Problem 28.6 (10 points):
a) (4 points) Find the eigenvalues λ_1, λ_2 and eigenvectors v_1, v_2 of the matrix

$$A = \begin{bmatrix} 9 & 1 \\ 2 & 8 \end{bmatrix} .$$

b) (6 points) Write down a closed-form solution for the discrete dynamical system

$$\begin{aligned} x(t+1) &= 9x(t) + y(t) \\ y(t+1) &= 2x(t) + 8y(t) \end{aligned}$$

for which $x(0) = 2$, $y(0) = -1$.

Solution:
a) Because the sum of the row entries are constant and equal to 10, the matrix has an eigenvalue $\boxed{10}$ with eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Because the sum of the diagonal elements is 17, which is the sum $\lambda_1 + \lambda_2$ of eigenvalues, we know that the other eigenvector is $\boxed{7}$. To compute the second eigenvector, we find the kernel of $A - I_2$ which is the kernel of

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} .$$

It is spanned by $\boxed{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}$.
We have now to write the initial condition as a sum of the eigenvectors

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} .$$

Solving this simple system of equations for the unknowns c_1, c_2 gives $c_1 = 1, c_2 = -1$ and so the closed-form solution

$$\vec{v}(t) = 1 \cdot 10^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \cdot 7^t \begin{bmatrix} -1 \\ 2 \end{bmatrix} .$$

Problem 28.7 (10 points):
a) (6 points) Find a closed-form solution to the system

$$\begin{aligned} x'(t) &= 9x(t) + y(t) \\ y'(t) &= 2x(t) + 8y(t) , \end{aligned}$$

for which $x(0) = 2$, $y(0) = -1$.

b) (4 points) Determine the stability of the linear system of differential equations:

$$\begin{aligned} x'(t) &= x(t) - 7y(t) - 9z(t) \\ y'(t) &= x(t) - 2y(t) + z(t) \\ z'(t) &= x(t) + 5y(t) + z(t) . \end{aligned}$$

Solution:
a) We can recycle the computation from the previous problem

$$\vec{v}(t) = 1 \cdot e^{10t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{7t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} .$$

b) The trace of the matrix A in that system $v' = Av$ is zero. This prevents all eigenvalues to have zero real part as the trace is the sum of the eigenvalues.

Problem 28.8 (10 points):
We consider the nonlinear system of differential equations

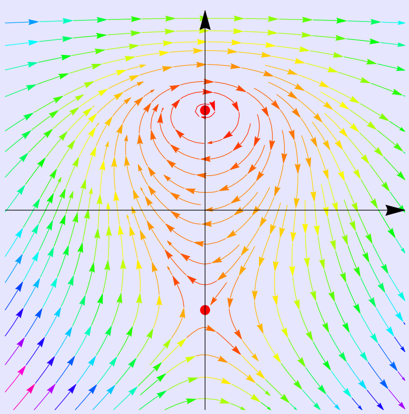
$$\begin{aligned} \frac{d}{dt}x &= x^2 + y^2 - 1 \\ \frac{d}{dt}y &= xy - 2x \ . \end{aligned}$$

a) (2 points) Find the nullclines and equilibrium points.

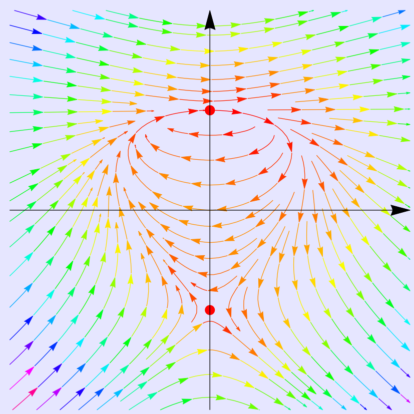
b) (3 points) Find the Jacobian matrix at each equilibrium point.

c) (3 points) Use the Jacobean matrix at an equilibrium to determine for each equilibrium point whether it is stable or not.

d) (2 points) Which of the diagrams A-D is the phase portrait of the system above? **Draw the nullclines and equilibrium points into the portrait!**



A



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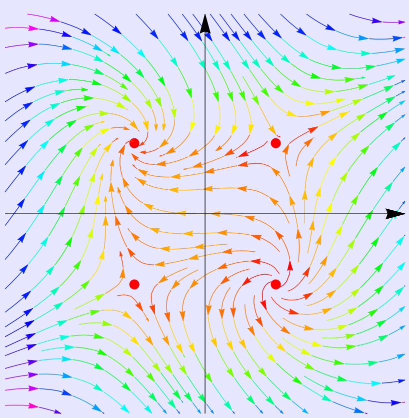

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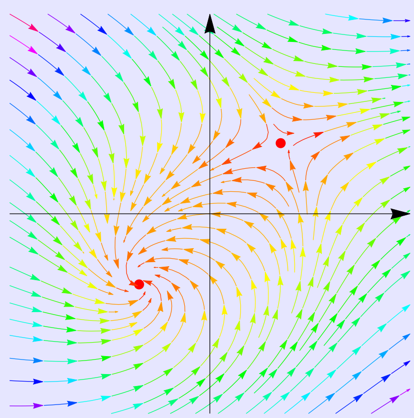

D

Solution:
a) The first null cline is a circle. The second is a union of the lines $x = 0, y = 2$. The intersection consists of two points $(0, 1)$ and $(0, -1)$.
b) The Jacobean matrix is $J = \begin{bmatrix} 2x & 2y \\ y-2 & -2x \end{bmatrix}$.
c) At the point $(0, 1)$ we have $J = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$ which has purely imaginary eigenvalues leading to circular elliptic point.
At the point $(0, -1)$ we get two real eigenvalues, one positive and one negative. This is a saddle.
d) A matches.

Problem 28.9 (10 points):
Which of the following are linear spaces? If the space is a linear space we need to see you checked its properties. If it is not, we want to see why it is not linear.

a) (2 points) The space of smooth functions f satisfying $f(x) = x + \sin(f(x))$.

b) (2 points) The space of smooth functions f satisfying $f(x) \geq -100$.

c) (2 points) The space of smooth functions f satisfying $f(x) + f(x)\sin(x) = 0$.

Which of the following are linear operators? If it is a linear operator, we need to see you have checked its properties. If it is not, we want to see a reason why it fails to be linear.

d) (2 points) The operator $T(f)(x) = f'(\cos(x))\sin(x)$.

e) (2 points) The operator $T(f)(x) = e^{f(x)} - 1$.

Solution:
a) no (f=0 fails)
b) no (scaling fails)
c) yes, check the three properties.
d) yes, check the three properties.
e) no, (scaling fails).

Problem 28.10 (10 points):
Cookbook or operator method. It is your choice! But we want to see protocol and steps, not just the answer!

a) (2 points) Find the general solution of the system $f''' = 24t$.

b) (2 points) Find the general solution of the system $f'' + 9f = 1$.

c) (3 points) Find the general solution of the system $f'' - 4f = 2t$.

d) (3 points) Find the general solution of the system $f'' + 10f' + 16f = 2t$.

Solution:
a) Here it is better to just integrate. $t^4 + C_1 t^2 + C_2 t + C_2$.
b) Cookbook $C_1 \cos(3t) + C_2 \sin(3t) + 1/9$.
c) Cookbook $C_1 e^{2t} + C_2 e^{-2t} - t/2$.
d) Cookbook $C_1 e^{-8t} + C_2 e^{-2t} + t/8 - 5/64$.

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