

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 30: Dirichlet's Proof

SEMINAR

30.1. The historical development of a mathematical topic not only gives background, it also illustrates the struggle in the search of a theory and often mirrors the difficulties which a modern student experiences, when learning the subject. We will spend some time in this proof seminar with the battle for proving that the Fourier series converges. Fourier claimed this to be true without justification. The proof given here came only later with Dirichlet. Be advised that this is not an easy proof as there are several bits and pieces which come together. It can be rewarding however to battle it and appreciate its difficulty.

30.2. Fourier theory is an important topic also because it was a turning point in understanding the concept of a function. The origin however came from a concrete problem. How can one describe the diffusion of heat? Fourier started to think about this problem while in the service of Napoleon during the campaign from 1798-1801 in Egypt. But the book on heat appeared only in 1822.

Problem A: Find the part in Fourier's book, where the Fourier series is introduced. The document which Fourier wrote in 1822 can be found on the website.

30.3. Before Fourier, one has seen functions only tied to their analytic expressions. Indeed one can see a Taylor series in the complex related to a Fourier series. Look at the series $\sum_{k=1}^n a_k z^k$. If we evaluate this on $|z| = 1$, we can plug in $z = e^{ix}$ and get $\sum_{k=1}^n a_k e^{ikx}$ which as a function of x has real and imaginary parts which are now given as Fourier series.

Problem B: What is the real part and imaginary part of $\sum_{k=1}^n a_k e^{ikx}$?

30.4. Because of this relation of Taylor series and Fourier series, one might think that Fourier series work only in analytic situations and fail for a function which has discontinuities. The surprise is that Fourier series can handle also discontinuous functions!

30.5. A major theorem about Fourier series deals with functions in \mathcal{X} , the space of piece-wise smooth functions on $[-\pi, \pi]$. It is a theorem due to **Peter Gustav Dirichlet** from 1829.

Theorem: The Fourier series of $f \in \mathcal{X}$ converges at every point of continuity. At discontinuities, it takes the middle value.

30.6.

Problem C: Try to understand as much as possible from the following proof of the theorem. During the proof seminar, you go through the main line of the proof. In the homework you flesh out some details.

[Part I] of the proof is a computation:

Let $S_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^n a_k \cos(kx) + b_k \sin(kx)$ denote the n 'th partial sum of f . By plugging in the formulas

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} f(y) dy \\ a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(ky) f(y) dy \\ b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(ky) f(y) dy, \end{aligned}$$

we get

$$S_n(x) = \int_{-\pi}^{\pi} D_n(x-y) f(y) dy$$

where

$$D_n(x-y) = \frac{1}{\pi} \left(\frac{1}{2} + \sum_{k=1}^n \cos(kx) \cos(ky) + \sin(kx) \sin(ky) \right) = \frac{1}{\pi} \left(\frac{1}{2} + \sum_{k=1}^n \cos(k(x-y)) \right).$$

30.7. [Part II]: the function $D_n(z)$ is called the **Dirichlet kernel**. The next lemma gives a simple closed-form expression for it, which does not involve a sum:

Lemma: $D_n(x) = \frac{1}{2\pi} \frac{\sin((n+\frac{1}{2})x)}{\sin(\frac{x}{2})}$.

Proof. There are three identities which tie things together. You verify them in the homework.

- a) $\frac{1}{\pi} \left(\frac{1}{2} + \sum_{k=1}^n \cos(kx) \right) = \frac{1}{2\pi} \sum_{k=-n}^n e^{ikx}$.
- b) $\frac{1}{2\pi} \sum_{k=-n}^n e^{ikx} = \frac{1}{2\pi} \frac{e^{i(n+1)x} - e^{-inx}}{e^{ix} - 1}$.
- c) $\frac{1}{2\pi} \frac{e^{i(n+1)x} - e^{-inx}}{e^{ix} - 1} = \frac{1}{2\pi} \frac{\sin((n+\frac{1}{2})x)}{\sin(x/2)}$.

This expression is understood by l'Hopital as $\frac{1}{\pi}(2n+1)$ for $x=0$. □

30.8. [Part III] of the proof gives a formula for the difference between $S_n(x)$ and $f(x)$. We see from the definition of $D_n(x) = \frac{1}{\pi} \left(\frac{1}{2} + \sum_{k=1}^n \cos(kx) \right)$ that $\int_{-\pi}^{\pi} D_n(y) dy = 1$. We can therefore write

$$f(x) = \int_{-\pi}^{\pi} D_n(y) f(x) dy.$$

By a change of variables,

$$S_n(x) = \int_{-\pi}^{\pi} D_n(x-y)f(y) dy = \int_{-\pi}^{\pi} D_n(y)f(x+y) dy .$$

Therefore,

$$S_n(x) - f(x) = \int_{-\pi}^{\pi} D_n(y)(f(x+y) - f(x)) dy .$$

30.9. Part IV introduces a function

$$F_x(y) = \frac{f(x+y) - f(x)}{2 \sin(y/2)} .$$

with the understanding that $F_x(0) = f'(x)$. Even so f is only piecewise continuous, the function $y \rightarrow F_x(y)$ is continuous and 2π -periodic. With this function, we can write

$$S_n(x) - f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} F_x(y) \sin((n + \frac{1}{2})y) dy .$$

30.10. Part V first uses a trig identity $\sin(n + \frac{1}{2})y = \cos(y/2) \sin(ny) + \sin(y/2) \cos(ny)$, then introduces two more continuous periodic functions

$$G_x(y) = F_x(y) \cos(y/2), H_x(y) = F_x(y) \sin(y/2) .$$

The n 'th Fourier coefficients of these functions are

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} G_x(y) \sin(ny) dy$$

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} H_x(y) \cos(ny) dy$$

Putting things together, we see

$$S_n(x) - f(x) = B_n + A_n .$$

30.11. Part VI is called the **Riemann-Lebesgue lemma**. It tells that for continuous functions g, h , we have in the limit $n \rightarrow \infty$

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos(nx) dx \rightarrow 0$$

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} h(x) \sin(nx) dx \rightarrow 0 .$$

we can then apply this in the above situation for $g(y) = G_x(y)$ or $h(y) = H_x(y)$ and see that $A_n \rightarrow 0$ and $B_n \rightarrow 0$. The Riemann-Lebesgue lemma follows from the **Bessel inequality**

$$a_0^2 + \sum_{k=1}^n a_k^2 + b_k^2 \leq \langle f, f \rangle = \|f\|^2 .$$

By the Pythagoras theorem, as $f(x) - S_n(x)$ and $S_n(x)$ are perpendicular, we have

$$\|f - S_n\|^2 + \|S_n\|^2 = \|f\|^2 .$$

This implies that $\|S_n\|^2 \leq \|f\|^2$ for all n as $\|S_n\|^2 = a_0^2 + \sum_{k=1}^n a_k^2 + b_k^2$ is a sum of non-negative terms, the infinite sum has to converge and a_n, b_n have to converge to zero.

HOMEWORK

This homework is due on Tuesday, 4/16/2019.

Problem 30.1: Verify

$$\frac{1}{\pi} \left(\frac{1}{2} + \sum_{k=1}^n \cos(kx) \right) = \frac{1}{2\pi} \sum_{k=-n}^n e^{ikx}.$$

Problem 30.2: Verify

$$\frac{1}{2\pi} \sum_{k=-n}^n e^{ikx} = \frac{1}{2\pi} \frac{e^{i(n+1)x} - e^{-inx}}{e^{ix} - 1}.$$

Problem 30.3: Verify

$$\frac{1}{2\pi} \frac{e^{i(n+1)x} - e^{-inx}}{e^{ix} - 1} = \frac{1}{2\pi} \frac{\sin((n + \frac{1}{2})x)}{\sin(x/2)}.$$

Problem 30.4: Verify that in the limit $x \rightarrow 0$, the expression

$$\frac{1}{\pi} \frac{\sin((n + \frac{1}{2})x)}{2 \sin(x/2)}$$

becomes $\frac{1}{2\pi}(2n + 1)$.

Problem 30.5: We have not yet seen what happens, with the convergence if x is a point of discontinuity. How do you settle the general case, in which several jump discontinuities can occur?

Here is a start: let us assume to have an odd function $g(x)$ which has a jump discontinuity at 0 with $g(x + 0) = a$, $g(x - 0) = -a$. The proof above shows that the Fourier series $\sum_n b_n \sin(nx)$ converges to $g(x)$ at every point $x \neq 0$. At $x = 0$, the series converges to 0, the middle point. Now, if $f(x)$ has a single discontinuity at 0 jumping by $2a$, look at $f(x) - g(x)$. This function is continuous and has a convergent Fourier series. The Fourier series of f therefore converges to the middle of the discontinuity.