Unit 31: Parseval's theorem

LECTURE

31.1. We have seen that every $f \in \mathcal{X}$ can be represented as a series

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) ,$$

where the Fourier coefficients a_n, b_n are were given by integrals

$$a_0 = \langle f, 1/\sqrt{2} \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)/\sqrt{2} \, dx,$$
$$a_n = \langle f, \cos(nt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx,$$
$$b_n = \langle f, \sin(nt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx.$$
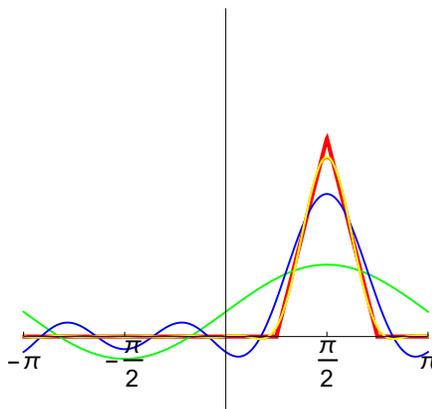

FIGURE 1.   A Fourier approximation of a function $f \in \mathcal{X}$ which is neither even nor odd. The function f is piecewise linear and continuous.

31.2. The inner product allowed us to define the length $||f||^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 \, dx$ of a vector. The following theorem is called the **Parseval's identity**. It is the **Pythagoras theorem** for Fourier series.

Theorem:
$$||f||^2 = a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 .$$

Proof. The function $g(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$ agrees with $f(x)$ except at finitely many points. This implies $||f||^2 = ||g||^2$. Let us compute the later using that $\cos(nx), \sin(nx)$ and $1/\sqrt{2}$ form an orthonormal basis:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} [\frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)][\frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)] \, dx$$

which is after foiling out $a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2$. $\qquad\square$

31.3. Here is an example: We have seen the Fourier series for $f(x) = x$ as

$$f(x) = 2(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \frac{\sin(4x)}{4} + \dots).$$

The coefficients $b_k = 2(-1)^{k+1}/k$ and so

$$4(\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \dots) = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{2\pi^2}{3}.$$

This can be written as

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots = \frac{\pi^2}{6}.$$

This is another solution of the Basel problem. (See Problem 18.3 in Math 22a).

31.4. There are different types of convergence for functions. From the Parseval's identity we have

$$||f - f_n|| \to 0.$$

We call this L^2 convergence. If f is smooth, then from the pointwise convergence we have a stronger **uniform convergence** using $||f - g||_\infty = \max|f(x) - g(x)|$.

> **Corollary:** If f is smooth, then $||f_n - f||_\infty \to 0$.

Proof. This follows from the Dirichlet proof on Fourier series and the Cantor-Heine Theorem (see Unit 8 in Math 22a). $\qquad\square$

31.5. What about the case with discontinuities? Here uniform convergence can fail. It is called the **Gibbs phenomenon**. Remember from the proof seminar, where we had

$$S_n(x) - f(x) = \int_{-\pi}^{\pi} D_n(y)(f(x+y) - f(x)) \, dy.$$

Now, in the case when $f(x) = \text{sign}(x)$ with $f(x) = 1$ for $x > 0$ and $f(x) = -1$ for $x < 0$, we can evaluate the error $S_n(x) - f(x)$ at $x = \pi/n$ and get

$$S_n(x) - f(x) = \int_{\pi/n}^{\pi} D_n(y) 2 \, dy.$$

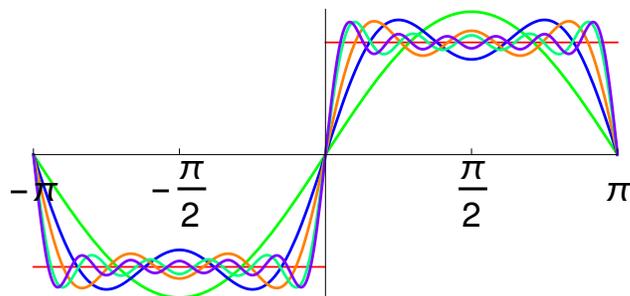This can be shown to have a non-zero limit for $n \to \infty$.

FIGURE 2. The Gibbs phenomenon: the Fourier approximation does not converge uniformly to f if f is not continuous. The series converges pointwise.

<div align="center">EXAMPLES</div>

31.6. Find the Fourier series of the function $f(x) = \begin{cases} 1 & , |x| < \pi/4 \\ 0 & , |x| \geq \pi/4 \end{cases}$. What is the sum of the squares of the Fourier coefficients? Answer: The function is even. It has a cos series. We compute $a_0 = (2/\pi)\pi/(4\sqrt{2}) = \frac{\sqrt{2}}{4}$ and $a_n = \frac{2}{n\pi}\sin(n\pi/4)$. The Fourier series is

$$f(x) = \frac{\sqrt{2}}{4}\frac{1}{\sqrt{2}} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(n\pi/4)\cos(nx) .$$

By Parseval, we have $a_0^2 + \sum_n a_n^2 = ||f||^2$ which is $(\pi/2)/\pi = 1/2$. We did not have to sum up the sum after all! Let us also illustrate how an AI computes things:

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FourierCosSeries[If[x^2 < Pi^2/4, 1, 0], x, 10]
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31.7. The function $g(x) = x^3 - \pi^2 x$ has the Fourier series

$$g(x) = \sum_{n=1}^{\infty} \frac{12(-1)^n}{n^3}\sin(nx) .$$

What is

$$\sum_{n=1}^{\infty} \frac{1}{n^6} ?$$

This number is called $\zeta(6)$, the value of the **Riemann Zeta function** at 6. Parseval's theorem $|g|^2 = \sum_{n=1}^{\infty} b_n^2$ shows that the result $144\zeta(6) = \sum_n b_n^2$ is

$$\frac{2}{\pi} \int_0^{\pi} (x^3 - \pi^2 x)^2 \, dx = \frac{16\pi^6}{105} .$$

From the Parseval identity we get

$$\zeta(6) = \frac{1}{144}\sum_n b_n^2 = \frac{1}{144}\frac{16\pi^6}{105} = \frac{\pi^6}{945} .$$

31.8. It is possible like this to get explicit expressions for $\zeta(2n)$ for even positive n. For odd n, such prepresentations are not known. The smallest, where one does not know it is the Apéry's constant $\zeta(3)$. A Parseval theorem type approach seems not to help in computing this constant.
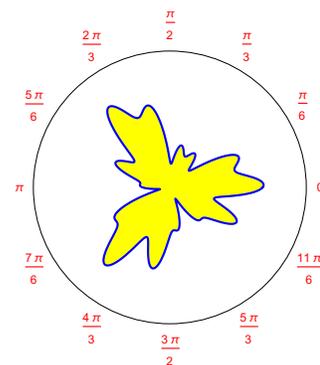
Homework

This homework is due on Tuesday, 4/23/2019.

Problem 31.1: Find the Fourier series of the function which is 22 on $[0, \pi/4]$ and zero everywhere else. What is the sum $a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2$?

Problem 31.2: a) Use Parseval to find the length $||f||$ for the **fairy godmother curve**
$f(x) = 10 + 2\cos(14x) + \cos(11x) + 2\sin(7x) - \cos(4x) + 5\cos(3x)$. b) What does $||f||$ mean geometrically for the polar curve $r(t) = f(t)$?

Problem 31.3: Compute both sides of the Parseval identity for $f(x) = x + |x|$.

Problem 31.4: Find $\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = 1/4 + 1/16 + 1/36 + \ldots$ from the known Basel problem formula of $\sum_n \frac{1}{n^2}$ and use this to compute the sum $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$ over the odd numbers. Written out, this is

$$\frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \cdots .$$

Problem 31.5: We have seen the Parseval identity

$$\langle f, f \rangle = a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 .$$

Why does this imply more generally that if f is a function with Fourier coefficients a_n, b_n and g is a function with Fourier coefficients c_n, d_n, then

$$\langle f, g \rangle = a_0 c_0 + \sum_{n=1}^{\infty} a_n c_n + b_n d_n ?$$

Either prove this directly analogously to what we did when proving the Parseval identity or then reduce it to the Parseval identity.

P.S. Here is a historical challenge: we know very little about **Marc-Antoine Parseval des Chenes**. The result is named after Parseval as there was a note written in 1799 which contains a statement looking similar. In the St-Andrews article of J.J. O'Connor and E.F. Robertson about Parseval, it is stated that *it would not be unfair to say that Parseval has fared well in having a well known result, which is quite far removed from his contribution, named after him. However he remains a somewhat shadowy figure and it is hoped that research will one day provide a better understanding of his life and achievements..*

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