LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

# Unit 32: Fourier Applications

## Lecture

**32.1.** Fourier theory has many applications. There are mathematical applications in **number theory**, **arithmetic**, **ergodic theory**, **probability theory** as well as applications in applied sciences like **signal processing**, **quantum dynamics**, **data compression** or **tomography**. In this lecture, we mention a few applications, sometimes a bit informally as subjects like probability theory, ergodic theory, number theory or inverse problems are subjects which each would fill courses by themselves.

**32.2.** In **probability theory**, a non-negative function $f$ which has the property that $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx = 1$ is called a **probability density function**. This is abbreviated often as PDF. The complex Fourier coefficients $c_n = \mathrm{E}[fe^{inx}]$ of $f$ form what one calls the **characteristic function** of the distribution. Why is this useful? If we have two distributions $f$ and $g$ representing **independent data**, then the **convolution**

$$f \star g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)g(y)\ dy$$

represents the distribution of the sum of the data. Here is the math:

**Lemma:** If $c_n$ and $d_n$ are the Fourier coefficients of $f$ and $g$, then $c_n d_n$ is the characteristic function of $f \star g$.

*Proof.* The $n$'th Fourier coefficient of $f \star g$ is

$$\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x-y)g(y)\ dy e^{-inx}\ dx\ .$$

A change of variables $z = x - y$ gives

$$\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(z)g(y)\ dy e^{-in(z+y)}\ dx\ .$$

This can be written as

$$[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(z)e^{-inz}\ dz][\frac{1}{2\pi}\int_{-\pi}^{\pi} g(y)e^{-iny}\ dy] = c_n d_n\ .$$

□

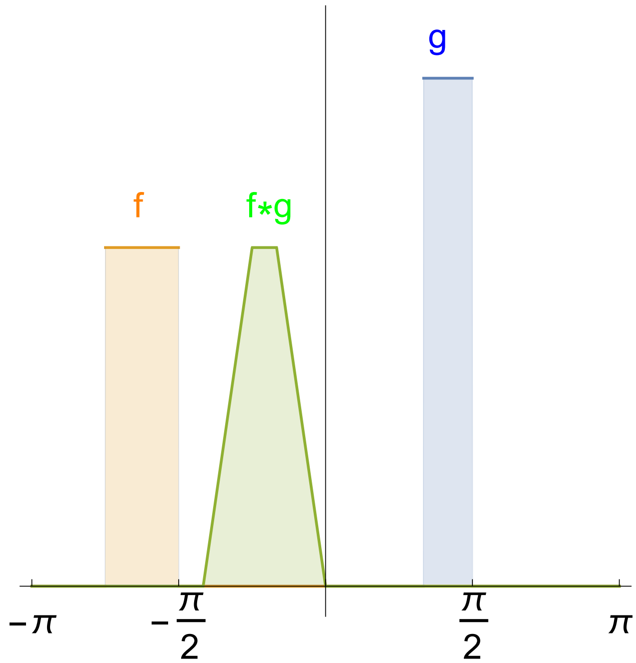



FIGURE 1. We see $f, g$ and its convolution $f \star g$. The Fourier coefficients of $f \star g$ is the product of the Fourier coefficients of $f$.

**32.3.** All this can be done also for distributions on $\mathbb{R}$ rather than $[-\pi, \pi]$. The characteristic function is then a **Fourier transform**. When dealing with **circular data**, then the **Fourier series** are important. Now, if $f$ is a constant distribution, then its Fourier data are $c_n = 0$ except $c_0 = 1$. For a probability distribution in general all $|c_n| < 1$ for $n \neq 0$ and $c_0 = 1$. We get now immediately the **central limit theorem for circular data** with data which have identical distribution:

**Theorem:** Given a sequence of independent circular data, then the distribution of the sum converges to the constant distribution.

*Proof.* Each density function $f_k$ has Fourier coefficients $c_n$ and the sum of $m$ independent data has the Fourier coefficients $c_n^m$. Since for $n \neq 0$, we have $|c_n| < 1$ we have $c_n^m \to 0$. In the limit we have a distribution which has only one non-zero Fourier coefficient $c_0 = 1$. This is the constant distribution. □

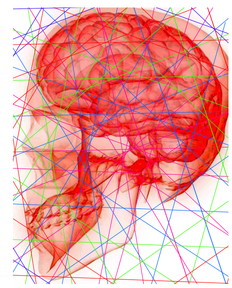
**32.4.** A similar analysis works also in the continuum case. There, the limiting distribution is the **standard normal distribution** $f(x) = 1/\sqrt{2\pi} e^{-x^2/2}$. It has the property that the convolution of $f(x)$ with itself is again a normal distribution but with variance 2. The **central limit theorem** now uses the **Fourier transform**.

**32.5.** In **ergodic theory**, which is also the mathematical frame work for "chaos", one studies the long term behavior of dynamical systems. Let us look at the transformation $T(x) = x + \alpha$ where $\alpha$ is some irrational multiple of $2\pi$. Given a function $f(x)$, what happens with **time average** $S_n/n = \frac{1}{n}[f(x) + f(x + \alpha) + \cdots + f(x + (n-1)\alpha)]$ in the limit $n \to \infty$. The expectation $\mathrm{E}[f]$ of $f$ is the **space average** $\int_{-\pi}^{\pi} f(x)\, dx/(2\pi)$. There is the following **ergodic theorem**

**Theorem:** Time average is space average $\frac{S_n}{n} \to \mathrm{E}[f]$ for $n \to \infty$

*Proof.* The Fourier coefficients of the sum is $c_n(1 + e^{i\alpha} + \cdots + e^{i(n-1)\alpha})$ which is $c_n(1 - e^{in\alpha})/(1 - e^{i\alpha})$. Because we divide by $n$, each Fourier coefficient converges to 0 except for the zero'th coefficient $c_0$ which is always $\mathrm{E}[f]$. $\square$

**32.6.** In **magnetic resonance imaging** one has the problem of finding the density function $g(x, y, z)$ of a three dimensional body from measuring the **absorption rate** along lines. One can reduce it to two dimensions by looking at **slice** $f(x, y) = g(x, y, c)$, where $z = c$ is kept constant. The **Radon transform**, introduced by Johann Radon in 1917, produces from $f$ another function



$$R(f)(p, \theta) = \int_{\{x\cos(\theta)+y\sin(\theta)=p\}} f(r(t))\ |r'(t)|dt$$

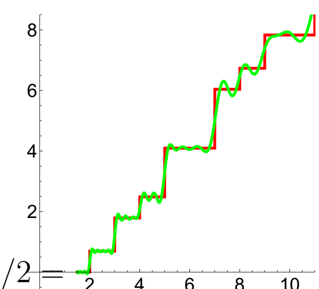
measuring the absorption along the line $L$ of polar angle $\alpha$ in distance $p$ from the center and where the line is parametrized by a curve $r(t)$. Reconstructing $f(x, y) = g(x, y, c)$ for different $c$ allows to recover the **tissue density** $g$ and so "see inside the body".

**Theorem:** The Radon transform can be diagonalized using Fourier theory

To do so, we need some regularity: we need that $\phi \to f(r, \phi)$ is piecewise smooth which then assures that the Fourier series $f(r, \phi) = \sum_n f_n(r)e^{in\phi}$ converges. We also need that $r \to f_n(r)$ has a Taylor series. The expansion $f(r, \phi) = \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\infty} f_{n,k}\psi_{n,k}$ with $\psi_{n,k}(r, \phi) = r^{-k}e^{in\phi}$ is an eigenfunction expansion with explicitly known eigenvalues $\lambda_{n,k}$. The **inverse problem** is subtle due to the existence of a **kernel** spanned by $\{\psi_{n,k} \mid (n+k) \text{ odd }, |n| > k\}$. In applied situations, one calls it an **ill posed problem**.

**32.7.** Fourier theory also helps to understand **primes**. For an integer $n$, let $\Lambda$ denote the **Mangoldt function** defined by $\Lambda(n) = \log(p)$ if $n = p^k$ is a power of some prime $p$ and $\Lambda(n) = 0$ else. Its sum $\psi(x) = \sum_{n\leq x} \Lambda(n)$ is called the **Chebyshev function**. Riemann indicated and Mangoldt proved first that it satisfies the **Riemann-Mangoldt formula**

$$\psi(x) = x - \sum_w \frac{x^w}{w} - \log(2\pi) - \frac{1}{2}\log(1 - x^{-2}) ,$$




where $w$ runs over the non-trivial roots of the **Riemann zeta function**

$$\zeta(s) = \sum_{n\geq 1} n^{-s} ,$$

and where $\log(2\pi) = \zeta'(0)/\zeta(0)$ comes from the simple pole at 1 and $\log(1 - x^{-2})/2 = \sum_{k=1}^{\infty} -x^{-2k}/(2k)$ is the contribution of the **trivial roots** $-2, -4, -6, \ldots$ of the zeta function and $f_j(x) = x^{w_j}/w_j = e^{\log(x)a_j + \log(x)ib_j}/w_j$ is the contribution from the nontrivial zeros $w_j$ (which are believed on the line $\mathrm{Im}(z) = 1/2$). Pairing complex conjugated roots $w_j = a_k + ib_j = |w_j|e^{i\alpha_j}, \overline{w}$ gives a sum of functions

$$f_j(x) = e^{\log(x)a_j} 2\cos(\log(x)b_j - \alpha_j)/|a_j + ib_j| .$$

The functions $f_j$ are the tunes of the **music of the primes**. There is a book and movie with this title by Marcus du Sautoy.

## Homework

This homework is due on Tuesday, 4/23/2019.

**Problem 32.1:** The piecewise linear continuous function $g$ has a graph connecting $(-\pi, 0), (-\pi/2, \pi/2 - 1), (0, -1/2), (\pi/2, \pi/2 - 1), (\pi, 0)$. It satisfies $g = f \star f$, where $f$ be the function which is $-1$ on $[-\pi/2, 0]$ and 1 on $[0, \pi/2]$. Use HW 32.5 to compute the Fourier coefficients of $g$.

**Problem 32.2:** a) In a magnetic resonance problem, we measure the density function $f(r, \phi) = \sum_n r^n \cos(n\phi)$. Find a closed-form for $f(r, \phi)$. **Hint:** the series is the real part of $\sum_{n=1}^{\infty} r^n e^{in\phi}$. You can assume $|r| < 1$.
b) Give an explicit expression for $\sum_{n=1}^{\infty} \frac{1}{2^n} \cos(nx)$.

**Problem 32.3:** There is a general principle which tells that the smoother a function is as faster the Fourier series decays.
Given a Fourier series $f(x) = \sum_n b_n \sin(nx)$ of a smooth function, can you give the Fourier series of derivative $f'(x)$? Conclude that for an odd $f \in C^\infty$ and any $k$, like $k = 22$, one has $b_n n^{22} \to 0$ as $n \to \infty$.

**Problem 32.4:** Something in number theory: Define the Fourier series $f(x) = \sum_p e^{ipx}/2^p$, where $p$ runs over all primes. Define the function $g(x) = f(x)^2$ and compute its Fourier coefficients $c_n$. Why is the Goldbach conjecture equivalent to the fact that all $c_{2n} \neq 0$ for $n > 1$?

**Problem 32.5:** Prove that if $f$ is an odd function with Fourier coefficients $b_n$, then $g = f \star f$ is even with Fourier coefficients $a_0 = 0$ and $a_n = -b_n^2/2$. Hint: use the theorem and hat for an odd function $b_n = -2\text{Im}(c_n)$ and $a_n = 2\text{Re}(c_n)$, where $c_n$ is the $n$'th complex Fourier coefficient of $f$. A key relation therefore is $2c_n = a_n - ib_n$.

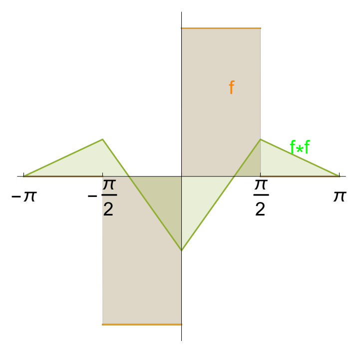



FIGURE 2. The convolution $g = f \star f$ in HW 32.1. We see both the odd step function $f$ and the even convolution $g$.