LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 34: Heat equation

Lecture

34.1. The partial differential equation

$$f_t = f_{xx}$$

is called the **heat equation**. It is an equation for an unknown function $f(t,x)$ of two variables t and x. The interpretation is that $f(t,x)$ is the temperature at **time** t and **position** x. In order to use Fourier theory, we assume that f is a function on the interval $[-\pi,\pi]$. The problem is: given an initial heat distribution $f(0,x)$, what is the situation $f(t,x)$ at a later time? The process does what one expects from heat. It produces **diffusion**.

34.2. What does the equation tell? We have a temperature distribution $x \to f(t,x)$. The rate of change in time of this temperature is the second space derivative of f. If x is a location, where $f(t,x)$ is concave down as a function of x, this means that f_t is negative and that the function will decrease there in the near future. If $f(t,x)$ is concave up, then this means that f_t is positive, meaning that f increases there. While the partial differential equation describes a motion of a function f the set-up is as before, where we looked at the motion $v(t)$ of a vector v.

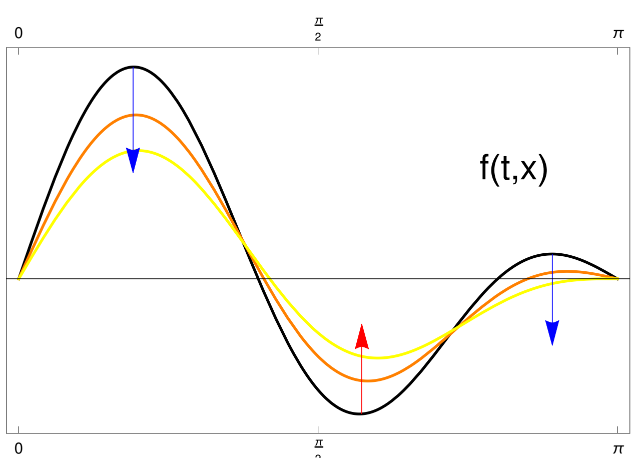

Figure 1. The heat equation describes the motion of a function. The evolution has a smoothing effect. The height of the function diffuses and settles to the average.

34.3. In order to use the **closed-form solution method** from the earlier part of the course, we write the heat equation as

$$f_t = D^2 f$$

and think of $A = D^2$ as a transformation or a matrix. Now remember what we did in the case of differential equations $x' = Ax$; we found the eigenvalues and eigenvectors of A. This is what we do here too. But we know already that $\cos(nx), \sin(nx)$ are eigenfunctions of D^2 to the eigenvalue $-n^2$. In the ordinary differential equation case, we also expressed $v(0) = c_1 v_1 + \cdots + c_n v_n$ as a sum of eigenfunctions, then wrote down the **closed-form solution** $v(t) = c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n$.

34.4. We do the same thing here and can use that the Fourier basis is an eigenbasis of D^2. This is the great discovery of Fourier:

Theorem: The solution to the heat equation is $f(t,x) = a_0 \frac{1}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n e^{-n^2 t} \cos(nx) + \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx)$.

Proof. For $t = 0$, we get the Fourier series of $f(0,x)$. A direct differentiation shows that $f_t = f_{xx}$. □

34.5. Fourier himself used the sin series. We could continue the function as an odd function on $[-\pi, \pi]$ and use only the sin-series. We could also continue the function as an even function on $[-\pi, \pi]$ and use the cos-series. The later is done in applications like JPG encoding of pictures as there, the average is important as it represents brightness. Let us look at an example of $f(0,x) = \sin(x)$ on $[0, \pi]$. If we continue that as an odd function, then this is just $\sin(x)$ and $f(t,x) = e^{-t^2} \sin(x)$ solves that equation. If we continue the function as an even function, then we deal with $f(x) = |\sin(x)|$.

34.6. A function of two variables $f(x,y)$ can be expanded into a Fourier series too. The Fourier basis is

$$\mathcal{B} = \left\{ \cos(nx)\cos(ny), \sin(nx)\sin(ny), \cos(nx)\sin(my), \frac{1}{\sqrt{2}}\cos(nx), \frac{1}{\sqrt{2}}\sin(nx), 1/2 \right\} .$$

It is here much more convenient to use the **complex basis** $\mathcal{B} = \{e^{i(nx+my)}\}$ and write

$$f(x,y) = \sum_{n,m} c_{n,m} e^{i(nx+my)}$$

with $c_{n,m} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} f(x,y) e^{-inx-imy} \, dxdy$. The real Fourier coefficients would include terms like $a_{nm} = \frac{1}{\pi^2} \int_{-\pi}^{\pi} f(x,y) \cos(nx) \cos(ny) \, dxdy$. Let $\mathcal{X}$ be the set of functions $f(x,y)$ with the property that for every x the function $g(y) = f(x,y)$ is piecewise smooth, then the Dirichlet theorem implies that the Fourier series converge. And we have

Theorem: The heat equation $f_t = f_{xx} + f_{yy}$ with initial condition $f(0,x,y)$ is solved by $f(t,x,y) = \sum_{n,m} c_{n,m} e^{-(n^2+m^2)t} e^{i(nx+my)}$.

Proof. For $t = 0$, is the Fourier series of $f(0,x,y)$. When differentiating each part $f_{nm} = e^{-(n^2+m^2)t} e^{i(nx+my)}$ with respect to t, this gives the same as when applying the Laplacian $\Delta f = f_{xx} + f_{yy}$ on f_{nm}. □

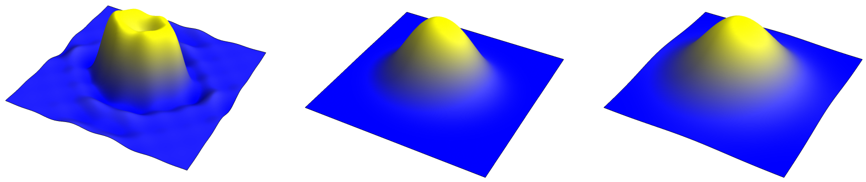

FIGURE 2. The evolution of a two dimensional heat equation is solved also explicitly using Fourier series.

EXAMPLES

34.7. Problem: Find the general solution of the modified heat equation $f_t = 3f_{xx}+f$, where $f(0)$ is 1 for $x \in [\pi/3, 2\pi/3]$ and 0 else. Let us assume that f stays zero at the boundary of $[0,\pi]$ and is continued in an odd way so that it has a sin-series. This equation is a driven heat equation which can model a **fire** which in which heat produces more fuel. **Solution:** f has a sin series with

$$b_n = \frac{2}{\pi}\int_{\pi/3}^{2\pi/3} \sin(nx)\, dx = \frac{2}{n\pi}[-\cos(2n\pi/3)+\cos(n\pi/3)] \; .$$

Now look the operator $A = 3D^2+1$ on the right hand side so that $f_t = Af$. What are the eigenvalues of A? Since the eigenvalues of D^2 are $-n^2$, the operator A has the eigenvalues $\lambda_n = -3n^2+1$. The closed-form solution is $f(t,x) = \sum_{n=1}^{\infty} b_n e^{(-3n^2+1)t}\sin(nx)$.

34.8. Problem: Let us take the same problem as before but increase the fuel strength feeding the fire $f_t = 3f_{xx}+5f$. What happens now is that

$$f(t,x) = \sum_{n=1}^{\infty} b_n e^{(-3n^2+5)t}\sin(nx) \; .$$

The high frequency parts still die out but there is one mode which explodes now exponentially because $\lambda_n = -3n^2+5$. The fire takes over. Let us write down the solution in a bit more intelligible way. The coefficient $b_n = -\cos(2n\pi/3)+\cos(n\pi/3)$ is 0 for even n, It is alternating 1 and -2 for odd n. We have the initial condition

$$f(0,x) = \frac{1}{\pi}(\frac{\sin(x)}{1} - \frac{2\sin(3x)}{3} + \frac{\sin(5x)}{5} - \cdots) \; .$$

The solution of the heat equation is now

$$f(t,x) = \frac{1}{\pi}(\frac{e^{2t}\sin(x)}{1} - \frac{e^{-7t}2\sin(3x)}{3} + \frac{e^{-22t}\sin(5x)}{5} - \dots) \; .$$

It is the e^{2t} part which renders the fire out of control.

Homework

This homework is due on Tuesday, 4/30/2019.

Problem 34.1: Solve the heat equation $f_t = 2019 f_{xx}$ on $[-\pi, \pi]$ with the initial condition $f(0,x) = 0$ if $x \in [-\pi/2, \pi/2]$ and $f(x) = \sin(2x)$ if $|x| \in [\pi/2, \pi]$.

Problem 34.2: Solve the partial differential equation $f_t = 3f_{xxxxxx} + 5f_{xx}$ with initial condition $f(0,x) = 22x$.

Problem 34.3: Solve the partial differential equation $f_t = -f_{xxxx} - f_{yyyy}$ with initial condition $f(0,x) = 2\cos(22x)\sin(33y) + 17\sin(12x)\sin(11y)$.

Problem 34.4: Solve the partial differential equation $f_t = f_{xx} + \sin(3t)$ with initial condition $f(0,x) = 3x + \cos(5x) + \sin(7x)$. **Hint:** First solve the homogeneous equation $f_t = f_{xx}$, then add a special solution which only depends on t.

The next problem deals with a very important partial differential equation. It looks like the heat equation and can be treated like the heat equation but its solutions behaves in a different way. It is that i which changes everything. Instead of $e^{-n^2 t}$ which goes to zero very fast, we have $e^{in^2 t}$ which is a wave.

Problem 34.5: a) The equation $if_t = f_{xx}$ is the **Schrödinger equation**. Assume $f(0,x) = \sin(17x) + \cos(12x)$, find the solution $f(t,x)$.
b) Write down the solution to the Schrödinger equation $if_t = f_{xx} + f_{yy} + f_{zz}$ with $f(0,x,y,z) = \sin(13x)\cos(15y)\sin(17z)$.

Figure 3. Firepit near the Harvard Science center. Contemplating heat.

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