Unit 35: Wave equation

LECTURE

35.1. The partial differential equation

$$f_{tt} = f_{xx}$$

is called the **wave equation**. It is an equation for an unknown function $f(t, x)$ of two variables t and x. The interpretation is that $f(t, x)$ is the string amplitude at **time** t and position x. Again, we assume that f is a function on the interval $[-\pi, \pi]$. One problem is: given an initial string position $f(0, x)$, what is $f(t, x)$ at a later time? Another problem is to give the initial velocity $f_t(0, x)$ and determine from this the string position at time t. We can also give both the initial position and velocity in which case we just add up the solution for the initial position and the solution for the initial velocity.
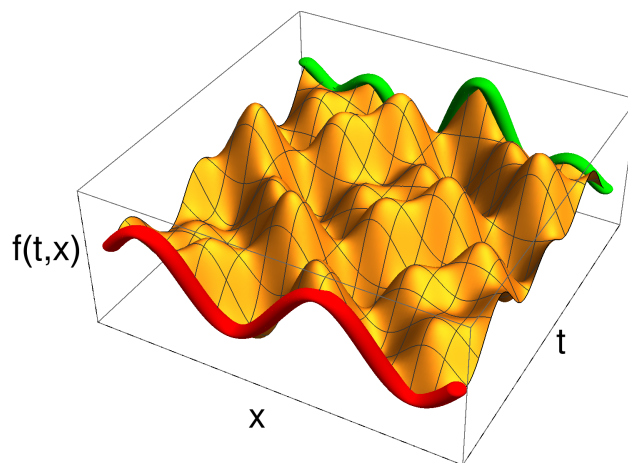

FIGURE 1.   We see a solution $f(t, x)$ of the wave equation $f_{tt} = f_{xx}$. The initial wave front is a sin function, but there is also an initial velocity given. We outlined $f(0, x)$ and $f(1, x)$.

35.2. What is the meaning of the wave equation? We can interpret the acceleration f_{tt} as a **force** acting on the string. By Newton's law acceleration is proportional to force. That force acts in such a way that the string wants to be flattened out. But as the system not only has position but also momentum, it does not just flatten out as in

the heat case. It is a **conservation of energy** which prevents the wave from going to zero without friction:

Lemma: The **energy** $H(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f_t^2/2 + f_x^2/2 \, dx$ is constant.

Proof. Using integration by parts (using that we are on a circle and so have no boundary) we have the general formula for differentiable functions

$$\int_{-\pi}^{\pi} f'(x)g'(x) \, dx = \int_{-\pi}^{\pi} -f(x)g''(x) \, dx \ .$$

We therefore have $\frac{d}{dt}H(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f_t f_{tt} + f_x f_{tx} \, dx$. Using the wave equation $f_{tt} = f_{xx}$ and the integration by parts formula $\int_{-\pi}^{\pi} f_x f_{tx} \, dx = \int_{-\pi}^{\pi} -f_{xx} f_t \, dx$, we get $\frac{1}{\pi} \int_{-\pi}^{\pi} f_t f_{xx} - f_{xx} f_t \, dx = 0$. $\qquad\square$

35.3. For the heat equation, the solution of $x'(t) = \lambda x$ was important. It was $x(t) = e^{\lambda t} x(0)$. For the wave equation, the solution of $x''(t) = -c^2 x$ is important. It is $x(t) = x(0)\cos(ct) + x'(0)\sin(ct)/c$. We usually have given things in the form $x''(t) = \lambda x$, so that $c = \sqrt{-\lambda}$. Now, if we have an initial condition which is a Fourier basis vector like $f = \cos(nx)$, where $D^2 f = (-n^2)f$, where $\lambda = -n^2$ and $c = n$, we have the solution $f(t, x) = \cos(ct)\cos(nx)$.

Theorem: If $f_t(0, x) = 0$ and $f(0, x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$, then $f(t, x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n \cos(nt)\cos(nx) + b_n \cos(nt)\sin(nx)$ solves the wave equation $f_{tt} = f_{xx}$.

Proof. We check that $f_t(0, x) = 0$ and that $f(0, x)$ agrees with the Fourier expansion of $f(0, x)$. $\qquad\square$

35.4. The solution to the **harmonic oscillator** $x''(t) = -c^2 x$ also has a contribution $x'(0)\sin(ct)/c$ which takes care of the **initial velocity**. This allows us to write down the **closed-form solution** if the initial velocity is given.

Theorem: If $f(0, x) = 0$ and $f_t(0, x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$, then $f(t, x) = t\frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} \frac{a_n}{n} \sin(nt)\cos(nx) + \frac{b_n}{n} \sin(nt)\sin(nx)$ solves the wave equation $f_{tt} = f_{xx}$.

Proof. We see that for $t = 0$, $f(0, x) = 0$ and that $f_t(0, x)$ agrees with the Fourier expansion of $f_t(0, x)$. $\qquad\square$

35.5. The energy can be written as $||f_x||^2/2 + ||f_t||^2/2$ which is a sum of a **potential energy** and **kinetic energy** of the string. It is custom to use the factor $1/2$ as we have in physics energy = mass times velocity squared divided by 2.

35.6. More generally, if we want to solve a partial differential equation $f_{tt} = Af$, where A is a function of D^2 we do:

35.7. For example, for the **driven wave equation** $f_{tt} = f_{xx} - f$ we first compute
the eigenvalues of the operator $A = D^2 - 1$. Since the eigenvalues of D^2 are $-n^2$,
the eigenvalues of A are $\lambda_n = -n^2 - 1$. This gives $c_n = \sqrt{n^2 + 1}$. If the initial wave
$f(0, x) = 4\sin(222x)$ and initial velocity $f_t(0, x) = 7\sin(365x)$, then we don't have to
compute the Fourier series as the functions are already Fourier series. The closed-form
solution for the initial position is

$$f(t, x) = 4\cos(\sqrt{222^2 + 1}t)\sin(222x)$$

The closed-form solution of the initial velocity is

$$f(t, x) = 7\sin(\sqrt{365^2 + 1}t)\sin(365x)/\sqrt{365^2 + 1} .$$

The solution of the problem is the sum

$$f(t, x) = 4\cos(\sqrt{222^2 + 1}t)\sin(222x) + 7\sin(\sqrt{365^2 + 1}t)\sin(365x)/\sqrt{365^2 + 1} .$$

35.8. Let us assume that $f(0, x, y)$ is a given function like $f(0, x, y) = \sum_{n,m} \frac{1}{n+m} \cos(nx)\cos(my)$.
and assume $f(0, t, x, y) = 0$. What is the solution of the wave equation

$$f_{tt} = f_{xx} + f_{yy} ?$$

The functions $\cos(nx)\cos(my)$ are eigenfunctions of $Af(x, y) = f_{xx} + f_{yy}$ with eigen-
values $\lambda_n = -n^2 - m^2$. The solution is

$$f(t, x, y) = \sum_{n,m} \frac{\cos(\sqrt{n^2 + m^2})}{n + m} \cos(nx)\cos(my) .$$

<div align="center">EXAMPLES</div>

35.9. What is the solution of the driven wave equation $f_{tt} = f_{xx} - f + 6t$ if $f(0, x) = \sum_n \frac{1}{n^3}\sin(nx)$. We first solve the **homogeneous problem**

$$f_{tt} = f_{xx} - f .$$

Since the right hand side is Af with $A = D^2 - 1$, which has the eigenvalues $-n^2 - 1$,
the solution is

$$f(t, x) = \sum_n \frac{\cos(\sqrt{n^2 + 1})}{n^3}\sin(x) .$$

A special solution which does not depend on x satisfies $f_{tt} = 6t$ which has the solution
t^3. The final solution is

$$f(t, x) = t^3 + \sum_n \frac{\cos(\sqrt{n^2 + 1})}{n^3}\sin(x) .$$

Homework

This homework is due on Tuesday, 4/30/2019.

Problem 35.1: A piano string is fixed at the ends $x = 0$ and $x = \pi$ and is initially undisturbed $f(0, x) = 0$. The piano hammer induces an initial velocity $f_t(x, 0) = g(x)$ onto the string, where $g(x) = \sin(3x)$ on the interval $[-\pi/2, \pi/2]$ and $g(x) = 0$ on $[\pi/2, \pi]$ or $[-\pi, -\pi/2]$. a) How does the string amplitude $f(t, x)$ move, if it follows the wave equation $f_{tt} = f_{xx}$? b) Now we replace the piano by a **harpsichord**, where the string is plucked. In that case $f(0, x) = g(x)$ is given and $f_t(0, x) = 0$.

Problem 35.2: A laundry line is excited by the wind. It satisfies the differential equation $f_{tt} = f_{xx} + \cos(t) + \cos(3t)$. Assume that the amplitude u satisfies initial position $f(0, x) = x$ and $f_t(0, x) = 4\sin(5x) + 10\sin(6x)$. Find the function $f(t, x)$ which satisfies the differential equation.

Problem 35.3: Solve the partial differential equation $f_{tt} = -f_{xxxx} + f_{xx}$ with initial condition $f_t(0, x) = x^3$ and $f(0, x) = x^3$.

Problem 35.4: a) Solve the wave equation $f_{tt} = 9f_{xx}$ on $[0, \pi]$ with the initial condition $f(0, x) = \max(\cos(x), 0)$.
b) Solve the wave equation $f_{tt} = f_{xx} + f_{yy}$ with $f(0, x, y) = 9\sin(3x)\cos(5y)$.

Problem 35.5: Verify that the energy $\frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f_t^2/2 + f_x^2/2 + f_y^2/2 \, dxdy$ is invariant under the evolution of the wave equation $f_{tt} = f_{xx} + f_{yy}$.
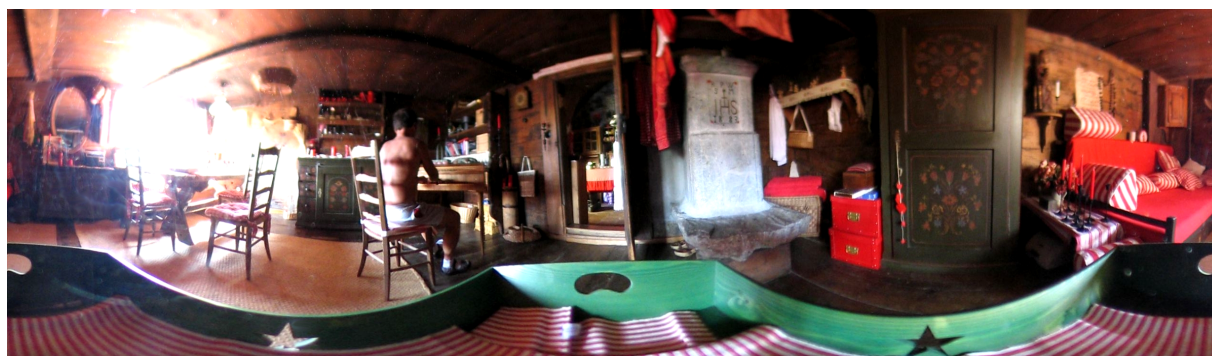

FIGURE 2. Oliver playing a harpsichord in the Swiss mountains. The wood stove to the right is from 1883, the house built in 1864. The 360 degree camera was placed in a tray. We see its boundary as a piecewise smooth function of the direction angle.