LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 36: Discrete PDE

Lecture

36.1. We have seen the Fourier theory allowed to solve the **heat equation**

$$f_t = -Lf \ ,$$

where $L = -D^2$ is the second derivative operator. The negative sign is added so that $-D^2$ has non-negative eigenvalues. The reason why things worked out so nicely was that the Fourier basis was an eigenbasis of D^2. Indeed, $L\sin(nx) = (-n^2)\sin(nx)$ and $L\cos(nx) = (-n^2)\cos(nx)$ and $L\frac{1}{\sqrt{2}} = 0$. We got a **closed-form solution** of the heat equation by writing the initial heat as a Fourier series then evolving each eigen function to get $f(t,x) = a_0\frac{1}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n e^{-n^2 t}\cos(nx) + b_n e^{-n^2 t}\sin(nx)$.

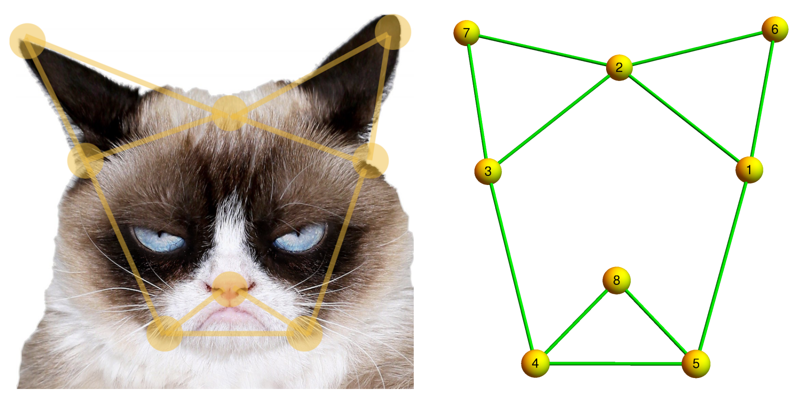

Figure 1. The "grumpy cat graph". The eigenvalues of the Kirchhoff Laplacian L are $\{3+\sqrt{5}, 3+\sqrt{3}, 4, 3, 3, 3-\sqrt{3}, 3-\sqrt{5}, 0\}$.

36.2. The same idea works also in a discrete framework when space is a graph. The analogue of the Laplacian is now the Kirchhoff matrix $L = A - B$, where A is the **adjacency matrix** and B is the diagonal matrix containing the vertex degrees. You have proven last semester that the eigenvalues are non-negative. The reason was that L could be written as d^*d for the gradient matrix d so that $\lambda(v,v) = (Lv,v) = (d^*dv,v) = (dv,dv)$ implying that $\lambda = (v,v)/(dv,dv) = |v|^2/|dv|^2 \geq 0$. The **discrete heat equation**

$$x' = -Lx$$

is now a discrete dynamical system we have seen before.

36.3. We can run a partial differential equation on any graph. Let's take the "Grumpy Cat Graph". It is especially fun to run the Schrödinger equation

$$if_t = Lf$$

on that graph. It is **Schrödinger's cat**. Grumpy cat has $v_0 = 8$ vertices and $v_1 = 11$ edges and $v_2 = 3$ triangles as ears and the snout. Its Euler characteristic $v_0 - v_1 + v_2 = 0$ is zero, one reason why the cat is so grumpy. It is also unhappy not knowing whether it is dead or alive and because his friend, "Arnold the cat" can live on a doughnut.

36.4. The Laplacian of the Grumpy cat graph encodes the graph because the entries -1 tell which vertices are connected.

$$L = \begin{bmatrix} 3 & -1 & 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 3 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 2 \end{bmatrix} .$$

The eigenvalues of L are $\{\lambda_1 = 3+\sqrt{5}, \lambda_2 = 3+\sqrt{3}, \lambda_3 = 4, \lambda_4 = 3, \lambda_5 = 3, \lambda_6 = 3-\sqrt{3}, \lambda_7 = 3-\sqrt{5}, \lambda_8 = 0\}$. We give the eigenvectors $v_3 = [1,1,1,-1,-1,-1,-1,1]$, $v_4 = [-1,0,0,0,-1,1,0,1]$, $v_5 = [0,0,-1,-1,0,0,1,1]$ and $v_8 = [1,1,1,1,1,1,1,1]$.

Theorem: For a connected graph, the solution $x(t)$ to the heat equation converges to a constant function which is the average value of $x(0)$.

Problem A: Prove this theorem. You can use that all eigenvalues of L are positive except one which is 0.

Problem A': Solve the heat equation for the grumpy cat with initial condition $f(0) = v_3 + 5v_4 + 2v_5$.

36.5. Let us now look at the **discrete wave equation**

$$f_{tt} = -Lf \; ,$$

where again L is the discrete Laplacian of a connected graph. Assume λ_k are the eigenvalues of L and v_k the eigenvectors.

Theorem: The function $f(t) = \sum_k c_k \cos(\sqrt{\lambda_k} t) v_k$ solves the discrete wave equation with initial condition $f(0) = \sum_k c_k v_k$.

Problem B: Verify this theorem by verifying that each part in the sum satisfies the wave equation.

Problem B': Solve the wave equation for the grumpy cat with initial condition $f(0) = v_3 + 5v_4 + 2v_5$.

Theorem: The function $f(t) = \sum_k c_k \sin(\sqrt{\lambda_k} t)/\sqrt{\lambda_k} v_k$ solves the wave equation with initial velocity $f'(0) = \sum_k c_k v_k$.

Problem C: Prove this theorem.

Problem C': Solve the wave equation for the grumpy cat with initial condition $f_t(0) = v_3 + 5v_4 + 2v_5$.

Problem D: Formulate the theorem for the discrete Schrödinger equation $if_t = Lf$.

Problem D': Solve the Schrödinger equation for the grumpy cat with initial condition $f(0) = v_3 + 5v_4 + 2v_5$.

36.6. Partial differential equations which are not linear are hard. An example is the **sine-Gordon equation**

$$f_{tt} = -Lf - c\sin(f) ,$$

where c is a constant. This can also be considered in the discrete, where L is the Kirchhoff matrix. One of the simplest examples is when the graph is the complete graph with 2 vertices. In that case

$$L = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

If $f = (x, y)$, then

$$\begin{aligned} x'' &= -x + y - c\sin(x) \\ y'' &= x - y - c\sin(y) \end{aligned}$$

This is a non-linear system if c is different from zero.

Problem E: Solve this system for $c = 0$ in the case when $(x(0), y(0)) = (2, 1)$ and $(x'(0), y'(0)) = (0, 0)$.

Problem F: Verify that the energy of the sine-Gordon equation, $H(x, y, x', y') = (x'^2 + y'^2)/2 + (x - y)^2/2 + c\cos(x) + c\cos(y)$, is constant.

Homework

This homework is due on Tuesday, 4/30/2019.

In the next three problems, we take G be the complete graph with 3 vertices.

Problem 36.1: Write down the discrete heat equation $f_t = -Lf$ and find the closed-form solution $f(t)$ with $f(0) = (0, 2, 1)$.

Problem 36.2: Write down the discrete wave equation $f_{tt} = -Lf$ and find the closed-form solution $f(t)$ with $f(0) = (0, 2, 1)$ and $f_t(0) = (0, 0, 0)$.

Problem 36.3: Write down the discrete Schrödinger equation $if_t = -Lf$ and find the closed-form solution $f(t)$ with $f(0) = (0, 2, 1)$.

Problem 36.4: Remember that if f is a function on vertices of a graph, then df is a function on the edge by $df((a, b)) = f(b) - f(a)$. Verify that the energy $H = \sum_e df(e)^2/2 + \sum_v f_t(v)^2/2$ is time invariant under the wave equation $f_{tt} = -Lf$. Hint: You can use that $L = d^*d$, where d is a $m \times n$ matrix, where n is the number of vertices and m the number of edges. Use that $(df, df) = (d^*df, f) = (Lf, f)$.

Problem 36.5: Pick a graph of your choice, write down the matrix L and write down a closed-form solution for
a) The discrete heat equation.
b) The discrete wave equation
c) The discrete Schrödinger equation.