LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 37: Last unit

37.1. In this last lecture, we look at some key points and connections. Of course, this is just an attempt. It is your task to do this yourself when reviewing the material. It can be helpful to see things from far also.

Objects and Arrows

37.2. It is important that you know what are **objects** of a theory and what are relations called **morphisms** between the objects here given by **transformations**. It is a fancy point of view to see objects and arrows between objects. The subject is called **category theory**. It is a field which builds bridges between different subjects. It is a bit abstract at first like set theory but it starts to get hold also in computer science. [1]

37.3. In linear algebra, we deal with the **category of vector spaces**. The objects are the **linear spaces**. Examples are $\mathbb{R}^n$ or $M(n,m)$ or **linear space of functions** like $C^\infty([-\pi,\pi])$. And then we looked at linear transformations which are implemented by matrices or operators.

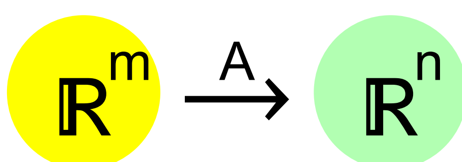

Figure 1. A matrix $A \in M(n,m)$ defines a linear map from $\mathbb{R}^m$ to $\mathbb{R}^n$. It is a morphism meaning that it respects the structure addition and scalar multiplication of the vector space.

Linear Spaces	**Linear Transformations**
Vector spaces	Matrices
Function spaces	Operators

37.4. Important examples of spaces are image and kernel of a transformation. Later in the course, we looked at **solution spaces of differential equations**. Also these spaces can be described in the form of kernels of transformations

[1] B. Milewski: Category Theory for Programmers, 2018

Time

to not only be able to survive in the short term but also to plan for the long term

37.5. It is pivotal for us to be able to predict the future. We would like to know the development of the weather or seasons. We need to plan ahead to avoid problems. It is a sign of intelligence to not only be able to survive in the short term but to also to worry about the long term prosperity. We might, in the future, change the trajectory of an asteroid targeting the earth, change the energy consumption habits to avoid a climate disaster, or to adopt a long term financial planning strategy for times when betting on exponential growth is no option any more. In order to make informed decisions, we have to be able to gather data, fit these data so that we can make a model or theory, then use this knowledge to predict the situation at a later time.

37.6. The science of time is called **dynamical systems theory**.

Discrete dynamical systems	$x(t+1) = Ax(t)$
Ordinary differential equations	$x'(t) = Ax(t)$
Partial differential equations	$f_t = Af$.

Solving equations

37.7. A major theme in this course was the problem to **solve equations**:

Systems of linear equations	$Ax = b$.
Find roots of a polynomial	$f_A(x) = 0$
Find least square solution	$(A^T A)^{-1} A^T b$
Find equilibrium points	$f(x, y) = 0, g(x, y) = 0$
Solve ordinary differential equations	$f'' = -c^2 f$.
Solve partial differential equations	$f_{tt} = f_{xx}$

Invariants

37.8. Invariants are quantities which do not change under coordinate change or do not change under time evolution. Here are examples:

Trace of a matrix
Determinant of a matrix
Eigenvalues of a matrix
Geometric multiplicity of eigenvalues
algebraic multiplicity of eigenvalues
energy
Markov property

37.9. A nice big picture had been painted by **Emmy Noether**. She related **invariant quantities** with **symmetries**. In linear algebra, symmetries pop up at various places. Examples:

Symmetry	Invariant
Similarities like $A \to S^{-1}AS$	Eigenvalue
even function	$b_n = 0$
odd function	$a_n = 0$

Building blocks

37.10. A nice idea is to split things up into smaller parts, solve the smaller parts, then put things together. **Linearity** makes this possible. If we have two solutions to a linear equation, we can add them and get a solution again.

37.11. In linear algebra, the situation is particularly nice if we have spaces which are invariant. If a transformation on $\mathbb{R}^4$ for example which preserves the xy and the zw plane is written in **block form**. If we can find a **basis** for which all invariant blocks are one dimensional, then we have **diagonalized the situation**. In that case we can completely solve the system using **closed-form solutions.**

37.12. We use that in one dimensions

equation	solution
$x(t+1) = \lambda x(t)$	$x(t) = \lambda^t x(0)$.
$x'(t) = \lambda x(t)$	$x(t) = e^{\lambda t} x(0)$.
$x''(t) = -c^2 x(t)$	$x(t) = \cos(ct)x(0) + \sin(ct)\frac{x'(0)}{c}$.

Examples

37.13. Good examples are pivotal for understanding and ideas and developments. Fourier theory has lots of open ends. Let us look at stranger series:

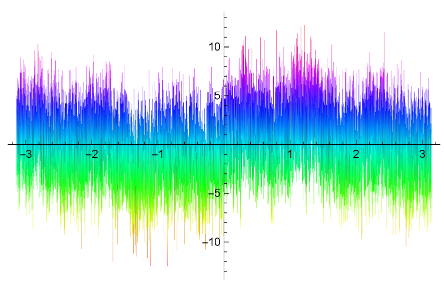

Figure 2. The Fourier series $\sum_{n=1}^{\infty} \sin(n!x)$ converges for every $x = \pi p/q$. We see its graph. It was first studied by Riemann.

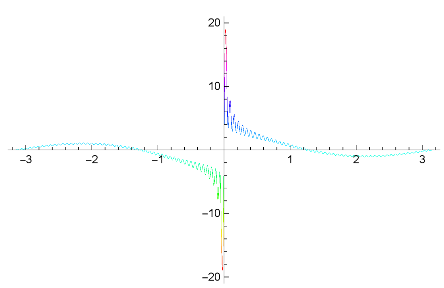

Figure 3. The function $\sum_{n=1}^{\infty} \sin(nx)/\log(n)$, an example due to Fatou. It converges everywhere, but is not even Lebesgue integrable.

37.14. Here is a challenge. Can you see what $\sum_{n=1}^{\infty} \frac{\sin(n)}{n}$ is? Hint. Try to find a formula for the Fourier sin series with $b_n = 1/n$.

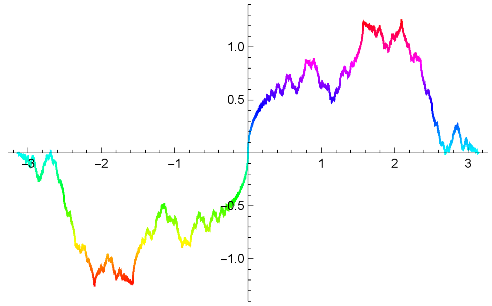

FIGURE 4. The function $\sum_{n=1}^{\infty} \sin(n^2 x)/n^2$ is due to Riemann who told his students that this might be a function with no derivative at any point. Hardy proved later that it is indeed nowhere differentiable except perhaps at points $\pi p/q$ with odd integers p and q. Only in 1970, it was shown that it is differentiable at those points.

History

37.15. The story of linear algebra, differential equations and Fourier theory cover an interesting time in mathematics of the 19th century. Between the time of Euler and Riemann, the understanding of functions completely changed.

37.16. The Dirichlet convergence theorem for example is not only an important mathematical result; it is also an important turning point in the history of mathematics as it led to a different level of rigor. To cite from the book of Kahane and Lemarié-Rieusset on Fourier series, "the article of Dirichlet on Fourier series is a turning point in the theory and also in the way mathematical analysis is approached and written."

37.17. Even Dirichlet, who worked with piece wise monotone functions had the impression that "these are all the functions encountered in nature". But more generalizations were necessary to deal more fractal type structures or have completeness as needed in quantum mechanics.

Beyond

37.18. After multivariable calculus and linear algebra a couple of interesting fields to explore like functional analysis, complex analysis, algebraic geometry or measure theory. One of the objects studied in functional analysis are Hilbert spaces and Banach spaces, which are all linear spaces but which also feature an inner product or norm. Fourier theory can then be formulated in a Hilbert space like $L^2(\mathbb{T})$ where the inner product what we have considered but where much for functions are allowed.

37.19. Mathematical subjects like statistics, discrete math, graph theory, theory of computation, dynamical systems, and numerical analysis all benefit from a solid knowledge of calculus and linear algebra. There are also immediate applications in other sciences like biology, chemistry, computer science, physics or astronomy. Of course, the engineering applications in computer graphics, artificial intelligence, economics, finance and even good old rocket science are enormous.

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