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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

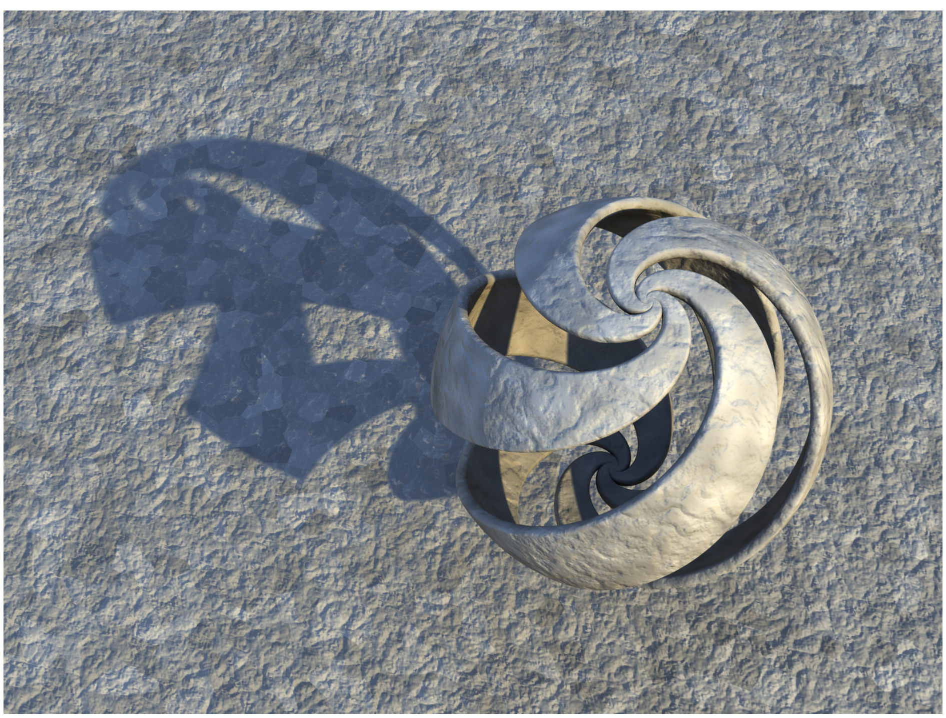
MATH 22B

Total:

Unit 38: Final Exam, 5/15/2019

Welcome to the final exam. Please don't get started yet. We start all together at 2:00 PM after reviewing some exam formalities. You can fill out the attendance slip already. Also, you can already enter your name into the larger box above.

- You only need this booklet and something to write. Please stow away any other material and any electronic devices. Remember the honor code.
- Please write neatly and give details. Except for problem 2 we want to see details, even if the answer should be obvious to you.
- Try to answer the question on the same page. There is additional space on the back of each page. If you must, use additional scratch paper at the end.
- If you finish a problem somewhere else, please indicate on the problem page where we can find it.
- You have 180 minutes for this 3-hourly.



Picture credit: rendered with Povray code of Tor Olav Kristensen.

Problems

Problem 38.1) (10 points):
a) (3 points) Write down the Laplacian $L \in M(2,2)$ of the complete graph with two vertices and find its two eigenvalues and eigenvectors.

b) (3 points) Write down a closed form solution of the **discrete Schrödinger equation** $if_t = Lf$ on this graph if the initial condition is $f(0) = [3,5]$.

c) (2 points) Prove that the set $X = \{f \in C^\infty(\mathbb{R}) \mid f(x) = -f(-x-1)\}$ is a linear space of functions.

d) (2 points) Prove that the map $T(f) = -f(-1-x)$ is a linear transformation on $C^\infty(\mathbb{R})$.

Problem 38.2) (10 points). Each question is one point:

a) Marc Kac asked a famous question in 1966. We have demonstrated the topic in class. What was the question?
b) What general solution method can be used to solve the differential equation $dx/dt = x^3t^2$?
c) What is the value of the golden mean ϕ? If you should have forgotten, write down a matrix for which it is the eigenvalue and compute the eigenvalue.
d) Which condition has to be satisfied so that a continuous dynamical system $x'(t) = Ax(t)$ is stable?
e) Write down the 5×5 Jordan block to the eigenvalue $\lambda = 4$.
f) Fill in second column of the 2×2 matrix $A = \begin{bmatrix} (1+i)/2 & .... \\ (1-i)/2 & .... \end{bmatrix}$ so that the matrix is in the group $SU(2)$.
g) Who published the book Liber Abaci in 1202?
h) You have proven the identity $QP - PQ = i$ for some operators Q and P acting on smooth functions. What is the name of this identity?
i) What is the name of the mathematician who gave a first rigorous proof that the Fourier series of a differentiable function converges?
j) What does the Parseval identity say?

Problem 38.3) (10 points):
In order to solve the system of equations

$$\left| \begin{array}{ccccccccccc} x & & & & & & & + & v & = 10 \\ & & y & + & z & & & & & = 6 \\ x & & & & & + & u & + & v & = 6 \end{array} \right|$$

we have to row reduce the following matrix:

$$B = \left| \begin{array}{ccccccc} 1 & 0 & 0 & 0 & 1 & | & 10 \\ 0 & 1 & 1 & 0 & 0 & | & 6 \\ 1 & 0 & 0 & 1 & 1 & | & 6 \end{array} \right| .$$

Carefully do that one step at a time. Even if you know how to do it faster, we want you to use one row reduction step at a time. Then, we want you to write down the general solution (x, y, z, u, v) of the system.

Problem 38.4) (10 points):
a) (8 points) Find the best function

$$f(x, y) = ax^4 + by^5 = z$$

which fits the data points $(0, 1, 1), (1, 1, 2), (1, 0, 4)$ using the least square method.

b) (2 points) Somewhere in the computation you had to invert a 2×2 matrix. What is the Jordan normal form of that matrix $A^T A$?

Problem 38.5) (10 points):
a) (4 points) Find all the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} .$$

If you should use some ingenuity to find the eigenvalues, give your reasoning. Only then, your genious will be appreciated! Without reasoning, we assume you were lucky.

b) (2 points) Find a matrix S such that $S^{-1}AS$ is a diagonal matrix B.

c) (1 point) What is the matrix B?

d) (1 point) Give the name of a theorem which assures without computation that A is diagonalizable.

e) (1 point) Is the system $x(t + 1) = Ax(t)$ stable or not?

f) (1 point) Is the system $x'(t) = Ax(t)$ stable or not?

Problem 38.6) (10 points):
The **Barycentric refinement process** for one-dimensional discrete geometries takes a graph with x vertices and y edges and produces a new graph with $x+y$ vertices and $2y$ edges. This defines a discrete dynamical system $T(x,y) = (x+y, 2y)$.

a) (3 points) Write down the 2×2 matrix A which implements this transformation T.

b) (3 points) We start with a graph having $x = 10$ vertices and $y = 11$ edges. How many vertices and edges are there after applying T two times?

c) (4 points) Find a closed form solution which gives $[x(t), y(t)]$ at time t with initial condition from b).'

Problem 38.7) (10 points):
a) (2 points) Find the QR decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

b) (2 points) Find the QR decomposition of the matrix

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} .$$

c) (1 point) Find the QR decomposition of the matrix AB, where A, B are given in a) and b).

d) (1 point) What are the eigenvalues of A?

e) (1 point) What are the eigenvalues of B?

f) (1 point) What is the determinant of AB?

g) (1 point) Why are the matrices AB and BA similar?

h) (1 point) What is the kernel of AB?

Problem 38.8) (10 points):

a) (2 points) What is the determinant of

$$A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} ?$$

b) (2 points) A 22×22 matrix B unknown to you has determinant 22. What is the determinant of the matrix

$$(B^{-22})^T ?$$

c) (6 points) Find the determinant of the following 22×22 matrix: (make sure to document what you are doing! You are allowed to write into the matrix. There is more to the matrix than you might think.)

$$\begin{bmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 8 & 0 & 0 & 8 & 8 & 8 & 8 & 0 & 8 & 8 & 8 & 8 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix}$$

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Problem 38.9) (10 points):
Solve the following differential equations. You can of course use any method you know but you have to document what you do.

a) (2 points) $f'(t) + 3f(t) = e^{-2t}$.

b) (2 points) $f''(t) - 2f'(t) + f(t) = t$

c) (2 points) $f''(t) + 4f(t) = \cos(t)$

d) (2 points) $f''(t) + 4f(t) = \cos(2t)$

e) (2 points) $f'''(t) - f''(t) - f'(t) + f(t) = 1$

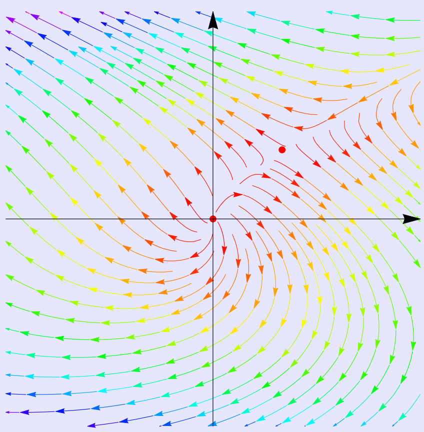
Problem 38.10) (10 points):
Analyze the solutions $(x(t), y(t))$ for the following nonlinear dynamical system

$$\frac{d}{dt}x = x - y^2$$
$$\frac{d}{dt}y = y - x$$

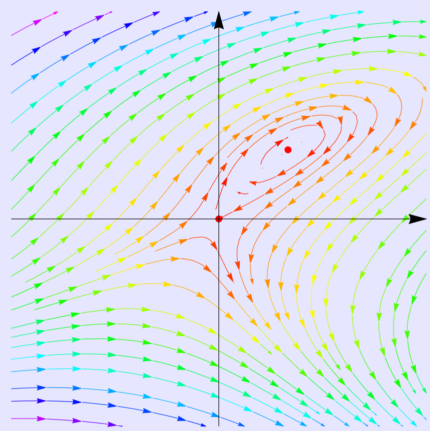
a) (3 points) Find the equations of the null-clines and find all the equilibrium points.

b) (4 points) Analyze the stability of all the equilibrium points.

c) (3 points) Which of the phase portraits A,B,C,D below belongs to the above system?



A



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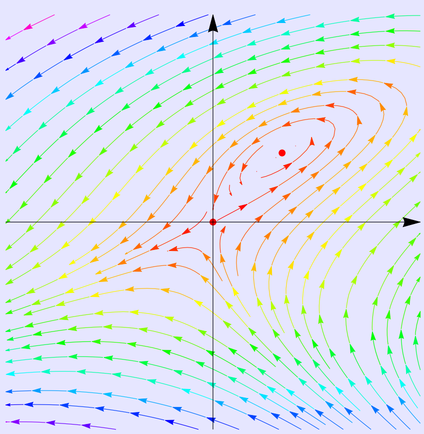

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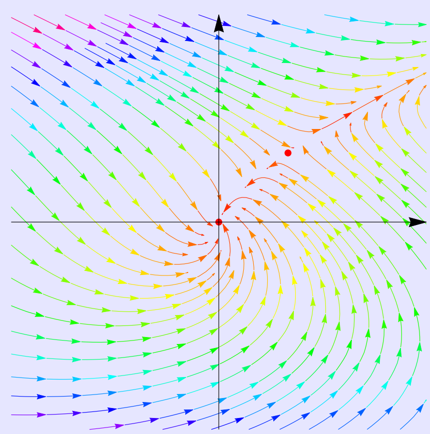
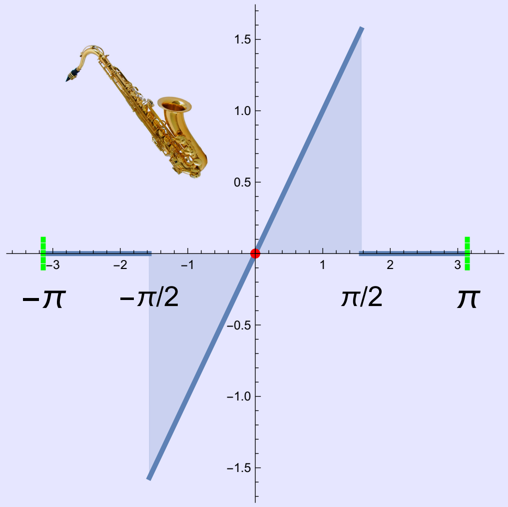


D

Problem 38.11) (10 points):
We try to model the sound of a **saxophone**. Its hull function is roughly given as a 2π periodic function which is on $[-\pi, \pi]$ defined as follows: the function is equal to x on $[-\pi/2, \pi/2]$ and $f(x)$ is 0 on $(\pi/2, \pi]$ and on $[-\pi, -\pi/2)$. You see the graph below. Your task is to find the Fourier series of this function.



Problem 38.12) (10 points):
In order to find out whether **grumpy cat** is dead or alive, you solve the **non-linear Schrödinger equation**

$$f_t = 3i f_{xxxx}$$

with initial condition $f(0, x) = \sin^2(x) + 7\sin(10x) + 9\cos(11x)$.

Problem 38.13) (10 points):

You are a scientist investigating **"monster waves"**. Also called **"rogue waves"**. The subject is also of interest to surfers. [This problem was written while watching again the classic "Point break" (the 1991 one, the 2015 remake has great stunts too):]

Find the solution of the nonlinear wave equation

$$f_{tt} = 2 f_{xx} + 3 f_{xxxxxx} + 12 t^2$$

for which

$$f(0, x) = 9 \sin(4x) + 8 \cos(6x)$$

and

$$f_t(0, x) = 7 \sin(13x) + \sum_{n=1}^{\infty} \frac{1}{n^4} \cos(nx) \; .$$