Linear Algebra And Vector Calculus

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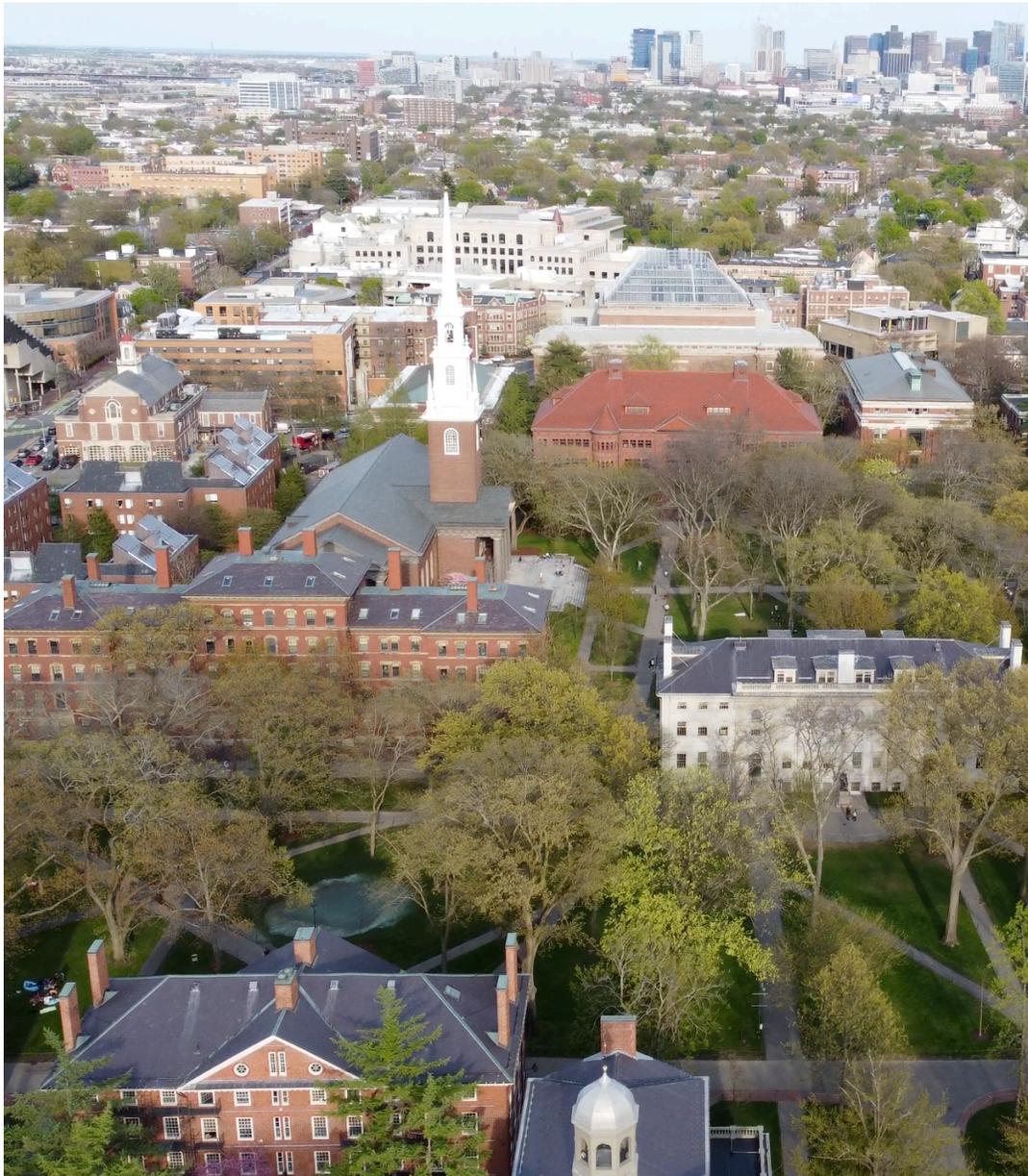

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Unit 1: Pythagorean theorem

Introduction

1.1. In this first lecture, we look at one of the most important theorems in mathematics, the **theorem of Pythagoras**. The historical roots of the theorem are mesmerizing: the first examples of identities like $5^2 + 12^2 = 13^2$ already appeared in Sumerian mathematics. Triples of numbers like $(5, 12, 13)$ are called **Pythagorean triples**. The theorem itself is much more than that. The theorem not only lists a few examples for evidence but states and proves that for all triangles, the relation $a^2 + b^2 = c^2$ holds if and only if the triangle is a right angle triangle. Without exaggeration, the Pythagorean theorem is one of the most beautiful and most important theorems. It has cameos in various other parts of mathematics. In harmonic analysis for example, it tells that the square of the length of a periodic function is the sum of the squares of its Fourier coefficients. In probability theory it tells that if two random variables X, Y are uncorrelated, then the variance of $X + Y$ is the sum of the variance of X and Y.
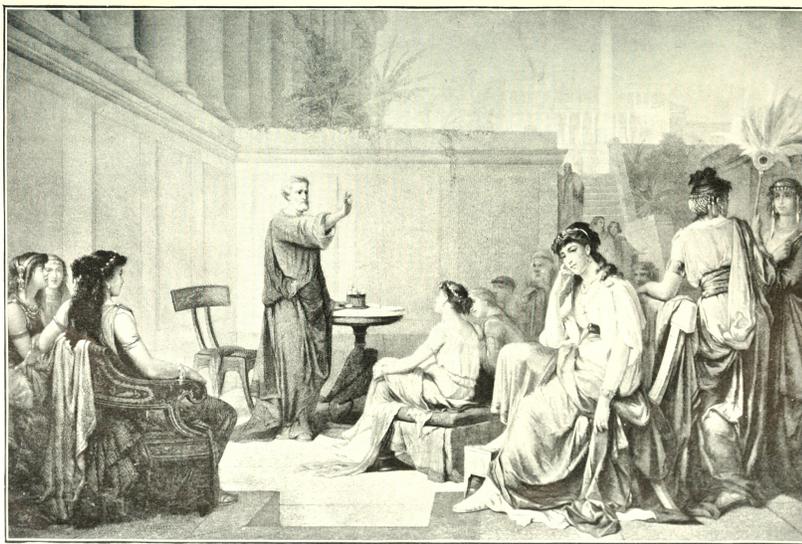

FIGURE 1. The picture appears in "The story of the greatest nations", (1910) and shows Pythagoras teaching mathematics.

1.2. We use here the theorem also while introducing **vectors** and **linear spaces**. The language of matrices is not only a matter of notation, but also allows for a slightly more sophisticated approach to vector calculus in which one distinguishes between column vectors and row vectors. Unlike in standard vector analysis courses, this is possible when working closer to linear algebra. Traditionally, many sources define a vector as a quantity with "magnitude" and "direction". This is highly problematic as a "movie" qualifies for this notion: it has a length and has a director. But we don't need to mock this with a pun: the zero vector 0 is a quantity which would not qualify as a vector because the zero vector does not have a direction. Because of such problems, one usually defines a vector as a quantity defined by two points A, B in space, writes $\vec{AB}$ and thinks about the vector as a translation from A to B or as an "arrow" starting from A and ending at B. Now, one has the difficulty that two parallel vectors of equal length are identified. One actually uses equivalence classes to get from affine space to linear space. The modern point of view is that one can attach a linear space of vectors at every point and think of $\vec{AB}$ as a vector attached to the point A. We will see for example the concept of a gradient field, which attaches at every point a row vector. Force fields are examples.

1.3. In any case, introducing spaces of matrices early has advantages also in a time where **data analysis** is recognized as an important tool. **Relational data bases** are founded on the concept of matrices. Most familiar are **spread sheets** which are two-dimensional arrays in which data are organized. More recently, such concept are also displaced with more sophisticated data structures like **graph data bases**. Still, a graph can also be described by matrices. Given two nodes x, y of the network, write in the matrix entry A_{xy} tells how they are related. In the simplest case put a 1 if the nodes are connected and 0 if they are not connected. In any case, **data** are always arrays of more basic quantities. The memory structure of a computer is organized as an array. As Alan Turing has shown, all computations we have formalized can be done on a one dimensional tape with entries 0 and 1. Modern computer storage devices are essentially such Turing tapes, but organized in a more sophisticated way, using partitions or sectors similarly as matrices are organized with rows and columns.

LECTURE

1.4. A finite rectangular array A of real numbers is called a **matrix**. If there are n rows and m columns in A, it is called a $n \times m$ matrix. We address the entry in the i'th row and j'th column with A_{ij}. A $n \times 1$ matrix is a **column vector**, a $1 \times n$ matrix is a **row vector**. A 1×1 matrix is called a **scalar**. Given a $n \times p$ matrix A and a $p \times m$ matrix B, the $n \times m$ matrix AB is defined as $(AB)_{ij} = \sum_{k=1}^{p} A_{ik} B_{kj}$. It is called the **matrix product**. The **transpose** of a $n \times m$ matrix A is the $m \times n$ matrix $A_{ij}^T = A_{ji}$. The transpose of a column vector is a row vector.

1.5. Denote by $M(n, m)$ the set of $n \times m$ matrices. It contains the **zero matrix** O with $O_{ij} = 0$. In the case $m = 1$, it is the **zero vector**. The **addition** $A+B$ of two matrices in $M(n, m)$ is defined as $(A+B)_{ij} = A_{ij}+B_{ij}$. The **scalar multiplication** λA is defined as $(\lambda A)_{ij} = \lambda A_{ij}$ if λ is a real number. These operations make $M(n, m)$ a **vector space = linear space**: the addition is **associative**, **commutative** with a unique

additive inverse $-A$ satisfying $A - A = 0$. The multiplications are **distributive**: $A(B + C) = AB + AC$ and $\lambda(A + B) = \lambda A + \lambda B$ and $\lambda(\mu A) = (\lambda\mu)A$.

1.6. The space $M(n, 1)$ is also called $\mathbb{R}^n$. It is the n-dimensional **Euclidean space**. The vector space $\mathbb{R}^2$ is the **plane** and $\mathbb{R}^3$ is the **physical space**. These spaces are dear to us as we draw on paper and live in space. The **dot product** between two column vectors $v, w \in \mathbb{R}^n$ is the matrix product $v \cdot w = v^T w$. Because the dot product is a scalar, the product is also called the **scalar product**. In the matrix product of two matrices A, B, the entry at position (i, j) is the dot product of the i'th row in A with the j'th column in B. More generally, the **dot product between** two arbitrary $n \times m$ matrices can be defined by $A \cdot B = \text{tr}(A^T B)$, where the **trace** of a matrix is the sum of its diagonal entries. This means $\text{tr}(A^T B) = \sum_{i,j} A_{ij} B_{ij}$. We just take the product over all matrix entries and add them up. The dot product is distributive $(u + v) \cdot w = u \cdot w + v \cdot w$ and **commutative** $v \cdot w = w \cdot v$. We can use it to define the **length** $|v| = \sqrt{v \cdot v}$ of a vector or the **length** $|A|$ of a matrix, where we took the positive square root. The sum of the squares is zero exactly if all components are zero. The only vector satisfying $|v| = 0$ is therefore $v = 0$.

1.7. An important key result is the **Cauchy-Schwarz inequality**.

Theorem: $|v \cdot w| \leq |v||w|$

Proof. If $w = 0$, there is nothing to prove as both sides are zero. If $w \neq 0$, then we can divide both sides of the equation by $|w|$ and so achieve that $|w| = 1$. Define $a = v \cdot w$. Now, $0 \leq (v - aw) \cdot (v - aw) = |v|^2 - 2av \cdot w + a^2|w|^2 = |v|^2 - 2a^2 + a^2 = |v|^2 - a^2$ meaning $a^2 \leq |v|^2$ or $v \cdot w \leq |v| = |v||w|$. $\qquad\square$

1.8. It follows from the Cauchy-Schwarz inequality that for any two non-zero vectors v, w, the number $(v \cdot w)/(|v||w|)$ is in the closed interval $[-1, 1]$. There exists therefore a unique **angle** $\alpha \in [0, \pi]$ such that $\cos(\alpha) = (v \cdot w)/(|v||w|)$. If this angle between v and w is equal to $\alpha = \pi/2$, the two vectors are **orthogonal**. If $\alpha = 0$ or π the two vectors are called **parallel**. There exists then a real number λ such that $v = \lambda w$. The zero vector is considered both orthogonal as well as parallel to any other vector.

1.9. Two vectors v, w define a (possibly degenerate) **triangle** $\{0, v, w\}$ in Euclidean space $\mathbb{R}^n$. The above formula defines an angle α at the point 0 (which could be the zero angle). The **side lengths** $a = |v|, b = |w|, c = |v - w|$ of the triangle satisfy the following cos **formula**. It is also called the **Al Kashi identity**.

Corollary: $c^2 = a^2 + b^2 - 2ab\cos(\alpha)$

Proof. We use the definitions as well as the distributive property (FOIL out): $c^2 = |v - w|^2 = (v - w) \cdot (v - w) = v \cdot v + w \cdot w - 2v \cdot w = a^2 + b^2 - 2ab\cos(\alpha)$. $\qquad\square$

1.10. The case $\alpha = \pi/2$ is particularly important. It is the **Pythagorean theorem**:

Theorem: In a right angle triangle we have $c^2 = a^2 + b^2$.

EXAMPLES

1.11. The dot product $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$ is $[1,3,1] \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = 1 - 6 - 1 = -6$. We have $|v| = \sqrt{11}, |w| = \sqrt{6}$ and angle $\alpha = \arccos(-6/\sqrt{66})$.

1.12. The dot product of $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 \\ 4 & -1 \end{bmatrix}$ is $\mathrm{tr}(A^T B) = 6 + 2 + 8 + (-1) = 15$. The length of A is $\sqrt{12}$, the length of B is 5. The angle between A and B is $\alpha = \arccos(15/(5\sqrt{12})) = \arccos(\sqrt{3}/2) = \pi/6$.

1.13. $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ are perpendicular because $\mathrm{tr}(A^T B) = 0$. The angle between them is $\pi/2$. The length of A is $a = \sqrt{10}$. The length of B is $b = \sqrt{4} = 2$. The length of $A + B = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ is $c = \sqrt{14}$. We confirm $a^2 + b^2 = c^2$. Note that $AB \neq BA$. Multiplication is not commutative.

1.14. Find the angles in a triangle of length a=4,b=5 and c=6. Answer: Al Kashi gives $2 \cdot 4 \cdot 5 \cos(\gamma) = 4^2 + 5^2 - 6^2 = 5$ so that $\gamma = \arccos(5/40)$. Similarly $2 \cdot 4 \cdot 6 \cos(\beta) = 27$ so that $\gamma = \arccos(27/48)$ and $2 \cdot 5 \cdot 6 \cos(\alpha) = 45$ so that $\alpha = \arccos(45/60)$.
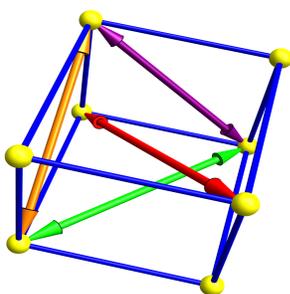
ILLUSTRATION

FIGURE 2. A cuboid of integer side length a, b and c such that $a^2 + b^2, a^2 + c^2, b^2 + c^2$ are squares is an **Euler brick**. Its side diagonals are now integers. The smallest one $(a, b, c) = (44, 117, 24)$ was found in 1719. If also $a^2 + b^2 + c^2$ is a square, meaning that the space diagonal is an integer too, we have a **perfect Euler brick**. Nobody has found one. It is a famous open problem due to Euler, whether there exists one.

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[1]Knill, 2009: http://www.math.harvard.edu/~knill/various/eulercuboid/lecture.pdf

FIGURE 3. This **Povray scene** was generated by a method which involves a lot of vector calculus and linear algebra: this open source **ray tracer** bounces around light in the virtual scene and computes the reflections. A camera then captures the photons, similarly as a real camera does. Textures are implemented by images, here a postcard of Harvard square from 1930. It is a image file encoding three 1688×1104 matrices R,G,B, red, green and blue values at each pixel. The scene is an "homage" to the novel "On Time and the River" by Thomas Wolfe who was a Harvard undergraduate here from 1920-1922 (notice the 22!)

1.15. Mathematics is not only **eternal**, but also **infinite**. To illustrate this, look at the **"Eternals" problem** [2]. Define the **Babylonian graph** B in which the positive integers are the vertices and where (a, b) are connected, if $a^2 + b^2$ is a perfect square. Every edge in B belongs to a **Pythagorean triple** $a^2 + b^2 = c^2$. One can ask which type of sub-graphs appear, how many connected components there are, whether the diameter is infinite, or how large closed loops can become. Hundreds of questions could be asked. **Embedded triangles** K_3 in B for example are **Euler bricks**! Are there **embedded tetrahedra** K_4, cliques of numbers (a, b, c, d) for which every pair is a Pythagorean triple? This would be an **Eulerian tesseract**. Is there one? Before proving anything, we have a **data problem**. Experiment!
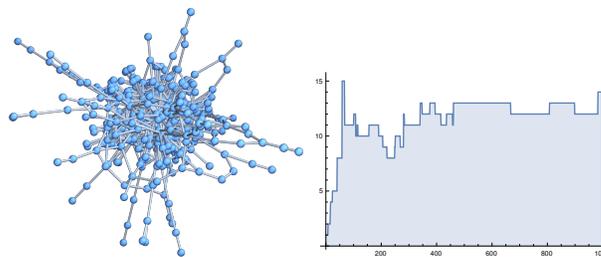

FIGURE 4. To the left we see the largest component B_{1000}. An experiment like **ListPlot[Table[GraphDiameter[Babylonian[n]],n,1000]]** gives the diameter of the largest component $B_1(n)$ of $B(n)$. We have $\text{Diam}(B_1(5000)) = 18$, $\text{Diam}(B_1(10000)) = 29$.

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[2]This problem has been communicated to us by Ajak, who knows thousands of years of mathematics

Homework

Problem 1.1: Use the definitions to find the angle α between the vector $v = [1, 1, 0, -3, 2, 1]^T$ and $w = [1, 1, 9, -3, -5, -3]^T$ in $\mathbb{R}^6$. If we think about v, w as data, the value $\cos(\alpha)$ is the **correlation** between the two data points v and w. If the cosine is positive, the data have positive correlation. If the cosine is negative, they have negative correlation.

Problem 1.2: Given the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

a) Find A^T, then build $B = A + A^T$ and $C = A - A^T$. The first matrix is called **symmetric**, the second is called **anti-symmetric**.
b) Compute AA^T and $A^T A$. Then evaluate $\text{tr}(A^T A)$ and $\text{tr}(AA^T)$.
c) Why are these two numbers computed in b) the same? Is it true in general for two $n \times m$ matrices that $\text{tr}(A^T B) = \text{tr}(B^T A)$? (There is a short verification using the sum notation).

Problem 1.3: a) Verify the triangle identity $|v-w| \le |v|+|w|$ in general by FOILing out $(v-w) \cdot (v-w)$, then generate an example of two vectors with integer coordinates in the plane $\mathbb{R}^2$, where one can apply this. Draw the situation.
b) Verify that if v and w have the same length, then $(v-w)$ and $(v+w)$ are perpendicular. Describe the situation in b) geometrically in a sentence.

Problem 1.4: Write the vector $F = [2, 3, 4]^T$ as a sum of a vector parallel to $v = [1, 1, 1]^T$ and a vector perpendicular to v. If we interpret F as a **force** acting on a kite of mass 1 and v as the velocity then $F \cdot v$ has an interpretation as power, the rate of change of the energy of the kite. The vector parallel to v would by Newton be the acceleration of the kite.

Problem 1.5: a) Find two vectors in $\mathbb{R}^2$ for which all coordinate entries are 1 or -1 and which are both perpendicular to each other.
b) Design four vectors in $\mathbb{R}^4$ for which all coordinate entries are 1 or -1 which are all perpendicular to each other.
Optional: can you invent a strategy which allows you for example to find 16 vectors in $\mathbb{R}^{16}$ which are all perpendicular to each other and have still entries in $\{-1, 1\}$?
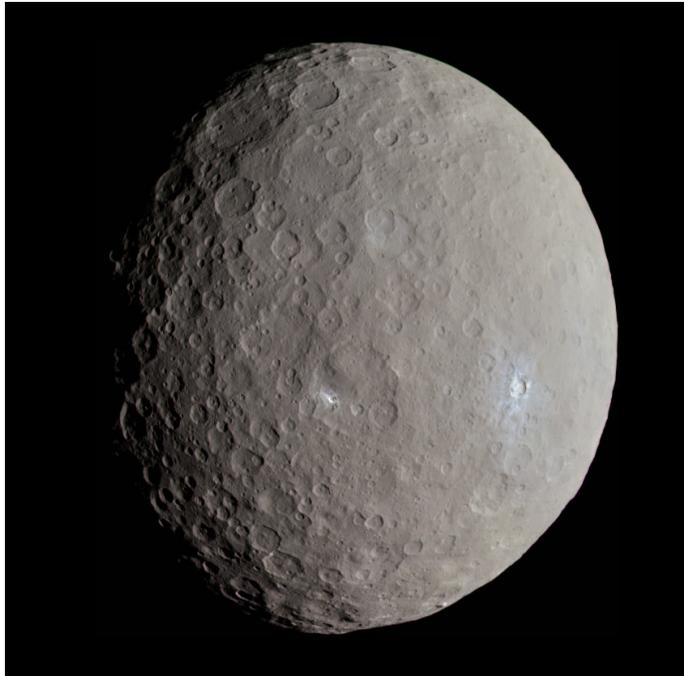
Unit 2: Gauss-Jordan elimination

Introduction

Figure 1. A picture of the dwarf planet Ceres taken in 2015 by the NASA Dawn space craft.

2.1. Systems of linear equations have already been tackled four thousand years ago by Babylonian mathematicians. [1] They were able to solve simple systems of equations of two unknown like $x + y = 12, x - y = 2$. Chinese mathematicians, in the "Nine chapter of the mathematical art", pushed this to systems of three equations and also to related number theoretical setting appearing in the form of the Chinese remainder theorem. The problem of solving systems of equations appeared also in analysis, like when maximizing functions with constraints. The method of determinants, pioneered

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[1]I checked with Ajak who also hinted, that Phastos might have leaked Gauss-Jordan elimination. The opening scene in the newest Marvel's movie shows the eternals arriving here 5000 BC.

by Leibniz, led with Gabriel Cramer to explicit solution formulas of systems or equations. The modern approach of solving systems of equation uses a clear cut elimination process. This process is wicked fast and was formalized by Carl Friedrich Gauss. It is the method we still are using today. It also allows to compute determinants effectively.

2.2. Elimination was of course used long before Gauss. We learn it early on as ordinary elimination. For example, solve for one variable and put it into the rest to have a system with less unknown. What Gauss did was to write down a formal elimination process. This was around 1809. He called the ordinary elimination "eliminationem vulgarem". The process came not out of the blue. The work on rather applied problems must have led to it. For example, Gauss in 1801 was able in a few weeks to predict the path of the minor planet **Ceres** from 24 measurements published in the summer of 1801. It is reported that Gauss needed more than 100 hours to determine the 5 parameters for the orbit. Exercises like this definitely motivated Gauss for writing down later more formal procedures allowing to solve linear problems or more general data fitting problems. The name "Gaussian elimination" was used first by George Forsythe, while Alan Turing described it in the way we teach it today. We will see here formally that the process determines in a unique way. [2]

Lecture

2.3. If a $n \times m$ matrix A is multiplied with a vector $x \in \mathbb{R}^m$, we get a new vector Ax in $\mathbb{R}^n$. The process $x \to Ax$ defines a **linear map** from $\mathbb{R}^m$ to $\mathbb{R}^n$. Given $b \in \mathbb{R}^n$, one can ask to find x satisfying the **system of linear equations** $Ax = b$. Historically, this gateway to **linear algebra** was walked through much before matrices were even known: there are Babylonian and Chinese roots reaching back thousands of years. [3]

2.4. The best way to solve the system is to **row reduce** the **augmented matrix** $B = [A|b]$. This is a $n \times (m+1)$ matrix as there are $m+1$ columns now. The **Gauss-Jordan elimination** algorithm produces from a matrix B a **row reduced matrix** rref(B). The algorithm allows to do three things: **subtract a row from another row**, **scale a row** and **swap two rows**. If we look at the system of equations, **all these operations preserve the solution space**. We aim to produce **leading ones** ①, which are matrix entries 1 which are the first non-zero entry in a row. The goal is to get to a matrix which is in **row reduced echelon form**. This means: A) every row which is not zero has a leading one, B) every column with a leading 1 has no other non-zero entries besides the leading one. The third condition is C) every row above a row with a leading one has a leading one to the left.

2.5. We will practice the process in class and homework. Here is a theorem

> **Theorem:** Every matrix A has a unique row reduced echelon form.

Proof. [4] We use the method of **induction** with respect to the number m of columns in the matrix. The **induction assumption** is the case $m = 1$ where only one column exists. By condition B) there can either be zero or 1 entry different from zero. If there

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[2] J.F. Grcar, Mathematicians of Gaussian Elimination, Notices of the AMS, 58, 2011

[3] For more, look at the exhibit on the website of the 2018 Math 22a.

[4] The proof is well known: i.e. Thomas Yuster, Mathematics Magazine, 1984

is none, we have the zero column. If it is non-zero, it has to be at the top by condition C). We are in row reduced echelon form. Now, let us assume that all $n \times m$ matrices have a unique row reduced echelon form. Take a $n \times (m+1)$ matrix $[A|b]$. It remains in row reduced echelon form, if the last column b is deleted (see lemma). Remove the last column and row reduce is the same as row reducing and then delete the last column. So, the columns of A are uniquely determined after row reduction. Now note that for a row of $[A|b]$ without leading one at the end, all entries are zero so that also the last entries agree. Assume we have two row reductions $[A'|b']$ and $[A'|c']$ where A' is the row reduction of A. A leading $①$ in the last column of $[A'|b']$) happens if and only if the corresponding row in A was zero. So, also $[A', c']$ has that leading $①$ at the end. Assume now there is no leading one in the last column and $b'_k \neq c'_k$. We have so x, a solution to the equation $A'_{kq}x_q + A'_{k,q+1}x_{q+1} + \ldots A'_{k,m}x_m = b'_k$. Since solutions to equations stay solutions when row reducing, also $A'_{kq}x_q + A'_{k,q+1}x_{q+1} + \cdots + A'_{k,m}x_m = c'_k$. Therefore $b'_k = c'_k$. $\qquad \square$

2.6. A separate lemma allows to break up a proof:

> **Lemma:** If $[A|b]$ is row reduced, then A is row reduced.

Proof. We have to check the three conditions which define row reduced echelon form. $\qquad \square$

2.7. It is not true that if A is in row reduced echelon form, then any sub-matrix is in row reduced echelon form. Can you find an example?

<div align="center">EXAMPLES</div>

2.8. To row reduce, we use the three steps and document on the right. To save space, we sometimes report only after having done two steps. We circle the **leading** $①$. Note that we did not immediately go to the leading $①$ by scaling the first. It is a good idea to **avoid fractions** as much as possible.

3	4	5	6	7	$\rightarrow$ row 3
20	30	40	50	60	
1	2	3	4	4	$\rightarrow$ row 1

$①$	2	3	4	5	
2	3	4	5	6	$-R_1$
3	4	5	6	7	$-R_2$

$①$	2	3	4	5	$+2R_2$
0	-1	-2	-3	-4	$*(-1)$
0	0	0	0	0	

$①$	2	3	4	5	
20	30	40	50	60	$*1/10$
3	4	5	6	7	

$①$	2	3	4	5	
1	1	1	1	1	$-R_1$
1	1	1	1	1	$-R_2$

$①$	0	-1	-2	-3	
0	$①$	2	3	4	
0	0	0	0	0	

2.9. Finish the following **Suduku problem** which is a game where one has to fix matrices. The rules are that in each of the four 2×2 sub-squares, in each of the four rows and each of the four columns, the entries 1 to 4 have to appear and so add up to 10 $\begin{bmatrix} 2 & 1 & x & 3 \\ 3 & y & z & 1 \\ 4 & 3 & a & 2 \\ b & c & d & e \end{bmatrix}$. We have the equations $2 + 1 + x + 3 = 10, 3 + y + z + 1 = 10, 4 + 3 + a + 2 = 10, b + c + d + e = 10$ for the rows, $2 + 3 + 4 + b =$

$10, 1 + y + 3 + c = 10, x + z + a + d = 10, 3 + 1 + 2 + e + 10$ for the columns and $2 + 1 + 3 + y = 10, x + 3 + z + 1 = 10, 4 + 3 + b + c = 10, a + 2 + d + e = 10$ for the four squares. We could solve the system by writing down the corresponding augmented matrix and then do row reduction. The solution is $\begin{bmatrix} 2 & 1 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 4 & 3 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

ILLUSTRATIONS

The system of equations

$$\begin{array}{ccccccccc} x & & & & + & u & & & = & 3 \\ & y & & & & & + & v & & = & 5 \\ & & z & & & & & + & w & = & 9 \\ x & + & y & + & z & & & & & = & 8 \\ & & & & u & + & v & + & w & = & 9 \end{array}$$

is a **tomography** problem. These problems appear in **magnetic resonance imaging**. A precursor was was X-ray Computed Tomography (CT) for which Allen MacLeod Cormack got the Nobel in 1979 (Cormack had a sabbatical at Harvard in 1956-1957, where the idea hatched). Cormack lived until 1998 in Winchester MA. He originally had been a physicist. His work had tremendous impact on medicine.
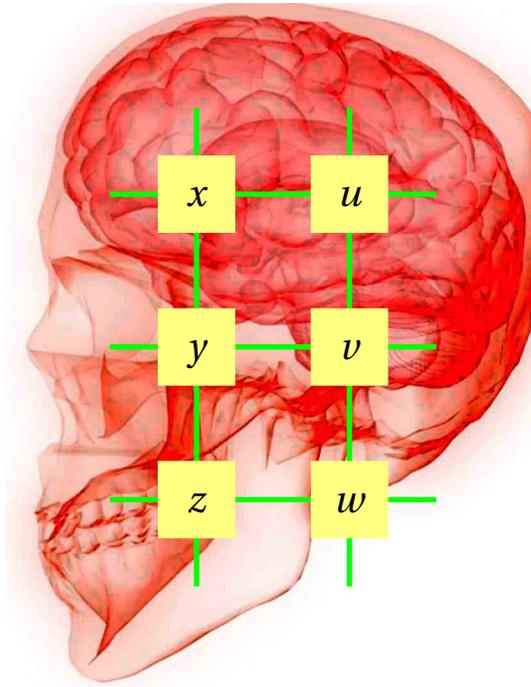

FIGURE 2. A MRI scanner can measure averages of tissue densities along lines. MRI (Magnetic Resonance Imaging) is a radiology imaging technique that avoids radiation exposure to the patient). Solving a system of equations allows to compute the actual densities and so to do the magic of "seeing inside the body".

We build the augmented matrix $[A|b]$ and row reduce. First remove the sum of the first three rows from the 4th, then change the sign of the 4'th column:

$$\begin{bmatrix} ① & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & ① & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & 1 & 9 \\ 1 & 1 & 1 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 1 & 1 & 1 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} ① & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & ① & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & ① & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & ① & 1 & 1 & 9 \\ 0 & 0 & 0 & 1 & 1 & 1 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} ① & 0 & 0 & 0 & -1 & -1 & -6 \\ 0 & ① & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & ① & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & ① & 1 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we can read of the solutions. We see that v and w can be chosen freely. They are free variables. We write $v = r$ and $w = s$. Then just solve for the variables:

$$
\begin{aligned}
x &= -6 + r + s \\
y &= 5 - r \\
z &= 9 - s \\
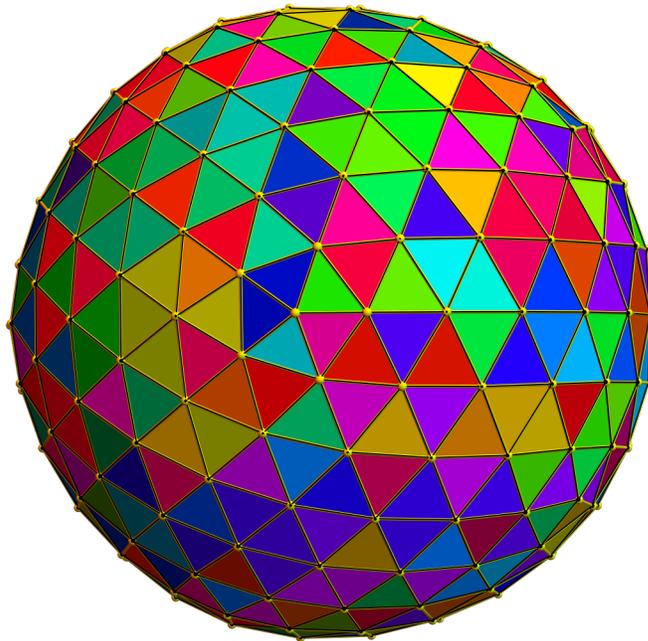u &= 9 - r - s \\
v &= r \\
w &= s
\end{aligned}
$$

FIGURE 3.   This is a polyhedron with F=540 faces. In the homework you compute from this knowledge alone the number of vertices V and the number of edges E. The equations which hold are $V - E + F = 2$ and $3F = 2E$.

HOMEWORK

Problem 2.1: For a **polyhedron** with v vertices, e edges and f triangular faces Euler proved his famous formula $V - E + F = 2$. An other relation $3F = 2E$ called a Dehn-Sommerville relation holds because each face meets 3 edges and each edge meets 2 faces. Assume the number F of triangles is $F = 540$. Write down a system of equations for the unknowns V, E, F in matrix form $Ax = b$, then solve it using Gaussian elimination.

Problem 2.2: Row reduce the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 0 \\ 1 & 2 & 3 & 0 & 0 \end{bmatrix}$.

Problem 2.3: In the "Nine Chapters on Arithmetic", the following system of equations appeared

$$3x + 2y + z = 39$$
$$2x + 3y + z = 34$$
$$x + 2y + 3z = 26$$

Solve it using row reduction by writing down an augmented matrix and row reduce.

Problem 2.4: a) Which of the following matrices are in row reduced echelon form?
$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.
b) Two $n \times m$ matrices in reduced row-echelon form are called **of the same type** if they contain the same number of leading l's in the same positions. For example, $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are of the same type. How many types of 2×2 matrices in reduced row-echelon form are there?

Problem 2.5: Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Compare $\mathrm{rref}(A^T)$ with $(\mathrm{rref}(A))^T$. Is it true that the transpose of a row reduced matrix is a row reduced matrix?

Unit 3: Definitions, Theorems and Proofs

INTRODUCTION

3.1. One of the truly amazing things about mathematics is that it is a frame work, where if something is established, it will remain truth for all eternity. The focus of most sciences changes rather quickly and frequently, entire paradigms change. Mathematics evolves too of course. But once established truths do not change. The theorem of Pythagoras we have proven in the first lecture is something which will still be true in a million years. The language how we describe a statement will almost certainly have completely changed in a future mathematical frame work. The statement of Pythagoras still will be valid.
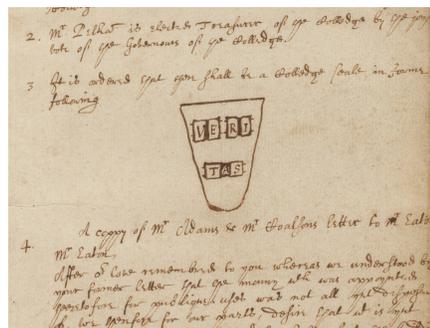

FIGURE 1. The first drawing of "Veritas". Veritas means "truth" Source: College Book 1, 1639-1795. UAI 5.5 Box 1, Harvard University Archives.

3.2. One of the most important sources for confusion are **sloppy definitions**. Mathematics has early on insisted on precise and unambiguous definitions. We have seen this in the first lecture. Defining a "vector" as a quantity with magnitude and direction is not only ambiguous and wrong (as it does not capture the zero vector), it lulls you into some sort of "understanding" as we all have intuition about magnitude like "length" and "direction" from daily life. It often happens even in "hard sciences" that sloppy definitions are used. Well, we have not be too arrogant. It turns out that coming up with precise and still elegant definitions is a rather difficult task in general. In physics, it took a long time to replace notions like "vis viva" and replace it with precise definitions like momentum or kinetic energy. It was Emily du Châtelet who

essentially contributed to clear up the definitions and distinguish momentum mv and energy $mv^2/2$

3.3. The backbone of mathematics are the theorems. These are statement which have been verified using a careful sequence of arguments, where each step is either using a basic logical step or then uses a previously established theorem. It is extremely important not to have any wrong theorem in this process. Otherwise, everything which is built upon it will fall. Mathematics is like a big computer program in which the individual procedures are the theorems. If one of the procedures is faulty, it can bring down the entire system. There is always the risk that a proof will turn out to be incomplete or wrong and history has shown this to be the case again and again. Most of the time, one can fix the statement. Sometimes, one can not fix it because the statement has counter examples. In that case one has to modify the statement or adapt the definitions so that it becomes true. Lakatos has in his famous book "proofs and refutations" illustrated this in the context of the Euler Gem formula $V - E + F = 2$ we have seen in the second lecture. This is a place where sloppy definitions for the notion of "polyhedron" led to wrong statements and the theorem had to be mended over time.

Lecture

3.4. Theorems are mathematical statements which can be verified by giving a proof. A **proof** assures that the theorem is true and remains valid also in the future. Let us look at an example of a theorem. It has already been known and proven by **Euclid of Alexandria**. It deals with **integers** and **primes**, positive integers larger than 1 which are only divisible by 1 or itself. The theorem tells that every positive integer is either 1 or prime or the product of two or more primes. To formulate the theorem more elegantly, we extend the notion of **product** and say that a prime is the product of $k = 1$ primes and that the number 1 is a product of $k = 0$ primes. Also we would say the number $20 = 2 * 2 * 5$ is the product of $k = 3$ primes, even so the prime 2 appears twice. This is similar to the water molecule $H_2O = H * H * O$ containing $k = 3$ atoms, as hydrogen H appears twice and oxygen O once. Now, like every molecule decomposes into atoms, every number decomposes into primes:

> **Theorem:** "Every integer $n \geq 1$ is a product of $k \geq 0$ primes".

This is a remarkable statement because there are infinitely many integers. We can not go therefore through an infinite list and check things for each. It could a priori happen that for some very large number, like the **Fermat number** $F_{1000} = 2^{(2^{1000})} + 1$, which can not even be written down in our universe, [1] the statement would fail.

3.5. In order that such a statement can be verified or refuted, one needs first of all to make sure that the objects are described by **clear definitions**. In the above sentence, this means that we need to know what the "integers" are, what a "product" is and what "prime numbers" are. This is already tricky in general. Most confusions which have happened historically in science (and still today!) are based on sloppy definitions. [2]

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[1] There are less than 2^{300} elementary particles available in our universe (as far as we know).

[2] Amuse yourself and try to find definitions of "entropy", "multiverse", "intelligence" or "life"

3.6. Once, the definitions of the ingredient of the statement is clear, it is helpful to clarify its **meaning**. We get intuition by looking at **examples**. We see for example that $100 = 2 * 2 * 5 * 5$ is indeed a product of prime numbers. We see also that 7 is a prime number. Examples are great but it is important at this stage to realize:

> **Principle:** Checking a statement by showing a few examples is not a proof.

We will come back to this later in the course.

Statement	Belongs to theorem
$3^2 + 4^2 = 5^2$	
$63 = [3, 4] \cdot [5, 12] \leq 5 * 13 = 65$	
$[0, 1, 0, 0, 1]$ can not be row reduced to $[0, 0, 1, 1, 0]$.	

3.7. One of the important proof techniques is the **principle of mathematical induction**. [3] It is mostly applied to integers but it can also be used for matrices as we have seen in the second lecture. The principle applies for statements $S(n)$ which depend on a number n.

> **Principle:** $S(1)$ and $S(n) \Rightarrow S(n + 1)$ implies $S(n)$ for all $n \geq 1$.

3.8. Here is an example:

> **Theorem:** S(n): $1 + 2 + 3 + \cdots + n = (n^2 + n)/2$.

Proof: the statement $S(n)$ is true for $n = 1$. Assume $S(n)$ is true. Now $S(n + 1)$ tells $1 + 2 + \cdots + n + (n + 1) = ((n + 1)^2 + (n + 1))/2$. Using the induction assumption, this means $(n^2 + n)/2 + (n + 1) = ((n + 1)^2 + (n + 1))/2$, which is true. We know therefore that the statement is true for all n.

---

[3] Already used by Plato and a second order axiom in the **Peano axiom system.**

3.9. Let us look at the theorem on primes above. In order to make this a statement which we can extend from n to $n + 1$, we modify the statement to

Theorem: S(n): Every $k \in \{2, 3, 4 \cdots n\}$ has a prime factorization.

3.10. $S(2)$ is true as $\{2\}$ only contains one number which is prime. Now assume $S(n)$ meaning that the statement is true for n, prove that $S(n + 1)$ is true. There are two cases: if $n + 1$ is prime, then $S(n+1)$ is true. If $n + 1$ is not prime, then $n = ab$ where a and b are numbers larger than 1 but smaller than n. By induction assumption, both a and b decay into primes: $a = p_1 p_2 \cdots, p_k$ and $b = q_1 q_2 \ldots, q_l$ where p_j and q_j are primes. Therefore, $n + 1 = p_1 p_2 \cdots p_k q_1 q_2 \cdots q_l$.

3.11. It is important to understand the statement and not to overreach it. We have not proven that every integer has a **unique** decomposition into prime factors. This was not known by Euclid (who might not even have thought about it). It was only proven 2000 years later by Gauss. A common mistake which happens in mathematical proofs is that one cites a theorem which is known but over reaches its scope or then that one forgets one of the assumptions.

Principle: Do not extend the scope of an already established fact without justification.

3.12. If you think such mistakes happen to rookies only, this is not the case. Leonard Euler, probably the greatest mathematician of all times once attempted a proof of Fermat's last theorem by working with extended number systems like $\mathbb{Z}[\sqrt{-3}]$ which are all the numbers of the form $a + \sqrt{-3}b$, where a, b are integers. You see, one can add and multiply such numbers like integers and remain in the class. Euclid's proof also shows that there is a prime factorization. But there can be different prime factorizations. An example is $4 = 2 * 2 = (1 + \sqrt{-3})(1 - \sqrt{-3})$. A similar mistake was done by Gabriel Lamé who announced in 1847 a proof of Fermat's last theorem telling that for $n \geq 3$, no solutions to $x^n + y^n = z^n$ exist unless $xyz = 0$. Lamé's genius idea was to decompose $x^n + y^n$ into linear factors using numbers satisfying $\xi^n = 1$, so called **roots of unity**. Also here, Euclid shows that a prime factorization exists, but it is also here not unique. The mistake was actually quite important. It led to a "theory of ideals" by Ernst Kummer which allowed to prove Fermat's last theorem in certain cases.

Principle: Mistakes can open new doors and find ideas. A creative search process can lead to mistakes at first.

4

3.13. Of course, we have to try to avoid mistakes in the final product at all costs. Euler certainly earned the right to make some mistakes by creating a lot of mathematics, which will remain true for all eternity. But mistakes can be much more basic. Here is a beautiful example due to Polya: [5]

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[4]see Mario Livio: Brilliant blunders, 2013
[5]George Polya: Induction and Analogy in Math, 1954 (Thanks to Jun Hou Fung for suggestion):

Theorem: S(n): In a collection of n horses, all have the same color.

Proof: The induction assumption is clear as for $n = 1$, all horses have the same color. Now assume that the statement is true for all groups of n horses. Take $n + 1$ horses and take the first away. These are n horses so that all have the same color. Now put the first back and take the last one away. Again we have n horses, so that all have the same color. Therefore all have the same color.

Problem D: What is wrong in the proof of Polya's horse theorem?

Here are some more amusements:

Theorem: Cats have nine tails.

3.14. Proof: No cat has no tail. A cat with a tail has a tail more than no cat. No cat has eight tails. Therefore, cats have nine tails.

3.15. For the following definition of "Prime numbers" we follow [6]:

A prime is a number with no divisors.
Boxes of chocolates always contain a prime number
so that, whatever the number of people present
somebody has to have that one left over.

3.16. Why do we start to do induction at $n = 1$ and not from the other end? The following song explains why: (just as a bit of background to appreciate the song: Aleph-Null $= \aleph_0$ is the cardinality of the **natural numbers** $\mathbb{N}$. $\aleph_1$ is the next larger cardinality. The cardinality of the **real numbers** $\mathbb{R}$ is $2^{\aleph_0}$ (as the Cantor diagonal argument shows that the real numbers can not be counted) which is the cardinality of all subsets of natural numbers. Cantor had shown that there are different infinities. A beautiful mind like Cantor's of course asked whether there is an infinity in between these two infinities.

The statement $2^{\aleph_0} = \aleph_1$ is the **continuum hypothesis** abbreviated CH. Work of Paul Cohen and Kurt Gödel in the sixties shows that one can not prove the statement nor its negation from ZFC set theory (an axiom system of our standard mathematics from which one can derive the Peano axioms including the principle of induction). Cantor had for a long time tried to prove CH, in vain. We know now that his efforts to prove this were doomed from the beginning. This possibility always exists. There is the possibility (very unlikely although) that we can not prove that every even number larger than 2 is a sum of two primes, even in the case if it would be true! [7]. The continuum hypothesis problem had been the first of Hilbert's problems of 1900.

Aleph-null bottles of beer on the wall,
Aleph-null bottles of beer,
You take one down, and pass it around,
Aleph-null bottles of beer on the wall.

---

[6] R. Ainsley: "Bluff your way in maths, 1990"
[7] See Apostolos Doxiadis: Uncle Petros and the Goldbach conjecture, Novel of 1992

3.17. And here is another Ainsley quote:

At the end of a proof you write Q.E.D,
which stands not for
Quod Erat Demonstrandum
as the books would have you believe, but
for Quite Easily Done.

HOMEWORK

Problem 3.1 Write down a proof by induction showing that $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$ for every integer $n \geq 1$.

Problem 3.2 Given a $n \times n$ matrix A, its **trace** is defined as the sum of the diagonal elements $\sum_k A_{kk}$. We can define in $M(n, m)$ the inner product $\mathrm{tr}(A^T B)$. First check that this inner product makes sense and that $A^T B$ is indeed a square matrix. Repeat each step of the proof of the Cauchy-Schwarz inequality and see that it still works.

Problem 3.3 Let us define a vector $v \star w = (v \cdot w)v/|v|^2$. It is called the **vector projection** of w onto v.
a) Is the operation $\star$ commutative?
b) Is the operation $\star$ associative?
c) Verify that v is perpendicular to $w - (v \star w)$.

Problem 3.4 Try to design yourself an elementary geometric proof of the Pythagorean theorem which does not use any algebra. First try this without looking it up. Then look up one the many proofs available and pick the one you like most and write or draw it out.

Problem 3.5 Given a $n \times m$ matrix A, assume that $\mathrm{rref}(A)$ has r leading 1 and that $\mathrm{rref}([A|b])$ has s leading 1. What condition on r and s and n and m implies that the system of equations $Ax = b$ has no solution? Experiment first with small examples.
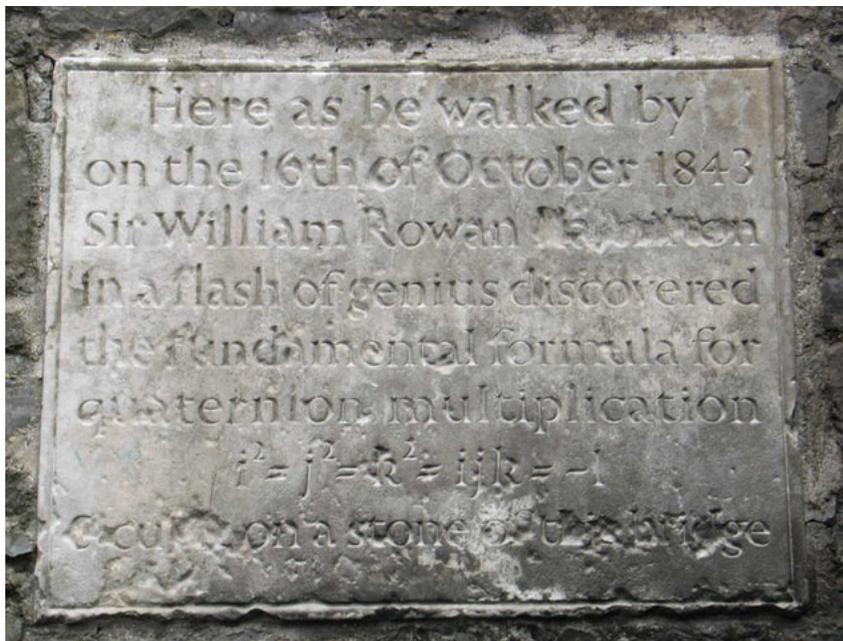
Unit 4: Cross product

FIGURE 1. The quaternion plaque at the Brougham bridge: here as he walked by on the 16th of October 1843 Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication $i^2 = j^2 = k^2 = ijk = -1$ and cut it on a stone of this bridge.

4.1. We have seen that we can multiply square matrices and get again a matrix. Wouldn't it be nice if we could also multiply two **vectors** and get a vector back. The dot product, which is the matrix product of a row vector with a column vector gave us a number. The matrix product of a column vector with a row vector would give us a square matrix. How can we design a product of column vectors which again gives us a column vector? This was the question which **William Rowan Hamilton** pondered for many years. The story goes that every morning, when he would come down to the breakfast table, his young son would ask "Dad, can you already multiply triplets?" to which William answered: "No Son, I do not know how to do that yet".

4.2. Eventually, Hamilton succeeded. The legend goes that while walking with his wife along the Royal Canal in Dublin, while crossing the Brougham bridge, he suddenly got the inspiration: one has to multiply quadruplets! These numbers would be written as $a + bi + cj + dk$ where i, j, k are symbols satisfying $i^2 = j^2 = k^2 = ijk = -1$. He was so happy that he would would devote the rest of his life with these numbers. Now it turns out that this algebra also produces a product of vectors which is called the **cross product**. It has a lot of nice properties like that the product of two vectors is perpendicular and that the length is related to area. It also has amazing applications in physics.

<div align="center">LECTURE</div>

4.3. The three dimensional space $\mathbb{R}^3$ is special. It is not only the only Euclidean space in which the Kepler problem is stable [1], it also features a **cross product** $v \times w$ which is in the same space. Such a product can be defined in $\mathbb{R}^n$ but it produces a vector in $\mathbb{R}^{n(n-1)/2}$. It happens that for $n = 3$ that the result is again in $\mathbb{R}^3$. The problem of "multiplying triplets" has been pondered by William Hamilton in the first half of the 19th century and is related to the fascinating story of **quaternions**. The discovery of quaternions was simultaneously the birth place of the dot and cross product.

4.4. The **cross product** of two vectors $v = [v_1, v_2, v_3]^T$ and $w = [w_1, w_2, w_3]^T$ is

$$
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix} .
$$

Take the dot product with v or w to see that $v \times w$ is perpendicular to both v and w. Obvious is also $v \times w = -w \times v$. The product is handy for constructions in $\mathbb{R}^3$. The vectors $v, w, v \times w$ are oriented like the first three fingers on the **right hand**: if v is the thumb, w is the pointing finger, then $v \times w$ is the middle finger. Let $v \cdot w = |v||w| \cos(\alpha)$:

Theorem: $|v \times w| = |v||w| \sin(\alpha)$ and $v \cdot (v \times w) = w \cdot (v \times w) = 0$.

Proof. We will verify in class by brute force the **Lagrange's identity** $|v \times w|^2 = |v|^2|w|^2 - (v \cdot w)^2$ which is also called **Cauchy-Binet** formula. Now use $|v \cdot w| = |v||w| \cos(\alpha)$ to get the result with $\cos^2(\alpha) + \sin^2(\alpha) = 1$. □

4.5. Given a triangle with side lengths a, b, c and angles α, β, γ, where α is opposite to a etc. We have the following sin-**formula**

Corollary: $\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}$.

Proof. We can use the theorem and express the area of the triangle as $ab \sin(\gamma)$ or $bc \sin(\alpha)$ or $ac \sin(\beta)$. By equating these three quantities and dividing out the common factor, we get the sin-formula. □

---

[1] by a theorem of Joseph Bertrand of 1873 and work of Sundman-von Zeipel

4.6. This is useful in applications as to define the area of the parallelogram as $|v \times w|$. That this is justified can be seen in two dimensions and:

Corollary: $|v \times w|$ is the **parallelogram area** spanned by v and w.

Proof. Use the formula $|v \times w| = |v||w|\sin(\alpha)$ and note that $|w|\sin(\alpha)$ is the height of the parallelogram spanned by v and w. The base length is $|v|$. $\qquad\square$

4.7. The scalar $u \cdot (v \times w)$ is called the **triple scalar product** of u, v, w. Its **sign** defines an **orientation** of the three vectors. It is also the **determinant** of the matrix

$$
\begin{bmatrix}
u_1 & v_1 & w_1 \\
u_2 & v_2 & w_2 \\
u_3 & v_3 & w_3
\end{bmatrix} .
$$

The absolute value of $u \cdot v \times w$ defines the **volume of the parallelepiped** spanned by u, v and w. Without the absolute value, we also speak of **signed volume**.

4.8. Side remark: In higher dimensions, the cross product is called **exterior product**. One uses $\wedge$ rather than $\times$ which is used in three dimensions. If $I = (i, j)$ is a choice of two elements in $\{1, 2, \ldots, n\}$ and v, w are two vectors in $\mathbb{R}^n$, then $(v \wedge w)_I = v_i w_j - v_j w_i$. The formula $|v \wedge w| = |v||w|\sin(\alpha)$ still holds and the proof is the same. We only need again to verify the **Cauchy-Binet** formula $|v|^2|w|^2 - (v \cdot w)^2 = |v \wedge w|^2$. But this is better done using matrices. If A is the matrix which contains v, w as columns, then $\det(A^T A) = \sum_P \det(A_P)^2$, where the sum on the right is over all 2×2 submatrices A_P of A. The expression $\det(A_P)$ is called a **minor**. Cauchy-Binet formula is super cool [2]. By the way, if we have k vectors and build $A \in M(n, k)$, a matrix which has these vectors as columns. Now, $\det(A^T A)$ is the volume of the parallelepiped spanned by these vectors. And Cauchy-Binet writes this as a sum of squares of k-dimensional volumes of projections which is in some sense a generalization of Pythagoras.

<div align="center">EXAMPLES</div>

4.9. What is the area of the triangle $A = (1, 1, 1)$, $B = (3, 5, 2)$ and $C = (2, 0, 3)$? We find the cross product between the vector $[2, 4, 1]^T$ going from A to B and the vector $[1, -1, 2]^T$ going from A to C. The cross product is

$$
\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ -3 \\ -6 \end{bmatrix} .
$$

Its length is $3\sqrt{14}$. The area of the triangle is half of it: $3\sqrt{14}/2$.

4.10. Find the volume of the parallelepiped with vertices $O = (0, 0, 0)$ and attached corners $A = (1, 1, 1)$, $B = (3, 4, 2)$ and $C = (2, 0, 3)$. The signed volume is

$$
\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 12 \\ -5 \\ -8 \end{bmatrix} = -1 .
$$

and take the absolute value. A negative number indicates that OA, OB, OC is left handed.

---

[2] O. Knill, Cauchy Binet for pseudo-determinants, Lin. Alg. and its Applications 459 (2014) 522-547

ILLUSTRATIONS

FIGURE 2. Swiss National Bank issued August 22, 2018 new 200 Frank bills. It shows the **right hand rule**: thumb $= v$, pointing finger $= w$, then $v \times w$ is the middle finger.
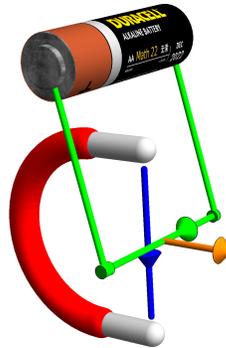

FIGURE 3. The **Lorentz force** F is a vector $F = qv \times B$ determined by the velocity v of a charged particle with charge q moving in a magnetic field B.
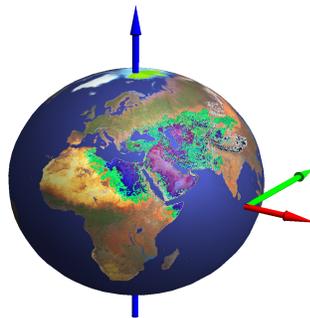

FIGURE 4. Given a particle of mass m at position r moving with the velocity r' then $L = mr \times r'$ is the **angular momentum**.

Homework

Problem 4.1: Find a vector w perpendicular to the vectors $u = [2, 3, 4]^T$ and $v = [3, 4, 7]^T$. Then use this result to find a vector x perpendicular to both v and w.

Problem 4.2: A **3D scanner** is used to build a 3D model of a face. It detects a triangle which has its vertices at $P = (2, 1, 1), Q = (1, 1, 0)$ and $R = (1, 2, 3)$. Find the area of that triangle as well as a vector perpendicular to the triangle. (*)

Problem 4.3: a) Find the volume of the parallelepiped which has the vertices $O = (0, 0, 0), P = (2, 3, 1), Q = (4, 3, 1), R = (6, 6, 2)$. $A = (1, 1, 1), B = (3, 4, 2), C = (5, 4, 2), D = (7, 7, 3)$.

Problem 4.4: Investigate which of the following formulas are always true for all vectors u,v,w,x,y. If it is true, either explain, cite a source (i.e. on the web), or a by hand or computer algebra verification. If it is not true, find a counter example.
a) $u \cdot (v \times w) = v \cdot (w \times u)$
b) $u \times (v \times w) = (u \times v) \times w$
c) $u \times (v + w) = u \times v + u \times w$
d) $u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$.
e) $(u \times v) \cdot (x \times y) = (u \cdot x)(v \cdot y) - (u \cdot y)(v \cdot x)$.

Problem 4.5: Given two vectors $p = [a, b, c]^T$ and $q = [u, v, w]^T$, build the matrices $P = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} Q = \begin{bmatrix} 0 & u & v \\ -u & 0 & w \\ -v & -w & 0 \end{bmatrix}$ Compare $p \times q$ and $QP - PQ$. Describe what you see. Try to formulate this as a theorem.

(*) The STL format which is used for 3D printing, has an extremely simple form. It consists of entries like

```
facet normal 0.15 -0.97 -0.20
outer loop
vertex -1.6996 -0.5597 -2.8360
vertex -1.8259 -0.5793 -2.8374
vertex -1.7232 -0.5399 -2.9509
endloop
endfacet
```

The first line gives the normal vector, then there is a loop with three vertices giving the triangle. There is obviously some redundancy as one could get the normal vector from the points using the cross product. But there is purpose: the redundant information makes working with the data structure faster, second, one can also look at situations, where the normal vector is not perpendicular to the surface, one can change the way how the is "shaded", like how light is reflected at the surface. Third, redundancy is always good to catch errors. Our genetic information in the DNA is stored in a highly redundant way. This allows error correction.

Unit 5: Surfaces

FIGURE 1.   The Klein bottle is a surface in three dimensional space. It can not be realized as a level surface of a function however as there are points of self intersections. But we can perfectly parametrize the surface.

5.1. Surfaces are **co-dimension one** objects in a space. They are important because they can divide up space. We can confine water in a bottle. This is not possible for co-dimension two. You can not confine water in a curve. Similarly, if you lived in 4-dimensional space, you could not store water in a two-dimensional surface. But things can get tricky already in three dimensions. There are two-dimensional closed surfaces which do not confine any space. Try to drink from a Klein bottle!

5.2. A surface can mathematically be described in two fundamentally different ways. It is either given as a **level surface** of a function on that space. Or then it can be the image of a map called parametrization. You know this from the earth, which is a sphere. We can either say that a sphere is the set of points which have a fixed

distance to its center point. Or then we can parametrize the sphere, for example using **longitude** and **latitude**. A plane through the 0 can be given either as the kernel $\{x \in \mathbb{R}^3 | Ax = 0\}$ of a 1×3 matrix A or then as the image $\{Ax | x \in \mathbb{R}^2\}$ of a 3×2 matrix. The first writes $ax + by + cz = 0$. The second writes the plane as $vs + wt$, where v, w are the column vectors of A and $x = [s, t]^T$ gives the parameters.

LECTURE

5.3. If A is a matrix, the solution space of a system of equations $A\mathbf{x} = b$ is called a **linear manifold**. It is the set of solutions of $A\mathbf{x} = 0$ translated so that it passes through one of the points. The equation $3x + 2y = 6$ for example describes a line in $\mathbb{R}^2$ passing through $(2, 0)$ and $(0, 3)$. The solutions to $Ax = 0$ form a **linear space**, meaning that we can add or scale solutions and still have again solutions. We can rephrase the just said in that a linear space is a linear manifold which contains 0. For example, for $x + 2y + 3z = 6$ we get a plane which is parallel to the plane $x + 2y + 3z = 0$. The former is a linear manifold (also called affine space), the later is a linear space. It is the solution space to $A\mathbf{x} = 0$ with $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and $\mathbf{x} = [x, y, z]^T$. Both planes are perpendicular to $n = [1, 2, 3]^T$. To find an equation for the plane through 3 points P, Q, R, define $n = PQ \times PR = [a, b, c]^T$ then write down $ax + by + cz = d$, where d is obtained by plugging in a point. The cross product comes handy.

5.4. The following important example deals with $A = [a_1, \ldots, a_m]$ in $M(1, m)$.

> **Theorem:** The vector $n = A^T$ is perpendicular to the plane $Ax = d$.

Proof. Given two points y, z in the plane. Then we have $Ay = d$ and $Az = d$. Then $x = y - z$ is a vector inside the plane. Now $A^T \cdot x = Ax = A(y - z) = Ay - Az = d - d = 0$. This means that x is perpendicular to the vector A^T. $\square$

In three dimensions, this means that the plane $ax + by + cz = d$ has a normal vector $A^T = n = [a, b, c]^T$. Keep this in mind, especially because $\mathbb{R}^3$ is our home.

5.5. This **duality result** will later will identified as a **fundamental theorem of linear algebra**. It will be important in data fitting for example. The **kernel** of a matrix A is the linear space of all solution $Ax = 0$. The kernel consists of all roots of A. The **image** of a matrix A is the linear space of all vectors $\{Ax\}$. We abbreviate $\ker(A)$ for the kernel and $\operatorname{im}(A)$ of the image. We will come back to this later.

> **Theorem:** The image of A^T is perpendicular to the kernel of A.

Proof. If x is in the kernel of A, then $Ax = 0$. This means that x is perpendicular to each row vector of A. But this means that x is perpendicular to the column vector of A^T. So, x is perpendicular to the image of A^T. This line of argument can be reversed to see that if x is perpendicular to the image of A^T, then it is in the kernel of A. $\square$

5.6. Given a function $f : \mathbb{R}^n \to \mathbb{R}$, the solution set $\{f(x_1, \cdots, x_n) = d\}$ is a **hyper surface**. We often say "surface" even so "surface" is reserved to $n = 3$. The simplest non-linear surfaces are **quadratic manifolds**

$$x \cdot Bx + Ax = d$$

defined by a symmetric matrix B and a row vector A and a scalar d. We assume that B is not the zero matrix or else, we are in the case of a linear manifold. We also can assume B to be symmetric $B = B^T$. For notation, we write $\text{Diag}(a, b, c) = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ and $1 = \text{Diag}(1, 1, 1)$.

5.7. Ellipsoids For $B = 1$ and $A = 0$ and $d = 1$ we get the **sphere** $|x|^2 = 1$. In $\mathbb{R}^2$, a sphere is a **circle** $x^2 + y^2 = 1$. In three dimensions we have the familiar sphere $x^2 + y^2 + z^2 = 1$. An more general ellipsoid with $B = \text{Diag}(1/a^2, 1/b^2, 1/c^2)$ is $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. By intersecting with $x = 0$ or $y = 0$ or $z = 0$, we see **traces**, which are all ellipses.
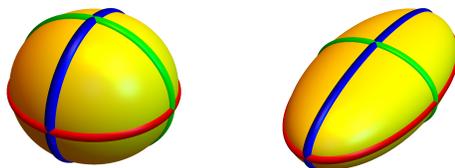
FIGURE 2.   The sphere $x^2 + y^2 + z^2 = 1$ and an example of an ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

5.8. Hyperboloids. For $B = \text{Diag}(1, 1, -1)$ and $d = 1$, we get a **one-sheeted hyperboloid** $x^2 + y^2 - z^2 = 1$. For $B = \text{Diag}(1, 1, -1)$ and $d = -1$, we get a **two-sheeted hyperboloid** $x^2 + y^2 - z^2 = -1$. A more general hyperboloid is of the form $x^2/a^2 + y^2/b^2 - z^2/c^2 = d$ with $d \neq 0$. The intersection with $z = 0$ gives in the one-sheeted case a circle, in the two-sheeted case nothing. The $x = 0$ trace or the $y = 0$ trace are both hyperbola.
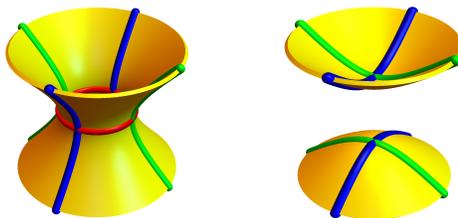

FIGURE 3.   The one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$ and the two-sheeted hyperboloid $x^2 + y^2 - z^2 = -1$.

5.9. Paraboloids. For $B = \text{Diag}(1, 1, 0)$ and $A = [0, 0, -1]$ and $d = 0$ we get the **paraboloid** $x^2 + y^2 = z$, for $B = \mathbf{Diag}(1, -1, 0)$ and $A = [0, 0, -1]$ and $d = 0$ we get the **hyperbolic paraboloid** $x^2 - y^2 = z$. We can recognize paraboloids by intersecting with $x = 0$ or $y = 0$ to see parabola. Intersecting the elliptical paraboloid $x^2 + y^2 = z$ with $z = 1$ gives an ellipse. Intersecting the hyperbolic paraboloid $x^2 - y^2 = z$ with $z = 1$ gives a hyperbola.
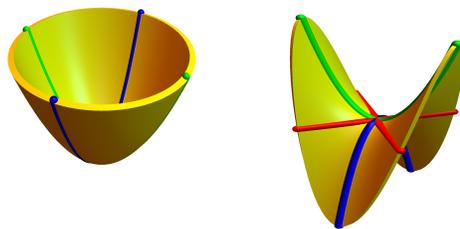
FIGURE 4. An elliptic paraboloid $z = x^2 + y^2$ and the hyperbolic paraboloid $z = x^2 - y^2$.

5.10. Special surfaces. If $B = \text{Diag}(1, 1, -1)$ and $d = 0$, we get a **cone** $x^2 + y^2 - z^2 = 0$. For $B = \text{Diag}(1, 1, 0)$ and $d = 1$ we get the **cylinder** $x^2 + y^2 = 1$.
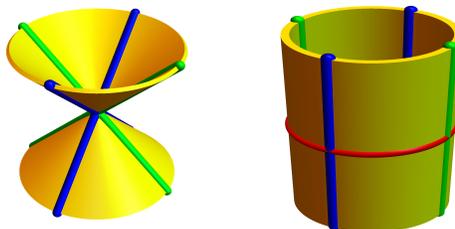

FIGURE 5. The cone $x^2 + y^2 = z^2$ and the cylinder $x^2 + y^2 = 1$.

5.11. Side remark: The 1-sphere $S^1 = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$ and the 3-sphere $S^3 = \{x^2 + y^2 + z^2 + w^2 = 1\} \subset \mathbb{R}^4$ carry a multiplication: S^1 is in the **complex numbers** $\mathbb{C} = \{x + iy\}$ and S^3 is in the **quaternions** $\mathbb{H} = \{x + iy + jz + kw\}$. The 1-sphere is the gauge group for **electromagnetism**, the 3-sphere (also called $SU(2)$) is responsible for the **weak force**. No other Euclidean sphere carries a multiplication for which $x \to x * y$ is smooth. Michael Atiyah once pointed out that this algebraic particularity might not be a coincidence and responsible for the structure of the **standard model of elementary particles** (one of the most accurate theories ever built by humanity). The strong force appears as one can let a set of 3×3 matrices $SU(3)$ act on $\mathbb{H}$. Atiyah suggested that gravity could be related to the **octonions** $\mathbb{O}$. There $S^7 = \{|x| = 1\} \subset \mathbb{R}^8$ carries still a multiplication, but it is no more associative. The list of normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$. [1]

5.12. Given a polynomial p of n variables, one can look at the surface $\{p(x) = 0\}$. It is called a **variety**.

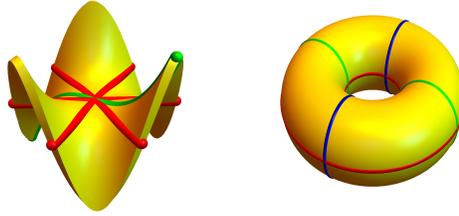[1] See the talk of 2010 of Atiyah ($https://www.youtube.com/watch?v = zCCxOE44M\_M$).

FIGURE 6. More examples of **varieties**, solution sets to polynomial equations. To the left we see **cubic surface** $x^3 - 3xy^2 - z = 0$ called the **monkey saddle**. To the right we see **torus** $(3 + x^2 + y^2 + z^2)^2 - 16(x^2 + y^2) = 0$ which is an example of a **quartic manifold**.
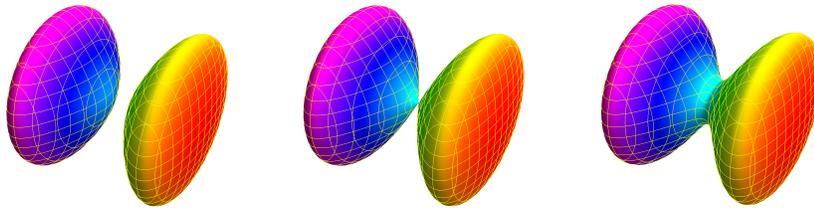

FIGURE 7. The variety $x^4 - x^2 + y^2 + z^2 = d$ for $d = -0.02, d = 0$ and $d = 0.02$.

EXAMPLES

5.13. Q: Find the plane Σ containing the line $x = y = z$ and the point $P = (3, 4, 5)$. A: Σ contains $Q = (0, 0, 0)$ and $R = (1, 1, 1)$ and so the vectors $v = [1, 1, 1]^T$ and $w = [3, 4, 5]^T$. The cross product between v and w is $[1, -2, 1]^T$. It is perpendicular to Σ. So, the equation is $x - 2y + z = d$, where d can be obtained by plugging in a point $(3, 4, 5)$. This gives $d = 0$ so that $x - 2y + z = 0$.

5.14. Can we identify the surface $x^2 + 2x + y^2 - 4y - z^2 + 6z = 0$? **Completion of the square** gives $x^2 + 2x + \boxed{1} + y^2 - 4y + \boxed{4} - z^2 + 6z - \boxed{9} = \boxed{1} + \boxed{4} - \boxed{9} = -4$. Now $(x + 1)^2 + (y - 2)^2 - (z - 3)^2 = -4$. This is a two-sheeted hyperboloid centered at $(-1, 2, 3)$.

5.15. Intersecting the **cone** $x^2 + y^2 = z^2$ with the plane $y = 1$ gives a hyperbola $z^2 - x^2 = 1$. Intersection with $z = 1$ gives a circle $x^2 + y^2 = 1$. Intersecting with $z = x + 1$ gives $y^2 = 2x + 1$, a parabola. Because bisecting a cone can give hyperbola, an ellipse or a parabola as cuts, one calls the later **conic sections**.

5.16. The case of **singular quadratic manifolds** is even richer: $x^2 - y^2 = 1$ is a **cylindrical hyperboloid**, $x^2 - y^2 = 0$ is a union of two planes $x - y = 0$ and $x + y = 0$. The surface $x^2 = 1$ is a union of two parallel planes, the surface $x^2 = 0$ is a plane.

Homework

Problem 5.1: a) What kind of curve is $2x^2 + 4x + 2y^2 + 2 = 0$?
b) What surface is $x^2 + y^2 - 4y + z^2 + 8z = 100$?
c) Let (x, y, z) be the set of points for which $|[x, y, z]^T \times [1, 1, 1]^T| = 1$.
Describe this set.

Problem 5.2: a) What kind of curves can you get when you intersect hyperbolic paraboloid $x^2 - y^2 = z$ with a plane? b) Explore what you get if you intersect the hyperboloid $S : x^2 + y^2 - z^2 = 1$ with the S rotated by 90 degrees around the x-axes.

Problem 5.3: Find explicit planes which when intersected with the hyperboloid $x^2 + 2y^2 - z^2 = 1$ produces an ellipse, or a hyperbola or a parabola.

Problem 5.4: Find the equation of a plane which is tangent to the three unit spheres centered at $(3, 4, 5), (1, 1, 1), (2, 3, 4)$.

Problem 5.5: Build a concrete function $f(x, y, z)$ of three variables such that some level surface $f(x, y, z) = c$ is a **pretzel**, a surface with three holes. Hint: the surface $g * h = 0$ is the union of the surfaces $g = 0$ and $h = 0$. Now, $g * h = c$ can produce surfaces in which things are glued nicely. If you should look up a surface on the web or literature, you have to give the reference. You can use the computer to experiment, or then describe your strategy in words.
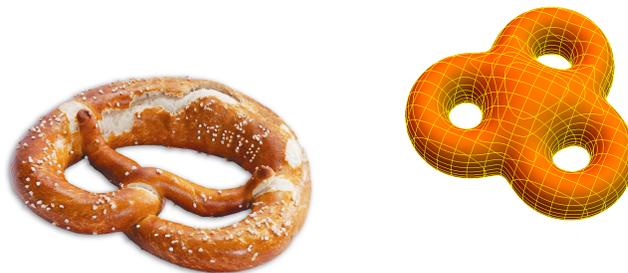

FIGURE 8. In the pretzel baked to the right we have used a polynomial $f(x, y, z)$ of degree 12. A problem in algebraic geometry would be to find the "smallest degree polynomial" which works and then find the most elegant polynomial.
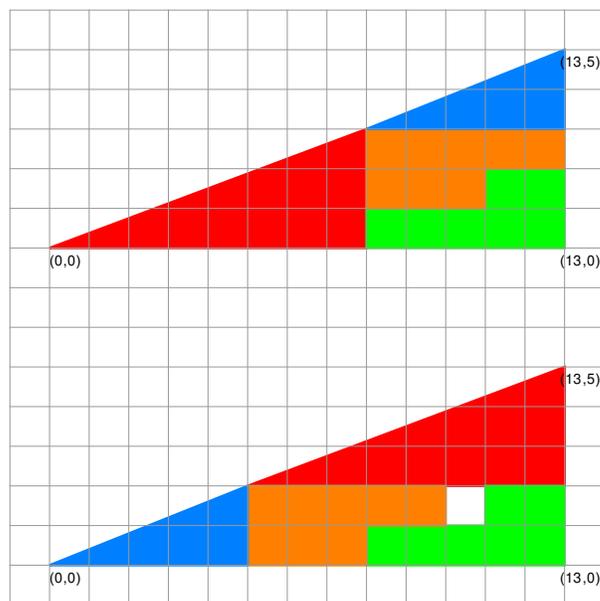
Unit 6: Visual proofs

INTRODUCTION

FIGURE 1. This is a famous visual proof that $65/2 = 63/2$. A triangle of area $65/2$ is cut into smaller pieces. After rearranging the pieces and only translating them, we get the same triangle with one square less.

6.1. Visual pictures are a great help for "seeing" why something is true. Some of the most beautiful proofs in mathematics can be seen to be true as such. Visual arguments also can be wrong and this is not only confined to geometric pictures. Especially when proving results in higher dimensions, where intuition from smaller dimensions come in, we can get into trouble.
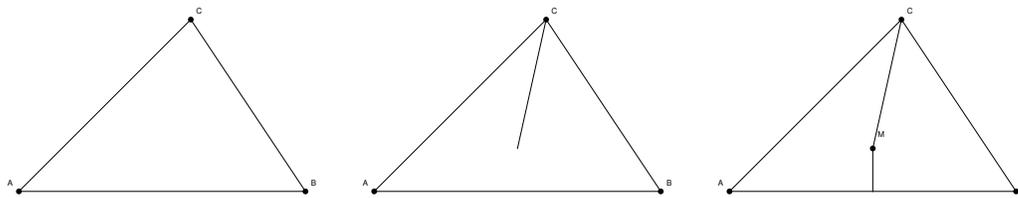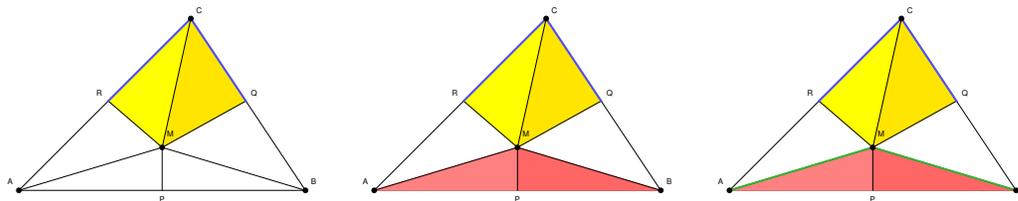
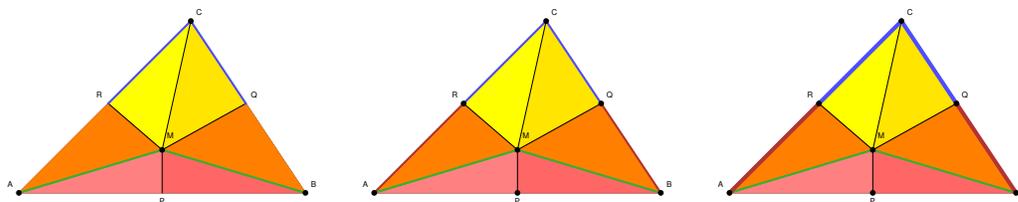Intersect the angle bisector at C with the perpendicular bisector of AB.
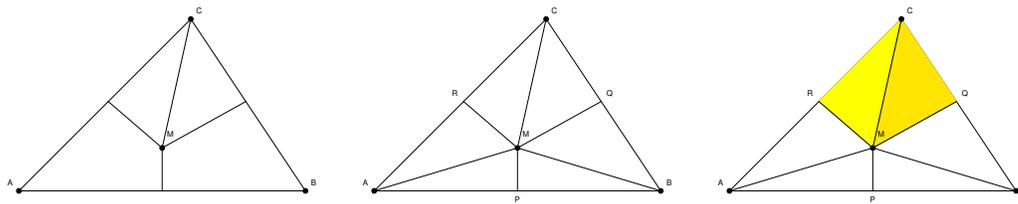
Draw the perpendicular to AC,BC and connect M with A and B.

Triangles MQC, MRC are congruent. Triangles MPB,MPA too.

So are MBA and AMR. Because AR=BQ and CQ=RC we have AC=BC.

FIGURE 2.   A proof that all triangles are isoscele. There is no mistake in the argument. All given steps are correct. Still, there is something wrong.

6.2. Geometric intuition and pictures allow to prove results visually. An example:
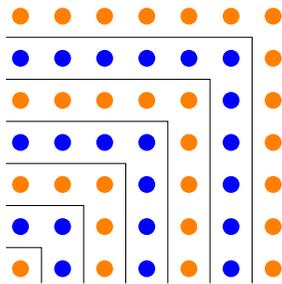

FIGURE 3. This is a proof without words.

1

> **Problem A:** What formula does Figure (3) prove?

6.3. By drawing a rectangle of side length a and b, we can see that the area $a * b$ is the same as the area $b * a$. For the cross product or matrices, this is wrong.
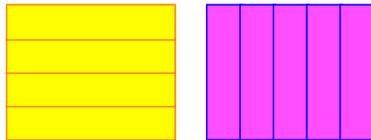
FIGURE 4. A Cuisenaire proof that $4 * 5 = 5 * 4$. Four yellow sticks of length 5 have the same area than 5 purple sticks of length 4.

6.4. Pictures help to get intuition about a mathematical result. The Pythagorean theorem was first proven geometrically. The visual proof we look at here could well have been the first which was found.

FIGURE 5. A visual proof of the Pythagorean theorem. It is probably one of the first proofs.

> **Problem B:** Use Figure (5) for a proof of the Pythagorean theorem. You can either describe in words, or label some parts of the picture. Remember that we want to show $c^2 = a^2 + b^2$.

FIGURE 6. A visual proof of $\sqrt{ab} \le (a+b)/2$.

2

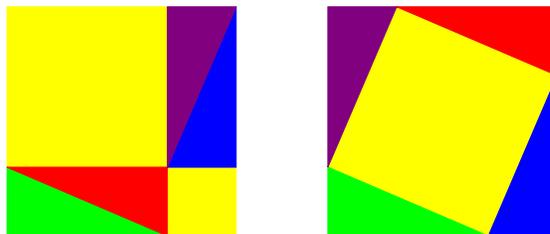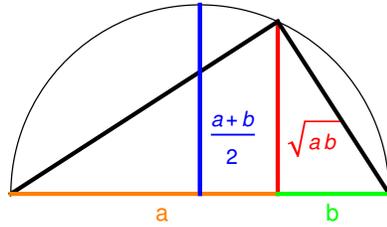**6.5.** The **geometric-algebraic inequality** assures that the geometric mean is smaller or equal than the algebraic mean. In order to appreciate that proof, we have first to verify an identity relating the lengths a, b cut by the altitude line and height h.

> **Problem C:** First check why the triangle in Figure 6 is a right angle. Then use Pythagoras three times to prove $ab = h^2$. Finally check the geometric-algebraic inequality.

6.6.

> **Theorem:** The radius of the inscribed circle in a $3:4:5$ triangle is 1

> **Problem D:** Use Figure (7) from the "9 Chapters" to prove the theorem.
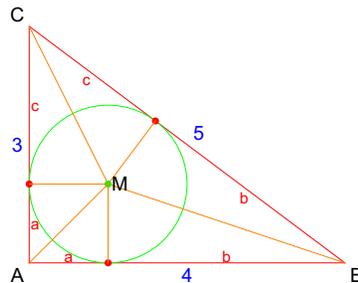
FIGURE 7. The 3-4-5 triangle. Can you use the picture to prove that a=1?

6.7. Find the formula for the volume of a tetrahedron given by 4 points A, B, C, D.
3

> **Problem E:** Use Figure (8) to prove that the volume is a sixth of the volume of the corresponding parallelepiped.

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[1]Cover of the book "Proofs without words"

[2]C. Gallant, Mathematics Magazine, 50(2), 1977, page 98

[3]"Illustrating Mathematics using 3D printers", by O. Knill and E. Slavkovsky.
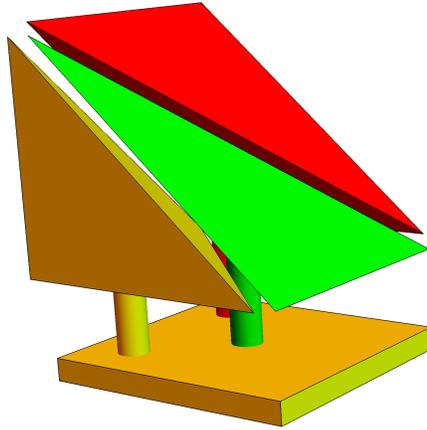
FIGURE 8.   The tetrahedron volume is 1/6 of a parallelepiped volume. Not only the Egyptians knew it, this figure can also be found in the "nine chapters". We build a statue which can be 3D printed.

Homework

Problem 6.1: The **3D Pythagoras theorem** states that the square of the area of ABC is the sum of the squares of the areas of the triangles OAB, OBC and OCA (which are each half of a rectangle). Use Figure (9) with $A = (a, 0, 0), B = (0, b, 0), C = (0, 0, c)$ to verify this theorem. Use the cross product to get the areas.
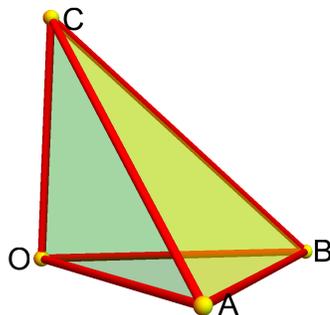
FIGURE 9. The 3D Pythagoras theorem.

Problem 6.2: a) Draw a picture with a planar figure explaining $(a + b)^2 = a^2 + 2ab + b^2$.
b) Draw a picture with a 3D figure explaining $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Problem 6.3: Find distance formulas which do not use any trig functions: a) For the distance of a point P to a line through two points A, B.
b) For the distance of a point P to a plane through three points A, B, C.
c) For the distance between the line through A, B and the line through C, D.

Problem 6.4: Design a visual proof for the Faulhaber formula $1^3 + 2^3 + 3^3 + ... + n^3 = (1 + 2 + 3 + ... + n)^2$ which is also called the Nichomachus theorem.

Problem 6.5: Look up the rules for quaternion multiplication $(u_0, u_1, u_2, u_3) \star (v_0, v_1, v_2, v_3)$ and verify that $(0, v_1, v_2, v_3) * (0, w_1, w_2, w_3) = (-v \cdot w, v \times w)$. Historically, this is an important identity as the dot and cross product have been introduced together in the form of quaternions.

Unit 7: Curves

INTRODUCTION

FIGURE 1.   The trajectory of a ball bouncing in an elliptical barrel is a curve. It is a one dimensional object because it can be described by one parameter. In this case we have a curve which is continuous but not smooth. Still, we can use calculus to describe properties of the curve like to see that it consists of pieces of parabola. By the way, on the left, with 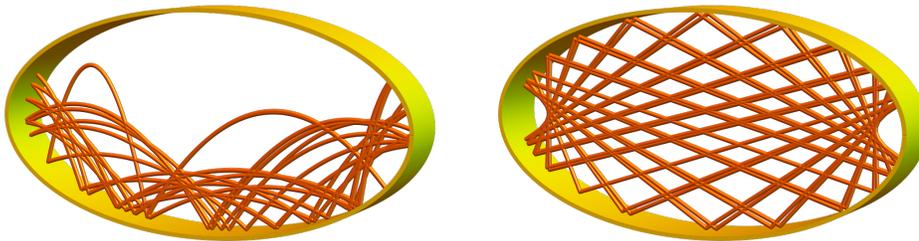gravity, this is a system we do not understand. The bouncing ball moves chaotically. We do not have the tools for example to tell where the ball is after 10^{100} bounces. To the right we see the situation without gravity. Now, we could determine where the ball is after 10^{100} bounces.

7.1. Many geometric objects can be assigned a **dimension**. This number tells how many parameters we need to describe the object. A point has dimension 0, a line has dimension 1, a plane has dimension 2. This is formalized in **linear algebra**. Given a matrix A, the number of leading 1 in rref(A) is the dimension of the image of A. The number of free variables (columns without leading 1 in rref(A) is the dimension of the kernel of A. For example, for $A = [1, 2, 3]$ which is already in row reduced form, we have one leading 1 and two free variables y and z. The equation 1 x + 2 y + 3 z = 0 describes a 2 dimensional object, a plane. If y, z are given, we can find x from the equation. The image of the column vector $v = A^T$ is the line spanned by this vector. This line is perpendicular to the plane and illustrates the fundamental theorem of linear algebra assuring the kernel of A being perpendicular to the image of A^T or equivalently the kernel of A^T being perpendicular to the image of A.

7.2. Curves are objects of dimension 1. For example, the line spanned by a vector v is written as the set of points $r(t) = tv = [t, 2t, 3t]^T$. We call this a **parametrization** of the line. The free variable t is called **time**. It determines, where we are located at a fixed time t. At time $t = 12$ for example we are positioned at the point $(12, 24, 36)$ corresponding to the vector $[12, 24, 36]^T$. [1] The vector v has the interpretation of a **velocity**. It tells us how fast we move on the line. Of course, replacing v with $3v$ would give us the same line but we would travel three times faster and would reach the point $(12, 24, 36)$ three times faster.

7.3. If the velocity can change direction and length, we can drive around on more interesting paths. The frame work is to take three continuous functions $x(t), y(t), z(t)$ and look at the path $(x(t), y(t), z(t))$ in space. We write this in vector notation as $r(t) = [x(t), y(t), z(t)]^T$. Now, because we get tired always to write the T pointing out that we use column vectors, we will just write $r(t) = [x(t), y(t), z(t)]$. Most of the time, we assume the functions to be differentiable but the case of a ping-pong ball bouncing off a table shows that also non-smooth curves can matter even in daily life. Curves can be very complicated. Take a ping-pong ball and place it into an elliptic container. The billiard path it traces is chaotic. In this lecture we look at curves given by parameterizations, learn how to take derivative to get the velocity or acceleration. We also learn how to integrate. This allows us to compute paths. We can for example compute where a ball falling in a gravitational field is at time t.

LECTURE

7.4. Given n continuous functions $x_j(t)$ of one variable t, we can look at the vector-valued function $r(t) = [x_1(t), \ldots, x_n(t)]^T$. We call it a **parametrized curve**. An example is $r(t) = [3 + 2t, 4 + 6t]$ which is a line through the point $(3, 4)$ and containing the vector $[2, 6]$. [2] If t is in the **parameter interval** $a \leq t \leq b$, then the image of r is $\{r(t) \mid a \leq t \leq b\}$, which defines a **curve** in $\mathbb{R}^n$. The curve **starts** at the point $r(a)$ and **ends** at the point $r(b)$. An other important example is the **circle** $r(t) = [\cos(t), \sin(t)]$, where t is in the interval $[0, 2\pi]$. Its image is a circle in the plane $\mathbb{R}^2$. The parametrization $r(t)$ contains more information than the curve itself: the parabolic curve $r(t) = [t, t^2]$ defined on $t \in [-1, 1]$ for example is the same as the curve $r(t) = [t^3, t^6]$ for $\in [-1, 1]$, but in the second parametrization, the curve is traveled with different speed. Curves in $\mathbb{R}^3$ can be admired in our physical space like $r(t) = [x(t), y(t), z(t)] = [t\cos(t), t\sin(t), t]$ which is a spiral. This particular curve is contained in the cone $x^2 + y^2 = z^2$.

7.5. If the functions $t \to x_j(t)$ are differentiable, we can form the derivative $r'(t) = [x_1'(t), \ldots, x_n'(t)]$. While this technically is again a curve, we think of $r'(t)$ as a vector attached to the point $r(t)$ and say that $r'(t)$ is **tangent** to $r(t)$. The length $|r'(t)|$ of the velocity is called the **speed** of r. If also higher derivatives of the functions $x_j(t)$ exist, we can form the second derivative $r''(t)$ called the **acceleration** or third derivative $r'''(t) = r^{(3)}(t)$ called the **jerk**. Then come **snap** $r^{(4)}(t)$, **crackle** $r^{(5)}(t)$ and **pop** $r^{(6)}(t)$ and the **Harvard** $r^{(7)}(t)$ introduced in the fall of 2016 in a multi-variable exam.

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[1] We can associate any vector v with a point. Think of the vector as connecting 0 with the point.

[2] To reduce clutter, we write row vectors $[2, 6, 1]$ rather than column vectors $[2, 6, 1]^T$

7.6. Given the first derivative function $r'(t)$ as well as the initial point $r(0)$, we can get back the function $r(t)$ thanks to the **fundamental theorem of calculus**. Because of **Newton's law** which tells that a mass point of mass m subject to a force field F depending on position and velocity satisfies the **Newtonian differential equation** $mr''(t) = F(r(t), r'(t))$, the following result is important:

Theorem: $r(t)$ is uniquely determined from $r''(t)$ and $r(0)$ and $r'(0)$.

Proof. In each coordinate we get $x_k'(t) = \int_0^t x_k''(s)\, ds + x_k'(0)$ and $x_k(t) = \int_0^t x_k'(s)\, ds + x_k(0)$. We have just applied twice the **fundamental theorem of calculus**. $\square$

A special case is if $r''(t)$ is constant. A special case is the **free fall situation**. The coordinate functions are then quadratic. Assume $r''(t) = [0, 0, -10]$, and $r'(0) = [0, 0, 0]$ and $r(0) = [0, 0, 20]$, then $r(t) = [0, 0, 20 - 5t^2]$. If you jump from 20 meters into a pool, you need $t = 2$ seconds to hit the water.

7.7. Given a curve $r(t)$ for which the velocity $r'(t)$ is never zero, we can form the **unit tangent vector** $T(t) = r'(t)/|r'(t)|$. If $T'(t)$ is never zero, we can then form $N(t) = T'(t)/|T'(t)|$, the **normal vector**. The vector $B = T \times N$ is called the **binormal vector**. The scalar $|T'(t)|/|r'(t)|$ is called the **curvature** of the curve.

Theorem: In $\mathbb{R}^3$, we have $K = |T'|/|r'| = |r' \times r''|/|r'|^3$.

Proof. We will do this computation in class. $\square$

7.8. Even if $r(t)$ is perfectly smooth, the curvature can become infinite. Lets look at the example $r(t) = [t^2, t^3, 0]$. Then $r'(t) = [2t, 3t^2, 0]$ and $r''(t) = [2, 6t, 0]$ and $r'(t) \times r''(t) = [0, 0, 6t^2]$. The curvature is $(6/t)(4 + 9t^2)^{-3/2}$ which has a singularity at $t = 0$.

7.9. Even when $r(t)$ is perfectly smooth and never zero, the normal vector can depend in a discontinuous way on t. Example: $r(t) = [t, t^3/3]$. Now $r'[t] = [1, t^2]$ and $T(t) = [0, t^2]/\sqrt{1 + t^4}$. We see that $T'(t)$ takes different signs in the second coordinate. After normalization we have $\lim_{t \to 0, t > 0} N(t) = [0, 1]$ and $\lim_{t \to 0, t < 0} N(t) = [0, -1]$. At the **inflection point** of the graph of the cube function, the concavity has changed from concave down to concave up. This has changed the direction of the normal vector N.

7.10. Side remark. We have looked at parametrized vectors only. If the entries $A_{ij}(t)$ of a matrix depend on times we have a matrix valued curve $A(t)$. This appears in differential equations, in quantum mechanics (operators moving in time) or - most importantly - in moving pictures! A movie is just a matrix valued curve.

7.11. Side remark. A planar curve $r(t) = [x(t), y(t)]^T$ in the plane defined on $t \in [0, 2\pi]$ is called a **simple closed curve** if $r(0) = r(2\pi)$ and there are no values $0 \le s \ne t < 2\pi$ for which $r(t) = r(s)$. For a smooth curve, meaning that the first two derivatives exist, we can look at the polar angle $\alpha(t)$ of the vector $r'(t)$. Define the **signed curvature** of the curve as $\kappa(t) = \alpha'(t)/|r'(t)|$. We have $|\kappa(t)| = K(t)$. The **Hopf Umlaufsatz** tells $\int_0^{2\pi} \kappa(t)\, dt = 2\pi$. In the case of the circle for example, $\kappa(t) = 1$.

7.12. Side remark. We can verify that any curve $r(t)$ parametrized on $[a, b]$ such that $r'(t) \neq 0$ for all $t \in [a, b]$ can be parametrized as $R(t)$ on $[a, b]$ such that $|R'(t)| = 1$ for all t. Proof: we look for a monotone function $s(t)$ such that the derivative of $r(s(t))$ has length 1. This means we want $|r'(s(t))|s'(t) = 1$. In other words, look for a function $s(t)$ such that $s'(t) = 1/|r'(s(t))| = F(s(t))$ and $s(a) = 0$. This is what we call a differential equation. There is a general existence theorem for differential equations (proven later) which assures that there exists a unique solution $s(t)$. End of proof. The result is very intuitive. You can drive from $r(a)$ to $r(b)$ along the curve traced by $r(t)$ by just keeping the speed 1. This gives your your new parametrization. Your new time interval will be $[0, L]$ where L is the arc length (the length of your trip). We will come to arc length computation in the next lesson.

7.13. Side remark. Continuous curves can be complicated: If you look at the pollen particle in a microscope, it moves erratically on a curve which is nowhere differentiable as it is constantly bombarded with air molecules which bounce it around. This is **Brownian motion**. There are also **Peano curves** or **Hilbert curves** $[0, 1] \to [0, 1]^2$ or space filling Hilbert curves $r(t) : [0, 1] \to Q = [0, 1]^3$ which cover every point of the **cube** Q. These curves define a continuous bijection from $[0, 1]$ to $[0, 1]^3$. (The inverse is not continuous. Still, the construction shows that there are the same number of points in $[0, 1]$ than in $[0, 1]^3$).
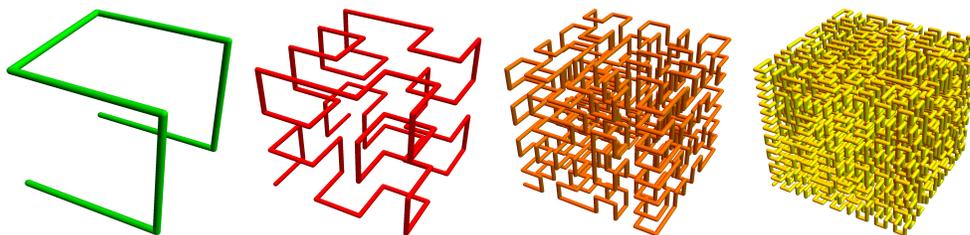

FIGURE 2. The four first stages in the construction of a space filling curve.

EXAMPLES

7.14. Assuming the **Newton equations** $mr''(t) = F(t)$, find the path $r(t)$ of a body of mass $m = 1/2$ subject to a force $F(t) = [\sin(t), \cos(t), -10]$ with $r(0) = [3, 4, 5]$ and $r'(0) = [1, 2, 7]$. Solution: we have $r''(t) = [2\sin(t), 2\cos(t), -20]$. Integration gives $r'(t) = [-2\cos(t), 2\sin(t), -20t] + [c_1, c_2, c_3]$. Fixing the constants gives $r'(t) = [3 - 2\cos(t), 2 + 2\sin(t), 7 - 20t]$. A second integration gives $r(t) = [3t - 2\sin(t), 2t - 2\cos(t), 7t - 10t^2] + [c_1, c_2, c_3]$ with other constants $C = [c_1, c_2, c_3]$. Comparing $r(0) = [0, -2, 0] + [c_1, c_2, c_3] = [3, 4, 5]$ gives $r(t) = [3 + 3t - 2\sin(t), 6 + 2t - 2\cos(t), 5 + 7t - 10t^2]$.

7.15. Let $r(t) = [L\cos(t), L\sin(t), 0]$. Then $r'(t) = [-L\sin(t), L\cos(t), 0]$ and $r''(t) = [-L\cos(t), -L\cos(t), 0]$ and $r'(t) \times r''(t) = [0, 0, L^2]$ and $|r'(t)| = L$ so that $|r'(t) \times r''(t)|/|r'(t)|^3 = 1/L$. A circle of radius L has curvature $1/L$!

7.16. A closed simple curve C in $\mathbb{R}^3$ is a **knot**. For any positive integer n, m we can look at the **torus knot** $r(t) = [(3 + \cos(mt))\cos(nt), (3 + \cos(mt))\sin(nt), \sin(mt)]$. The **total curvature** of a knot is defined as $\int_0^{2\pi} K(t) \, dt$. See Figure 3. [3]

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[3]A general theorem of Fay and Milnor assures that a knot of total curvature $\leq 4\pi$ is trivial.

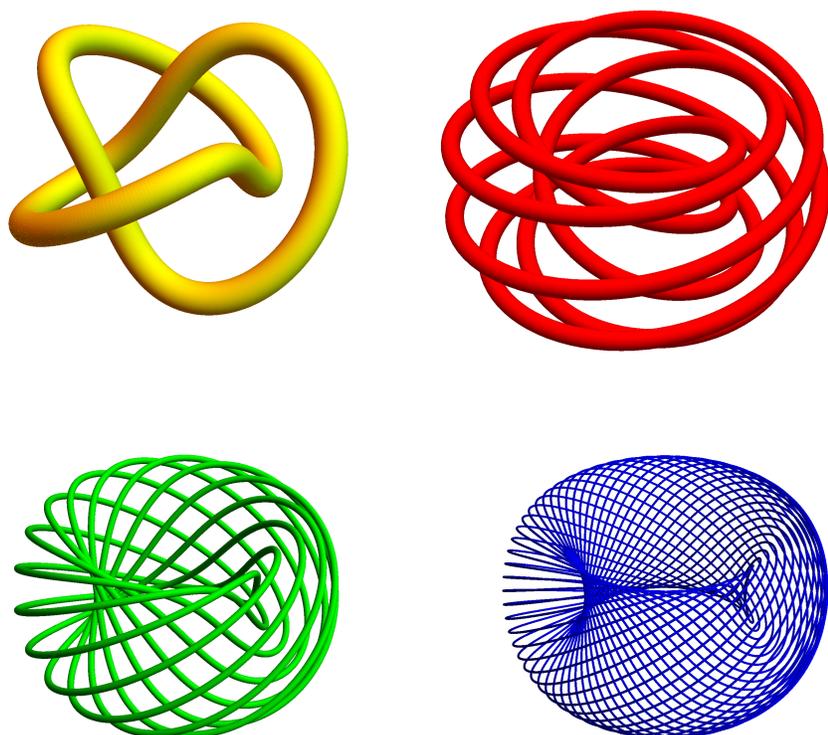FIGURE 3. Torus knots $T(2,3), T(7,3), T(12,13)$ and $T(30,43)$. Their total curvatures are $38.6, 245.6, 487.2, 2167.3$.
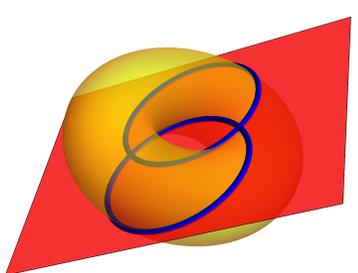

FIGURE 4. Villarceau circles obtained when slicing a bagel. Given a two surfaces, it can be tough to find the intersection.

Homework

Problem 7.1: You sit on a bench at $A = r(0) = [0, 0, 3]$ near the frozen Charles located between Winthrop and Elliot and chip stones aiming at $B = [0, -300, 15]$, a point near the Harvard business school. In order not to get into trouble, we assume everything happens in our imagination and that the stone is friction-free. You use a sling shot and throw with initial velocity $r'(0) = [0, -24, 61]$, assume the gravitational acceleration to be $r''(t) = [0, 0, -10]$ at all times and use meters for distance and seconds for time. At which point does the stone reach the 15 meter height mark while descending? [Optional: you like a challenge and want to bounce off on the ice surface at $C = [0, -200, 0]$ and reach the point B. What initial velocity v at A does achieve this?]

Problem 7.2: We want to produce a logo for a new company and experiment. Draw the curve $r(t) = [\cos(t), \sin(t)] + [\cos(11t), \sin(9t)]/4$ and find the velocity, acceleration, and curvature at $t = 0$.

Problem 7.3: Parametrize the curve $r(t)$ obtained by intersecting the cylinder $x^2/9 + y^2/4 = 1$ with the plane $z = x + 5y$.

Problem 7.4: Verify that the **torus knot** $r(t) = [x(t), y(t), z(t)] = [(2+\cos(mt))\cos(nt), (2+\cos(mt))\sin(nt), \sin(mt)]$ lives on the torus $(3 + x^2 + y^2 + z^2)^2 - 16(x^2 + y^2) = 0$.

Problem 7.5: You slice a bagel in a non-standard way. Let us assume that the bagel is given by $(x^2 + y^2 + z^2 + 16)^2 - 100(x^2 + y^2) = 0$. Verify that if we intersect this torus with the plane $3x = 4z$, then we get the **Villarceau circles** $r(t) = [4\cos(t), 3 + 5\sin(t), 3\cos(t)]$ as well as the circle $r(t) = [4\cos(t), -3 + 5\sin(t), 3\cos(t)]$.

FIGURE 5. The scenery for our imaginary dream sling shot experiment.

Unit 8: Arc length

INTRODUCTION
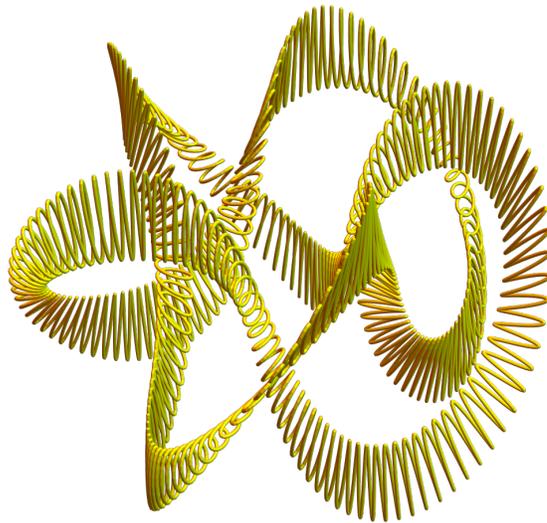
FIGURE 1.   A rather complicated knot curve $r(t)$. Despite its complexity, we can compute the length of the curve numerically by integrating $|r'(t)|$ over the parameter interval. In this case, the curve has diameter 14 and already a length of 1243. While a DNA double helix is 10 nanometer wide, the total length of a human DNA is about 2 meters long.

8.1. In this lecture we really get into calculus as we use both differentiation as well as integration to compute the length of curves. This unit is also a good point to brush up some integration techniques. The main theoretical result is that if $r'(t)$ is piecewise continuous, then we can compute the length. Mathematically we will see that the Riemann integral $\int_a^b f(t)\,dt$ exists for every continuous function.

8.2. In single variable calculus courses, one usually assumes that f is differentiable in which case the proof is much simpler. So, in some sense, we want to illustrate here also that calculus leads to real analysis which is close also to the core foundations of mathematics. Both when computing derivatives or integrals we are using a notion of "limit". When we compute the length of a curve we split it into small pieces and add up the lengths of these pieces. That his process gives a limiting finite end result is by no means trivial. If we look at the motion of a pollen particle in a fluid and compute the length by tracking smaller and smaller time intervals, the length actually diverges to infinity.
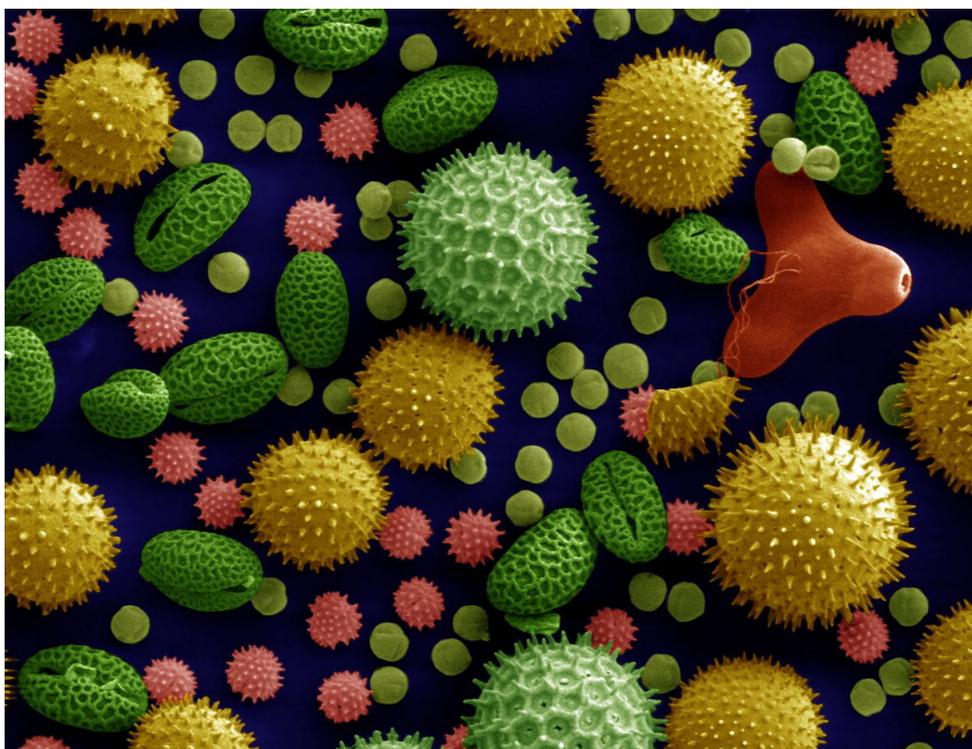
FIGURE 2. A colorized electron microscope scan of pollen grains from various plants, like sunflowers. This picture has been done by the Dartmouth Electron microscope facility and put into the public domain. By the way, these particles are also an inspiration for surfaces.

LECTURE

8.3. We assume in this lecture that curves are **continuously differentiable** meaning that the velocity is continuous. We would write $r \in C^1([a,b], \mathbb{R}^d)$. Given a parametrized curve $r(t)$ defined over an interval $I = [a,b]$, its **arc length** is defined as

$$L = \int_a^b |r'(t)| \, dt \ .$$

For $f(t) = |r'(t)|$ the integral is defined as the **lim sup** (we don't know yet whether lim exists),

$$\int_a^b f(t) \, dt = \limsup_{n \to \infty} \frac{S_n}{n} = \limsup_{n \to \infty} \frac{1}{n} \sum_{a \le \frac{k}{n} < b} f\left(\frac{k}{n}\right) \, .$$

This **Archimedes integral** is a special **Riemann integral**. It satisfies $\min(f) \le (b-a)^{-1} \int_a^b f(t) \, dt \le \max(f)$. The **intermediate value theorem** implies that there is $y \in [a, b]$ such that $f(y) = (b-a)^{-1} \int_a^b f(t) dt$. The minimum and maximum exists by **Bolzano's extreme value theorem**. Related to Bolzano is the **Heine-Cantor theorem** assuring that a continuous function f on a closed finite interval $[a, b]$ is **uniformly continuous**: there exists a function $M(t)$ satisfying $\lim_{t \to 0} M(t) = 0$ with $|f(x) - f(y)| \le M(|x - y|)$ for all $x, y \in [a, b]$. Stronger is **Lipschitz continuity**, which is $M(t) = M \cdot t$ for some constant M. The next proof shows in general that continuous functions are Riemann integrable; the limsup is actually a limit:

Theorem: Arc length exists and is independent of the parameterization.

Proof. (i) To see parameter independence, assume a time change $\phi(t)$ with a monotone smooth function $\phi : [a, b] \to [\phi(a), \phi(b)]$. If $r(t)$ on $[\phi(a), \phi(b)]$ and $R(t) = r(\phi(t))$ on $[a, b]$ are the two parametrizations and $f(t) = |r'(t)|$ and $F(t) = |R'(t)| = |r'(\phi(t))|\phi'(t)$, then by substitution, the arc length of $r(t)$ is $\int_{\phi(a)}^{\phi(b)} f(t) \, dt = \int_a^b f(\phi(t))\phi'(t) \, dt$ which is $\int_a^b F(t) \, dt$, the arc length of $R(t)$.
(ii) From (i) we can assume $[a, b] = [0, 1]$. By uniform continuity, there are $M_n \to 0$ such that if $|y - x| \le 1/n$, then $|f(y) - f(x)| \le M_n$. The **intermediate value theorem**, gives for every $I_k = [x_k, x_{k+1}] = [k/n, (k+1)/n] \subset [0, 1]$, a $y_k \in I_k$ such that $\int_{x_k}^{x_{k+1}} f(x) \, dx = f(y_k)/n$. Now, $\int_0^1 f(x) \, dx = (1/n) \sum_k f(y_k)$ and $|S_n/n - \int_0^1 f(x) \, dx| = (1/n)|\sum_k [f(x_k) - f(y_k)]| \le (1/n) \sum_k |f(x_k) - f(y_k)| \le 1/n \sum_k M_n = M_n \to 0$. $\square$

<div align="center">EXAMPLES</div>

8.4. The arc length of the circle $r(t) = [R \cos(t), R \sin(t)]$ with $t \in [0, 2\pi]$ is $\int_0^{2\pi} |r'(t)| \, dt = \int_0^{2\pi} R \, dt = 2\pi R$.

8.5. The arc length of the parabola $r(t) = [t, t^2/2]$ with $t \in [-1, 1]$ is $\int_{-1}^1 \sqrt{1 + t^2} \, dt$. We will do this integral in class. The result is $\sqrt{2} + \text{arcsinh}(1)$.

8.6. The arc length of the curve $r(t) = [\log(t), \sqrt{2}t, t^2/2]$ for $t \in [1, 2]$. It is

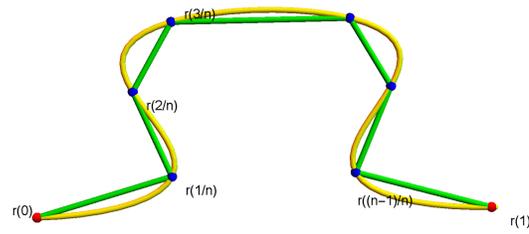$$\int_1^2 \sqrt{1/t^2 + t^2 + 2} \, dt = \int_1^2 (t + 1/t) \, dt = \log(2) + 3/2 \, .$$

FIGURE 3. A polygon approximation of a curve produces a Riemann sum approximation of the length integral.
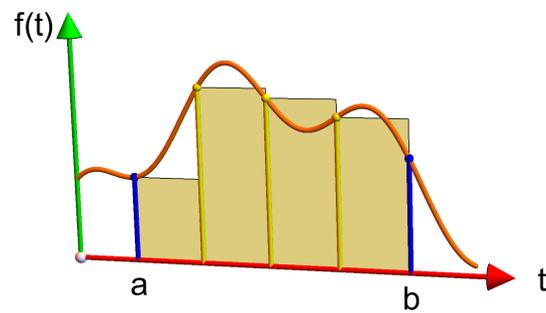

FIGURE 4. A Riemann sum approximation of a continuous function produces in the limit the "area under the curve".
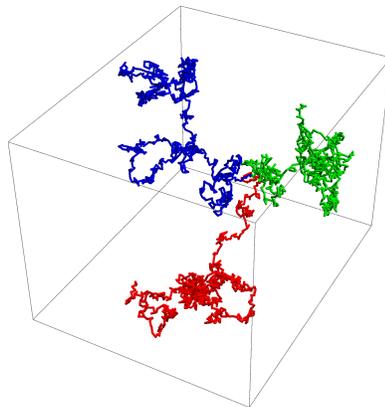
FIGURE 5. Brownian motion produces continuous paths which are not differentiable. The arc length integral does not exist.

Problem 8.1: Find the arc length of the curve
$$r(t) = [12t, 8t^{3/2}, 3t^2] \,,$$
where $t \in [0, 7]$.

Problem 8.2: Find the arc length of the cycloid
$$r(t) = [t - \sin(t), 1 + \cos(t)]$$
from 0 to 2π. The upside down cycloid is the solution to the famous **Brachistochrone problem**, the curve along which a ball descends fastest. Hint. You might want to use the double angle formula $2 - 2\cos(t) = 4\sin^2(\frac{t}{2})$.

Problem 8.3: Compute numerically the arc length of the knot $r(t) = [\sin(4t), \sin(3t), \cos(5t), \cos(7t)]$ from $t = 0$ to $t = 2\pi$. By drawing the first coordinates only and using color as the fourth coordinate, we can see that there are no non-trivial knots in $\mathbb{R}^4$. You can not tie your shoes in $\mathbb{R}^4$!

Problem 8.4: What is the relation between $|\int_0^1 r'(t) \ dt|$ and $\int_0^1 |r'(t)| \ dt$? Give an interpretation of both sides.

Problem 8.5: Find the arc length of the **catenary** $r(t) = [t, \cosh(t)]$, where $\cosh(t) = (e^t + e^{-t})/2$ is the **hyperbolic cosine** and $t \in [-1, 1]$. Hint. You can use the identity $\cosh^2(t) - \sinh^2(t) = 1$, where $\sinh(t) = (e^t - e^{-t})/2$ is the **hyperbolic sine**. We have $\cosh' = \sinh, \sinh' = \cosh$.

Galileo was the first to investigate the catenary. It is the curve, a freely hanging heavy rope describes, if the end points have the same height. Galileo mistook the curve for a parabola. It was Johannes Bernoulli in 1691, who obtained its true form after some competition involving Huygens, Leibniz and two Bernoullis. The name "catenarian" (=chain curve) was first used by Huygens in a letter to Leibnitz in 1690.
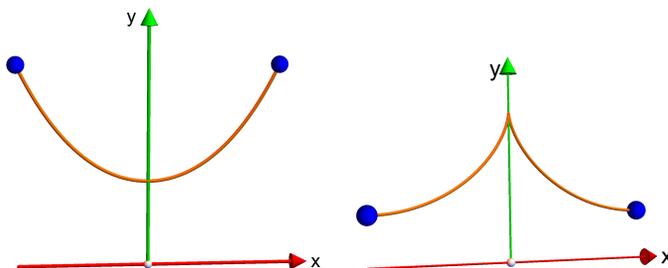
FIGURE 6. The catenary and the cycloid.

Unit 9: Intuition

Introduction
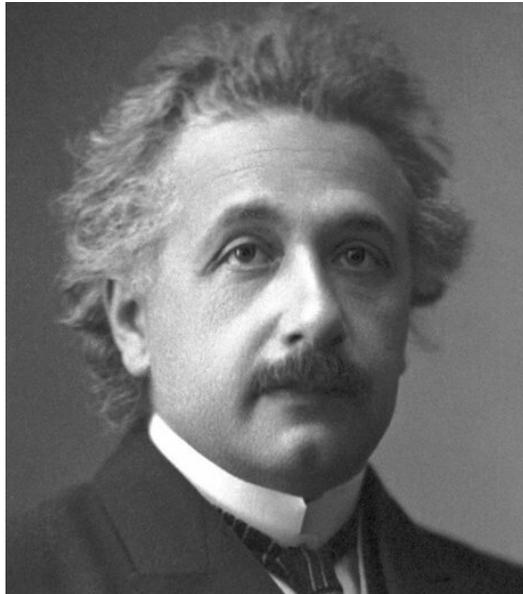
FIGURE 1. If you look up the word "intuition", you inevitably get to the Einstein quote **"The only real valuable thing is intuition"**. This widely quoted sentence seems not to have any source in Einsteins work. The quote appears in a book **"The psychology of conciousness"** of 1973 by Robert Even Ornstein. In the 1986 edition of the book, the quote is gone. It is questionable that Einstein should have said this: intuition without knowledge and technique mastery is of little value, as it is as vague as the "sixth sense" or "gut feeling". Photo: Nobel foundation archive. Einstein was awarded the prize in 1921 and received it in 1922.

9.1. Intuition is a mysterious notion in psychology. Ask a different school of psychology or go to a different culture and intuition will be understood in different ways. It is sometimes even linked to spirituality or religion. In mathematics, good intuition is usually thought of as the ability to "gain insight" or "see structures", sometimes an ability to be "creative". One attempt for definition is "understanding without conscious reasoning". Intuition can also be dangerous. An intuitive argument is not a proof for

example, even if it might lead to a rigorous proof eventually. Extremely intuitive thinkers are sometimes also self-delusional. This is illustrated by the existence of extremely intuitive proofs of major open problems, but they are usually simply wrong.

9.2. What is intuition then? René Descartes tried to formulate it in his treatise **"Rules for the Direction of the mind"** written between 1619 and 1628. Rule 12 in that document tells **"Finally we ought to employ all the help of understanding, imagination, sense and memory, first for the purpose of having a distinct intuition of simple propositions"**. For Descartes therefore, intuition has various components including understanding, imagination, sense and memory. This is a rather modern notion. A computer scientist can argue that computers already can be intuitive: the proof is by evidence only but we have seen in chess for example that computers have surpassed any human chess player. The last attempts by world chess champions to win against a machine failed. Since then, human-machine chess matches are all handicap variants, where the human is given a substantial advantage. And chess is a game where intuition is important. [1]

Seminar

9.3. Despite all what has been said in the introduction, it is important in mathematics to gain "intuition" about objects, about definitions and about theorems and proofs. One way to see intuition is to see it as a mnemonic device which allows to understand things in a way which is better to remember. It also gives us pointers where we have to be careful. Non-intuitive results can also lead to intuition in other areas. An example from probability theory is the non-intuitive result that if you have a class of 23 students, the probability that two have the same birthday is more than a half. This is the **birthday paradox**. Indeed, the probability that none of them has the same birthday is $(365/365)(364/365)\cdots(343/365) = 0.4927$. Now, once you have seen this, you have gained intuition about **conincidences**. They happen much more than we believe is reasonable. We can now take this to our advantage and design algorithms which gives us a result if two events collide. This has been used in cryptology for example. We can design algorithms which can pick a lock much faster than we believe is possible. [2]

9.4. Now, rather than telling how to get intuition, it is maybe better to look at cases where intuition fails. This can then turned around and allows us to elevate our intuitive understanding. We will illustrate pitfalls with intuition by showing that intuitive notions can also mislead. We can state "false theorems" which we would believe to be true but which are false. We start with the notion of "continuity" for which an intuitive definition tells: we can "draw the graph of a continuous function without having to lift the pen". Of course, we can not work with this definition to prove theorems. It is a good intuitive notion although and provides some sort of "proto understanding". If you want to probe your notion of understanding about continuity, ask yourself whether the function $f(x) = x\sin(1/x)$ is continuous or not everywhere.

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[1] "The Queen's gambit" illustrates that not only intuition but hard work, training and memory matters

[2] The Pollard rho method is such an application to factor large integers.

9.5. Starting with Cauchy and pushed heavily by Weierstrass, continuity is defined precisely using the infamous $\epsilon - \delta$ definition: f is continuous at x, if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - y| \leq \delta$, then $|f(x) - f(y)| \leq \epsilon$. Using more fancy mathematical quantifier notation $\forall$ (for all) and $\exists$ (exists) and $\Rightarrow$ (implies) and ϵ (is element of) you can impress your friends (and annoy readers and graders) by writing

$$\boxed{\forall \epsilon > 0 \exists \delta > 0 \forall y \in [a, b], |x - y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \epsilon \, .}$$

The fact that his definition is not intuitive at all and that most students just learn this "epsilontic" by **intimidation** is illustrated by the following variation by Ed Nelson [3] We make it our first exercise:

> **Problem A:** What does the following statement mean?

$$\boxed{\forall \delta > 0 \exists \epsilon > 0 \forall y \in [a, b], |x - y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \epsilon \, .}$$

9.6. In the Monday lecture we have seen how a polygonal approximation of a curve allows to compute the arc length of a curve. Here is a first "anti-theorem". Your task is to figure out what is wrong.

9.7. We compute the **circumference of a circle** by a polygonal approximation. The following statement uses the intuition that if a polygon is close to a curve, then its length is close to the curve:
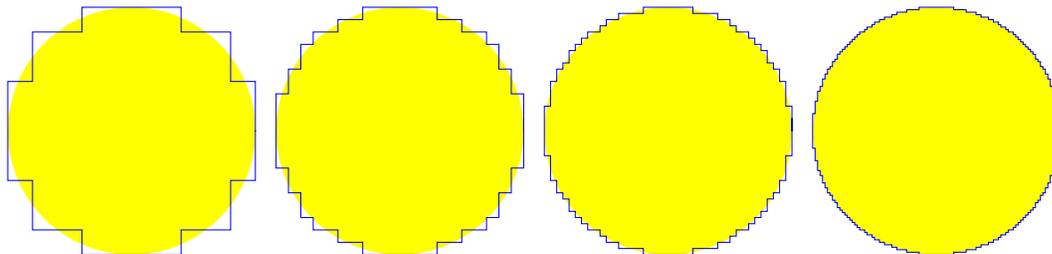
FIGURE 2. The circumference of a circle is 8.

9.8. This leads to the following anti-theorem: [4] A continuous planar curve is a function $t \to r(t) = [x(t), y(t)]$, where both functions $x(t), y(t)$ are continuous functions.

> **False Theorem:** The circumference of the unit circle is 8.

> **Problem B:** What is wrong with the argumentation?

9.9. We could also think that the arc length of a continuous curve is finite.

> **False Theorem:** The arc length of a continuous curve is finite.

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[3]E. Nelson, Internal set theory: A new approach to nonstandard analysis, 1977
[4]Again thanks to Jun Hou Fung for suggestion
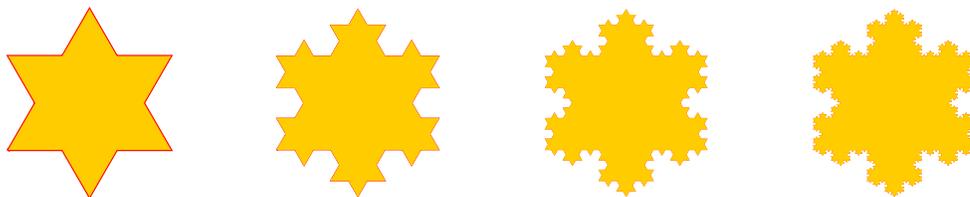
FIGURE 3. The first 4 approximations of the Koch snowflake.

Problem C: Find a formula for the length of the k'th Koch curve approximation if initially, the triangle has side length 1

9.10. If a curve $t \to r(t) = [x(t), y(t)]$ has the property that $x(t)$ and $y(t)$ stay bounded and have no jump discontinuities, we would think that the curve is continuous.

False Theorem: A bounded curve without jumps is continuous.

9.11. A counter example is the **devil comb** $r(t) = [t, \sin(1/t)]$ for $t \in [0, 1]$. It does not have a jump discontinuity and it is bounded. The function is not defined at $t = 0$ but we can define $r(0) = [0, 0]$ to make it defined anywhere on $[0, 1]$.

Problem D: Why is this function $r(t)$ not continuous at $t = 0$?

9.12. Finally, we could think:

False Theorem: A continuous function is differentiable at some point.

9.13. A counter example was given by Weierstrass. It is called the Weierstrass function. G.H. Hardy proved in 1916 that the function

$$f(x) = \sum_{n=1}^{\infty} a^{-n} \cos(a^n x)$$

does not have any point of differentiability if $a > 1$.
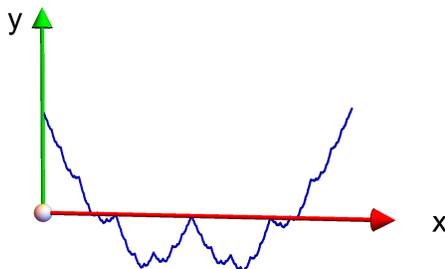
FIGURE 4. The Weierstrass function for $a = 2$, displayed on $[0, \pi]$.

Problem E: Show that $f(x) = \sum_{n=1}^{\infty} 2^{-n} \cos(2^n x) \in [-1, 1]$.
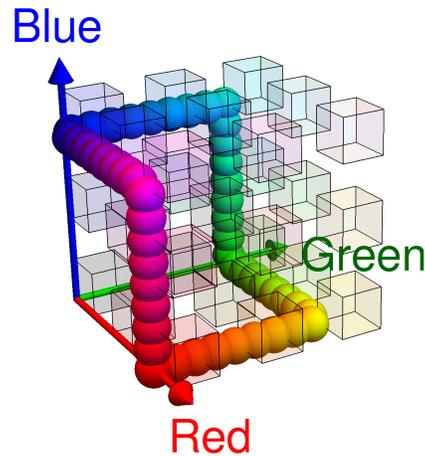
FIGURE 5. The "Hue curve" in color space. It will appear in the assignment.

9.14. Lets look at the **Morley theorem** in planar geometry. It tells that in any triangle, the angle trisector intersections are an equilateral triangle. Can you find a proof? Don't try. Without looking it up, finding an intuitive proof is very, very hard.
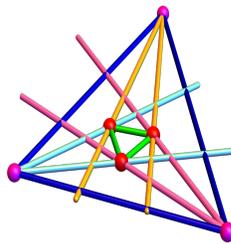
FIGURE 6. The **Morley theorem** has a short and intuitive proof. But it is hard to find.

9.15. Finally we come back to the question early on whether the function

$$f(x) = x \sin(1/x)$$

is continuous. The answer is yes. The function $g(x) = \sin(1/x)$ is not continuous as we can find arbitrary small $x = 1/(\pi/2 + k2\pi)$ for which the function is $g(x) = 1$ and arbitrary small $x = 1/(k2\pi - \pi/2)$ for which the function is $g(x) = -1$. For the function $f(x)$ however we can say that $|f(x)| \leq |x|$. So, if x is small, also $|f(x)|$ is small. If you want to check your intuition with formal statements: given any $\epsilon > 0$ we can find a $\delta > 0$ (namely $\delta = \epsilon$) such that if $|x| \leq \delta$ then $|f(x)| \leq \epsilon$.

Homework

Problem 9.1 Prove that there was a time in your life when the length of your largest tooth in millimeters was your height in meters.

Problem 9.2 Is the function $f(x, y) = (x^4 + y^4)/(x^2 + y^2)$ continuous everywhere if we assume $f(0, 0) = 0$? Intuition is a bit harder here as we divide 0 over 0 at the origin and are in two variables. In either case, give a reason for your answer. You can already use polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$. Coordinates will be talked about more next week.

Problem 9.3 Use the intermediate value theorem to derive Rolle's theorem using an "intuitive" argument: if f is continuously differentiable (meaning $f'(x)$ is differentiable) and $f(0) = f(1) = 0$, then there exists a point in $(0, 1)$ with $f'(x) = 0$.

Problem 9.4 A cylinder S of radius 1 and height 1 is approximated by a polyhedron with triangles of size ϵ. If S_n is the polygonal approximation. Does the surface area $|S_n|$ of the polyhedron and the surface area of the surface S satisfies $|S_n| \to |S|$? Give an example where the answer is yes.

Problem 9.5 As a continuation of 9.4, there is a Chinese lantern type construction which shows that $|S_n| \to |S|$ is wrong in general. Look the construction of the Schwarz lantern from 1880 up and describe it.

Unit 10: Coordinates

INTRODUCTION

10.1. Algebra is a powerful tool in geometry. In this lecture we circle back to the concept of coordinates and look also at other coordinate systems. We have introduced space as column vectors like $[1, 2, 3]^T$. We can think of it as an arrow from the origin to the point $(1, 2, 3)$. One speaks of the numbers appearing in $(1, 2, 3)$ as coordinates while the entries in $[1, 2, 3]^T$ are components of the vector. Most of the time we do not distinguish between the point $(1, 2, 3)$ and the vector $[1, 2, 3]^T$ as the two objects can clearly be identified naturally. We look in this lecture also at other coordinates like Polar and spherical coordinates. This will be important when doing integration.
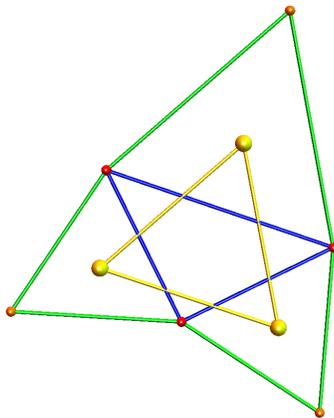
FIGURE 1. Napoleon's theorem tells that if we draw equilateral triangles over the sides of a triangle, their center of mass are on an equilateral triangle. A geometric proof is not so easy to find but using coordinates it is a direct calculation: for three complex numbers a, b, c, then $u = (a+b)/2+i(b-a)/3, v = (b+c)/2+i(c-b)/2, w = (a+b)/2+i(b-a)/3$ satisfy $|u - v| = |u - v| = |v - w|$. The result is famous because no other theorem has been rediscovered so many times. While Napoleon might never have discovered or proved it himself, he kept conversations with mathematicians like Lagrange or Fourier.
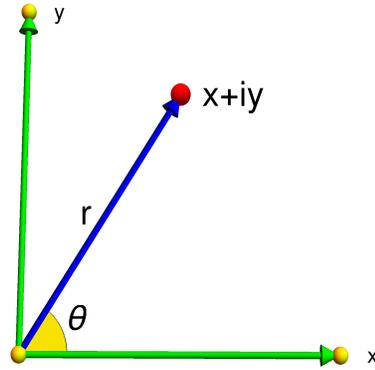
FIGURE 2. In the two-dimensional plane a point $(x, y) = (3, 4)$ can also be identified with the complex number $z = x + iy = 3 + 4i$ or the vector $[3, 4]^T$. The magnitude of the vector is $\sqrt{x^2 + y^2}$ and is defined to be the length of the complex number z. Multiplication rotates and scales. A multiplication with i rotates by 90 degrees.

LECTURE

10.2. It was René Descartes who in 1637 introduced **coordinates** and brought algebra close to geometry. [1] The **Cartesian coordinates** (x, y) in $\mathbb{R}^2$ can be replaced by other coordinate systems like **polar coordinates** (r, θ), where $r = \sqrt{x^2 + y^2} \geq 0$ is the **radial distance** to the $(0, 0)$ and $\theta \in [0, 2\pi)$ is the **polar angle** made with the positive x-axis. Since θ is in the interval $[0, 2\pi)$, it is best described in the complex notation $\theta = \arg(x + iy)$. The radius $r = |z| = \sqrt{x^2 + y^2}$ is the length of the complex number. The conversion from the (r, θ) coordinates to the (x, y)-coordinates is

$$x = r \cos(\theta)$$
$$y = r \sin(\theta)$$

The radius is $r = \sqrt{x^2 + y^2}$, where if non-zero, we always take the positive root. The angle formula $\arctan(y/x)$ only holds if x and y are both positive. The angle θ is not uniquely defined at the origin $(0, 0)$, most software just assumes $\arg(0) = 0$.

10.3. We can write a vector in $\mathbb{R}^2$ also in the form of a **complex number** $z = x + iy \in \mathbb{C}$ with symbol i. This is not only notational convenience. Complex numbers can be added and multiplied like other numbers and while $\mathbb{R}^2 = \mathbb{C}$, the later has a **multiplicative structure**. In order to fix that structure, one only needs to specify that $i^2 = -1$. This gives $(a + ib)(c + id) = ac - bd + i(ad + bc)$. We also have $|a + ib| = \sqrt{(a + ib)(a - ib)} = \sqrt{a^2 + b^2}$. An important **formula of Euler** link the exponential and trigonometric functions:

Theorem: $e^{i\theta} = \cos(\theta) + i \sin(\theta)$.

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[1]Descartes: La Géometrie, 1637 (1 year after the foundation of Harvard college)

10.4. The proof is to write the series definition on both sides. First recall the definitions of $e^x = 1 + x + x^2/2! + x^3/3! + ...$. If we plug in $x = i\theta$ we get $e^{i\theta} = 1 + i\theta - \theta^2/2! - i\theta^3/3! + \theta^4/4!...$ But this is $(1 - \theta^2/2 + \theta^4/4!...) + i(\theta - \theta^3/3! + \theta^5/5! - ...)$ which is $\cos(\theta) + i\sin(\theta)$. QED. If you prefer not to see the functions $\exp, \sin, \cos$ being **defined** as series, you can see them as **Taylor series** $f(x) = f(0) + f'(0)x + f''(0)/2!x^2 + ... = \sum_{k=0}^{\infty}(f^{(k)}(0)/k!)x^k$. By differentiating the functions at 0, we see then the connection.

10.5. The Euler formula implies for $\theta = \pi$ the magical formula

> **Theorem:** $e^{i\pi} + 1 = 0$

This formula is often voted the **"nicest formula in math"**. [2] It combines "analysis" in the form e, "geometry" in the form of π, "algebra" in the form of i, the additive unit 0 and the multiplicative unit 1.

10.6. The Euler formula allows to write any complex number as $z = re^{i\theta}$. Given an other complex number $w = se^{i\phi}$ we have $zw = rse^{i\theta+\phi}$ showing that the polar angles add and the radius multiplies. The Euler formula also allows to define the **logarithm** of any complex number as $\log(z) = \log(|z|) + i\arg(z) = \log(r) + i\theta$. We see now that going from (x, y) to $(\log(r), \theta)$ is a very natural transformation from $\mathbb{C} \setminus 0$ to $\mathbb{C}$. The exponential function $\exp : z \to e^z$ is a map from $\mathbb{C} \to \mathbb{C} \setminus 0$. It transforms the additive structure on $\mathbb{C}$ to the multiplicative structure because $\exp(z + w) = \exp(z)\exp(w)$.

10.7. In three dimensions, we can look at **cylindrical coordinates** (r, θ, z) which are just polar coordinates in the first two coordinates. A cylinder of radius 2 for example is given as $r = 2$. The torus $(3 + x^2 + y^2 + z^2)^2 - 16(x^2 + y^2) = 0$ can be written as $3 + r^2 + z^2 = 4r$ or more intuitively as $(r - 2)^2 + z^2 = 1$, a circle in the $r - z$ plane.
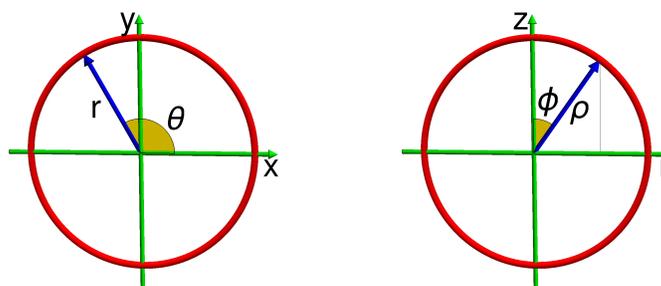
FIGURE 3. Key pictures to derive cylindrical and spherical coordinates.

10.8. The **spherical coordinates** (ρ, θ, ϕ), where $\rho = \sqrt{x^2 + y^2 + z^2}$. The angle θ is the polar angle as in cylindrical coordinates and ϕ is the angle between the point (x, y, z) and the z-axis. We have $\cos(\phi) = [x, y, z] \cdot [0, 0, 1]/|[x, y, z]| = z/\rho$ and $\sin(\phi) = |[x, y, z] \times [0, 0, 1]|/|[x, y, z]| = r/\rho$ so that $z = \rho\cos(\phi)$ and $r = \rho\sin(\phi)$ and therefore

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[2]D. Wells, Which is the most beautiful?, Mathematical Intelligencer, 1988

$$\begin{aligned} x &= \rho \sin(\phi)\cos(\theta) \\ y &= \rho \sin(\phi)\sin(\theta) \\ z &= \rho \cos(\phi) \end{aligned}$$

where $0 \le \theta < 2\pi, 0 \le \phi \le \pi$ and $\rho \ge 0$.

10.9. A **coordinate change** $x \to f(x)$ in the plane is a map $f : \mathbb{R}^2 \to \mathbb{R}^2$. A point (x_1, x_2) is mapped into (f_1, f_2). We write ∂_{x_k} for the **partial derivative** with respect to the variable x_k. For example $\partial_{x_1}(x_1^2 x_2 + 3x_1 x_2^3) = 2x_1 x_2 + 3x_2^3$.

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} , \quad df\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \partial_{x_1} f_1(x) & \partial_{x_2} f_1(x) \\ \partial_{x_1} f_2(x) & \partial_{x_2} f_2(x) \end{bmatrix} ,$$

where df is a matrix called the **Jacobian matrix**. The determinant is called the **distortion factor** at $x = (x_1, x_2)$.

10.10. For polar coordinates, we get

$$f\begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix} , df\begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix} .$$

Its distortion factor of the Polar map is r. We will use this when integrating in polar coordinates.

10.11. If $f(z) = z^2 + c$ with $c = a + ib, z = x + iy$ is written as $f(x, y) = (x^2 - y^2 + a, 2xy + b)$, then df is a 2×2 **rotation dilation matrix** which corresponds to the complex number $f'(z) = 2z$. The algebra $\mathbb{C}$ is the same as the algebra of rotation-dilation matrices.

10.12. A **coordinate change** $x \to f(x)$ in space is a map $f : \mathbb{R}^3 \to \mathbb{R}^3$. We compute

$$f\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} , df\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \partial_{x_1} f_1(x) & \partial_{x_2} f_1(x) & \partial_{x_3} f_1(x) \\ \partial_{x_1} f_2(x) & \partial_{x_2} f_2(x) & \partial_{x_3} f_2(x) \\ \partial_{x_1} f_3(x) & \partial_{x_2} f_3(x) & \partial_{x_3} f_3(x) \end{bmatrix} .$$

We wrote $x = (x_1, x_2, x_3)$. Its determinant $\det(dT)(x)$ is a volume distortion factor.

10.13. For spherical coordinates, we have

$$f\begin{bmatrix} \rho \\ \phi \\ \theta \end{bmatrix} = \begin{bmatrix} \rho\sin(\phi)\cos(\theta) \\ \rho\sin(\phi)\sin(\theta) \\ \rho\cos(\phi) \end{bmatrix} , df\begin{bmatrix} \rho \\ \phi \\ \theta \end{bmatrix} = \begin{bmatrix} \sin(\phi)\cos(\theta) & \rho\cos(\phi)\cos(\theta) & -\rho\cos(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) & \rho\cos(\phi)\sin(\theta) & \rho\cos(\phi)\cos(\theta) \\ \cos(\phi) & -\rho\sin(\phi) & 0 \end{bmatrix} .$$

The distortion factor is $\det(df(\rho, \phi, \theta)) = \rho^2 \sin(\phi)$.

EXAMPLES

10.14. The point $(x, y) = (-1, 1)$ corresponds to the complex number $z = -1 + i$. It has the polar coordinates $(r, \theta) = (\sqrt{2}, 3\pi/4)$. As we have $z = re^{i\theta}$, we check $z^2 = (-1 + i)(-1 + i) = -2i$ which agrees with $(re^{i\theta})^2 = r^2 e^{2i\theta} = 2e^{6\pi i/4}$.

10.15. a) $(x, y, z) = (1, 1, -\sqrt{2})$ corresponds to spherical coordinates $(\rho, \phi, \theta) = (2, 3\pi/4, \pi/4)$.
b) The point given in spherical coordinates as $(\rho, \phi, \theta) = (3, 0, \pi/2)$ is the point $(0, 3, 0)$.

10.16. a) The set of points with $r = 1$ in $\mathbb{R}^2$ form a circle.
b) The set of points with $\rho = 1$ in $\mathbb{R}^3$ form a sphere.
c) The set of points with spherical coordinates $\phi = 0$ are points on the positive z-axis.
d) The set of points with spherical coordinates $\theta = 0$ form a half plane in the yz-plane.
e) The set of points with $\rho = \cos(\phi)$ form a sphere. Indeed, by multiplying both sides with ρ, we get $\rho^2 = \rho \cos(\phi)$ which means $x^2 + y^2 + z^2 = z$, which is after a completion of the square equal to $x^2 + y^2 + (z - 1/2)^2 = 1/4$.

10.17. For $A \in M(n,n)$, $f(x) = Ax + b$ has $df = A$ and distortion factor $\det(A)$.

10.18. Find the Jacobian matrix and distortion factor of the map $f(x_1, x_2) = (x_1^3 + x_2, x_2^2 - \sin(x_1))$. Answer: Write both the transformation and the Jacobian:

$$ f \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1^3 + x_2 \\ x_2^2 - \sin(x_1) \end{bmatrix} , \ df \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1^2 & 1 \\ -\cos(x_1) & 2x_2 \end{bmatrix} . $$

The Jacobian matrix is $\det(df(x)) = 6x_1^2 x_2 + \cos(x_1)$.

ILLUSTRATIONS

10.19. Let $T : \mathbb{C} \to \mathbb{C}$ be defined as $z \to z^2 + c$, where $z = x + iy$. The set of all $c = a + ib$ for which the iterates $T^n(0)$ stay bounded is the **Mandelbrot set** M. For $c = -1$ we get $T(0) = -1, T^2(0) = T(-1) = 0$ so that $T^n(z)$ is either 0 or -1. The point $c = -1$ is in M. The point $c = 1$ gives $T(0) = 1, T^2(0) = 1^2 = 1 = 2, T^3(0) = 2^2 + 1 = 5$. Induction shows that $T^n(0)$ does not converge. The point $c = 1$ is not in M.

10.20. If T is the transformation in $\mathbb{R}^3$ which is in spherical coordinates given by $T(x) = x^2 + c$, where x^2 has spherical coordinates $(\rho^2, 2\phi, 2\theta)$ if x has (ρ, ϕ, θ). It turns out that $T(x) = x^8 + c$ gives a nice analogue of the Mandelbrot set, the **Mandelbulb**.
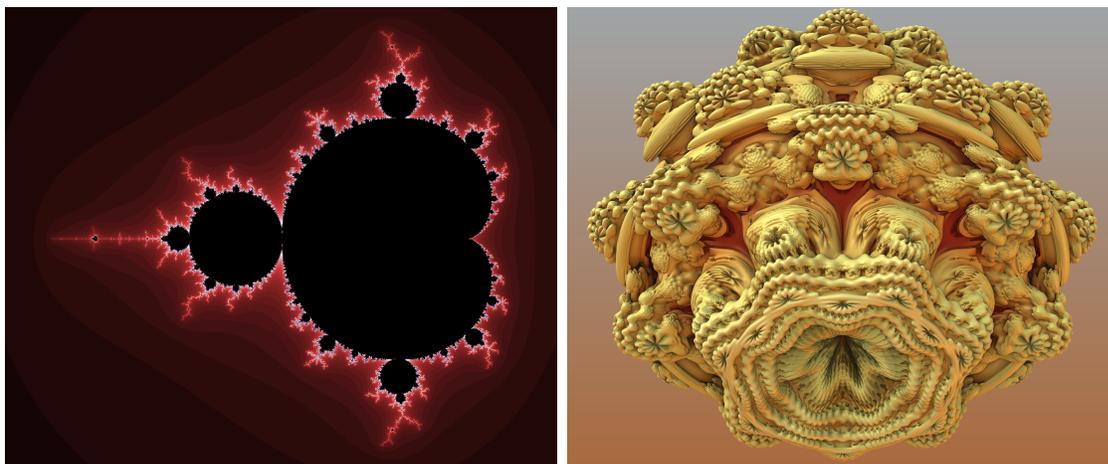
FIGURE 4. The **Mandelbrot set** $M = \{c \in \mathbb{C} | T(z) = z^2 + c$ has bounded $T^n(0)\}$. There is a similar construction in space $\mathbb{R}^3$ which uses spherical coordinates. This leads to the **Mandelbulb set** $B = \{c \in \mathbb{R}^3 \mid T(x) = x^8 + c$ has bounded $T^n(0)$ $\}$, where x^8 has spherical coordinates $(\rho^8, 8\phi, 8\theta)$ if x has spherical coordinates (ρ, ϕ, θ).

Homework

Problem 10.1: a) Find the polar coordinates of $(x, y) = (-1, \sqrt{3})$.
b) Which point has the polar coordinates $(r, \theta) = (2, \pi/4)$?
c) Find the spherical coordinates of the point $(x, y, z) = (1, 1, \sqrt{2})$.
d) Which point has the spherical coordinates $(\rho, \theta, \phi) = (3, \pi/2, \pi/3)$?

Problem 10.2: a) Compute $T_c^n(0)$ for $c = (1 + i)$ for $n = 1, 2, 3$. Is $1 + i$ in the Mandelbrot set?
b) What is the "eye for an eye" number i^i? (You can use $z^w = e^{w \log(z)}$).

Problem 10.3: a) Which surface is described as $r = z$?
b) Describe the hyperbola $x^2 - y^2 = 5$ in polar coordinates.
c) Which surface is described as $\rho \sin(\phi) = \rho^2$?
d) Describe the hyperboloid $x^2 + y^2 - z^2 = 1$ in spherical coordinates.

Problem 10.4: a) Compute the Jacobian matrix and distortion factor of the coordinate change $T(x, y) = (2x + \sin(x) - y, x)$ (**Chirikov map**).
b) Compute the Jacobian matrix and distortion factor of the coordinate change $T(x, y) = (1 - 1.4x^2 - y, 0.3x)$. (Classical **Hénon map**).
P.S. When you do the coordinate change of the Chiriov map again and again, one can observe **chaos**. In the case of the Hénon map, one sees a **strange attractor**, a fractal object which similarly as the Koch curve encountered last week has a dimension larger than 1.

Problem 10.5: a) Verify that the Mandelbrot set M is contained in the set $|c| \leq 2$. As a reminder, this means you have to show that then $0 \to c \to c^2 + c \to (c^2 + c)^2 + c...$ escapes to infinity.
b) Optional: Use the same argument to see that the Mandelbulb set B is contained in the set $|c| \leq 2$.

Unit 11: Parametrization

Introduction

11.1. We have seen that when parametrizing curves $r(t)$, we have much more control than when looking at curves given by equations. It would be difficult to describe a helix $r(t) = [\cos(t), \sin(t), t]$ in terms of equations for example. For surfaces also, it is good to have as many coordinates as the dimension. We live on a two dimensional sphere $x^2 + y^2 + z^2 = 1$ but do not use the x, y, z coordinates to describe a point on the surface. We use two coordinates longitude) and latitude Euler used first the parametrization $[x, y, z] = [\cos(t)\cos(s), \sin(t)\sin(s), \sin(s)]$ where t, s are angles. You can check quickly that $x^2 + y^2 + z^2$ adds up to 1 so that whatever angles t, s we chose, we always are on the sphere.
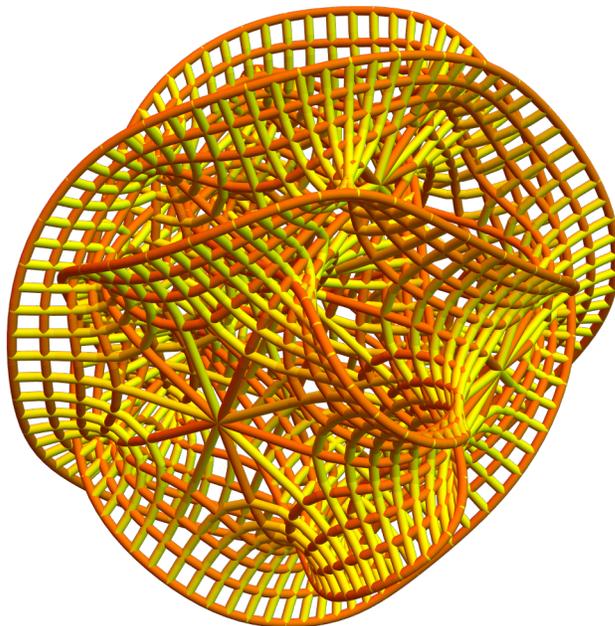

FIGURE 1. This surface is an example of a Calabi-Yau surface. It is parametrized $r(u, v)$. We drew out some grid curves, where u is constant or v is constant.

LECTURE

11.2. A map $r : \mathbb{R}^m \to \mathbb{R}^n$ is called a **parametrization**. We have seen maps r from $\mathbb{R}$ to $\mathbb{R}^n$, which were **curves**. Then we have seen maps $f : \mathbb{R}^n \to \mathbb{R}^n$ which were **coordinate changes**. In each case we defined the **Jacobian matrix** $df(x)$. In the case of the curve $r : \mathbb{R} \to \mathbb{R}^n$, it was the **velocity** $dr(t) = r'(t)$. In the case of coordinate changes, the Jacobian matrix $df(x)$ was used to get the **volume distortion factor** $\det(df(x)) = \sqrt{\det(df^T df)}$. Today, we look at the case $m < n$. In particular at $m = 2, n = 3$. As in the case of curves, we use the letter r to describe the map. The image of a map $r : R \subset \mathbb{R}^m \to \mathbb{R}^n$ is then a **m-dimensional surface** in $\mathbb{R}^n$. The **distortion factor** $||dr||$ defined as $||dr||^2 = \det(dr^T dr)$ will be used later to compute **surface area**. [1]
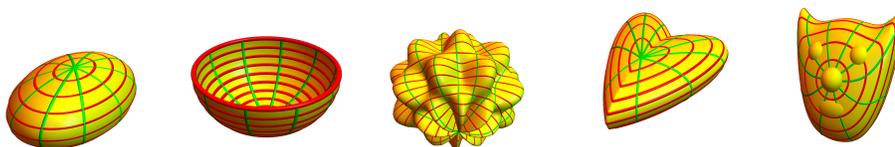
FIGURE 2. An ellipsoid, half an ellipsoid, a bulb, a heart and a cat.

11.3. We mostly discuss here the case $m = 2$ and $n = 3$, as we ourselves are made of two-dimensional surfaces, like cells, membranes, skin or tissue. A map $r : R \subset \mathbb{R}^2 \to \mathbb{R}^3$, written as $r(\begin{bmatrix} u \\ v \end{bmatrix}) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$ defines a two-dimensional surface. In order to save space, we also just write $r(u,v) = [x(u,v), y(u,v), z(u,v)]$. In computer graphics, the r is called uv**-map**. The uv-plane is where you draw a texture. The map r places it onto the surface. In geography, the map r is called (surprise!) a **map**. Several maps define an **atlas**. The curves $u \to r(u,v)$ and $v \to r(u,v)$ are called **grid curves**.

11.4. The parametrization $r(\phi, \theta) = [\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)]$ produces the **sphere** $x^2 + y^2 + z^2 = 1$. The full sphere has $0 \leq \phi \leq \pi$, $0 \leq \theta < 2\pi$. By modifying the coordinates, we get an **ellipsoid** $r(\phi, \theta) = [a\sin(\phi)\cos(\theta), b\sin(\phi)\sin(\theta), c\cos(\phi)]$ satisfying $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. By allowing a, b, c to be functions of ϕ, θ we get "bumpy spheres" like $r(\phi, \theta) = (3 + \cos(3\phi)\sin(4\theta))[\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)]$.

11.5. Planes are described by linear maps $r(x) = Ax + b$ with $A \in M(3,2)$ and $b \in M(3,1)$. The Jacobian map is $dr = A$. Let r_u, r_v be the two column vectors of A. Actually, r_u is a short cut for $\partial_u r(u,v)$, which is the velocity vector of the **grid curve** $u \to r(u,v)$.

---

[1]Distinguish $||A||^2 = \det(A^T A)$ and $|A|^2 = \text{tr}(A^T A)$ in $M(n,m)$. They only agree for $m = 1$.

11.6. An example is the parametrization $r(u,v) = [u + v - 1, u - v + 3, 3u - 5v + 7]$.
In this case $b = \begin{bmatrix} -1 \\ 3 \\ 7 \end{bmatrix}$, $r_u = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ $r_v = \begin{bmatrix} 1 \\ -1 \\ -5 \end{bmatrix}$ and $A = dr = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 3 & -5 \end{bmatrix}$. We see

$A^T A = \begin{bmatrix} 11 & -15 \\ -15 & 27 \end{bmatrix}$ which has determinant 72. We also have

$$|r_u \times r_v|^2 = \left| \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \\ -5 \end{bmatrix} \right|^2 = \left| \begin{bmatrix} -2 \\ 8 \\ -2 \end{bmatrix} \right|^2 = 72$$

11.7. The previous computation suggests a relation between the normal vector and the fundamental form $g = dr^T dr$. In three dimensions, the distortion factor of a parametrization $r : \mathbb{R}^2 \to \mathbb{R}^3$ can indeed always be rewritten using the cross product:

> **Theorem:** $\det(dr^T dr) = |r_u \times r_v|^2$.

Proof. As $dr^T dr = \begin{bmatrix} r_u \cdot r_u & r_u \cdot r_v \\ r_v \cdot r_u & r_v \cdot r_v \end{bmatrix}$, the identity is the **Cauchy-Binet identity**
$|r_u \times r_v|^2 = |r_u|^2 |r_v|^2 - |r_u \cdot r_v|^2$ which boils down to $\sin^2(\theta) = 1 - \cos^2(\theta)$, where θ is the angle between r_u and r_v. This is the angle between the grid curves you see on the pictures.
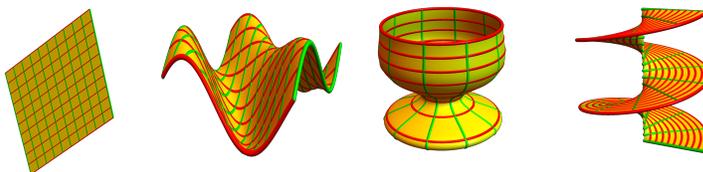
FIGURE 3. A plane, graph, surface of revolution and helicoid.

EXAMPLES

11.8. For the **unit sphere** $r(\phi, \theta) = [\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)]$ and $A = dr$:

$$g = A^T A = \begin{bmatrix} \cos(\phi)\cos(\theta) & \cos(\phi)\sin(\theta) & -\sin(\phi) \\ -\sin(\phi)\sin(\theta) & \sin(\phi)\cos(\theta) & 0 \end{bmatrix} \begin{bmatrix} \cos(\phi)\cos(\theta) & -\sin(\phi)\sin(\theta) \\ \cos(\phi)\sin(\theta) & \sin(\phi)\cos(\theta) \\ -\sin(\phi) & 0 \end{bmatrix}$$

This is $g = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2(\phi) \end{bmatrix}$ and $\sqrt{\det(g)} = \sin(\phi)$ is the distortion factor.

11.9. An important class of surfaces are **graphs** $z = f(x,y)$. Its most natural parametrization is $r(x,y) = [x, y, f(x,y)]$, where the map r just lifts up the bottom part to the elevated version. An example is the elliptic paraboloid $r(x,y) = [x, y, x^2 + y^2]$ and the hyperbolic paraboloid $r(x,y) = [x, y, x^2 - y^2]$. We could of course have written also $r(u,v) = [u, v, u^2 - v^2]$.

11.10. A **surface of revolution** is parametrized like $r(\theta, z) = [g(z)\cos(\theta), g(z)\sin(\theta), z]$. Note that we can use any variables. In this case, $u = \theta, v = z$ are used. An example is the **cone** $r(\theta, z) = [z\cos(\theta), z\sin(\theta), z]$ or the **one-sheeted hyperboloid** $r(\theta, z) = [\sqrt{z^2 + 1}\cos(\theta), \sqrt{z^2 + 1}\sin(\theta), z]$.

11.11. The **torus** is in cylindrical coordinates given as $(r - 3)^2 + z^2 = 1$. We can parametrize this using the polar angle θ and the polar angle centered at center of the circle as $r(\theta, \phi) = [(3 + \cos(\phi))\cos(\theta), (3 + \cos(\phi))\sin(\theta), \sin(\phi)]$. Both angles θ and ϕ go from 0 to 2π. We see now also the relation with the **toral coordinates**.

11.12. The **helicoid** is the surface you see as a staircase or screw. The parametrization is $r(\theta, p) = [p\cos(\theta), p\sin(\theta), \theta]$. How can we understand this? The key is to look at grid curves. If $p = 1$, we get a curve $r(\theta) = [\cos(\theta), \sin(\theta), \theta]$ which we had identified as a **helix**. On the other hand, if you fix θ, then you get lines.

11.13. Side remark. The **first fundamental form** $g = dr^T dr$ is also called a **metric tensor**. In **Riemannian geometry** one looks at a manifold M equipped with a metric g. The simplest case is when g comes from a parametrization, as we did here. In physics, we know that it is **mass** which deforms space-time. The quantity $||g||^2 = \det(g)$ is a multiplicative analogue of $|g|^2 = \text{tr}(g)$. For an invertible positive definite square matrix A, we will later see the identity $\log\det(A) = \text{tr}\log(A)$ which illustrates how both determinant and trace are pivotal numerical quantities derived from a matrix. Trace is **additive** because of $\text{tr}(A+B) = \text{tr}(A)+\text{tr}(B)$ and determinant is **multiplicative** $\det(AB) = \det(A)\det(B)$ as we will see later.

11.14. To summarize, we have seen so far that there are two fundamentally different ways to describe a manifold. The first is to write it as a level surface $f = c$ which is a **kernel** of a map $g(x) = f - c$. A second is to write it as the **image** of some map r.

ILLUSTRATION

FIGURE 4. "Veritas on Earth and the Moon" theme (rendered in Povray).
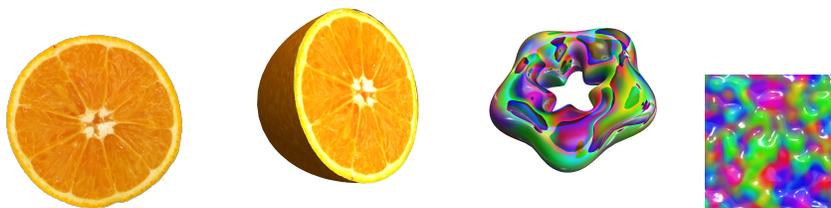
FIGURE 5. A fruit and math-candy© math-candy.com (rendered in Mathematica)

Problem 11.1: Parametrize the upper part of the two sheeted hyperboloid $x^2 + y^2 - z^2 = -1, z > 0$ as a surface of revolution.

Problem 11.2: a) Parametrize the plane $x + 2y + 3z - 6 = 0$ using a map $r : \mathbb{R}^2 \to \mathbb{R}^3$. b) Now find the matrix $A = dr$ and compute $g = A^T A$ as well as the distortion factor $\sqrt{\det(A^T A)}$. c) Also compute r_u, r_v and $r_u \times r_v$ and then compute $|r_u \times r_v|$. You should get the same number.

Problem 11.3: Given a parametrization $r(\theta, \phi) = [(7 + 2\cos(\phi))\cos(\theta), (7 + 2\cos(\phi))\sin(\theta), 2\sin(\phi)]$ of the 2-torus, find the implicit equation $g(x, y, z) = 0$ which describes this torus.

Problem 11.4: Parametrize the hyperbolic paraboloid $z = x^2 - y^2$. What is the first fundamental form $g = dr^T dr$ which is $g = \begin{bmatrix} r_x \cdot r_x & r_x \cdot r_y \\ r_y \cdot r_x & r_y \cdot r_y \end{bmatrix}$?. What is the distortion factor $\sqrt{\det(g)}$?

Problem 11.5: The matrix $g = dr^T dr$ is also called the **first fundamental form**. If $r : \mathbb{R}^4$ to $\mathbb{R}^4$ is a parametrization of **space time** then g is the **space time metric tensor**. The matrix entries of g appear in **general relativity**. Now for some reasons, physics folks use Greek symbols to access matrix entries. They write $g_{\mu\nu}$ for the entry at row μ and column ν. This appears for example in the **Einstein field equations**

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} .$$

We just want you to look up the equation and tell from each of the variables, what it is called and whether it is a matrix, a scalar function or a constant.

Unit 12: Creativity

INTRODUCTION

12.1. You have seen a couple of proofs so far. You might wonder, how was the proof found? For example, we have seen the Cauchy-Schwarz inequality $|v \cdot w| \leq |v||w|$. How did one get to the idea to first assume $|w| = 1$ then define $a = v \cdot w$ and look at $0 \leq (v - aw) \cdot (v - aw) = |v|^2 - a^2$? In this case the crucial input comes from a visual picture as we can see $v - aw$ as a vector perpendicular to the vector w. If you are given the problem to prove the Cauchy-Schwarz inequality without looking up the proof, this is a very difficult task. It is difficult because it needs an idea. Getting ideas is what creativity is about.
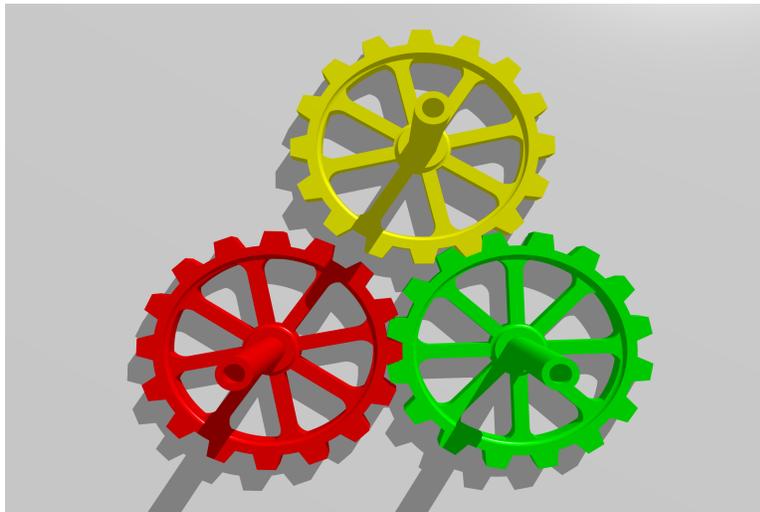
FIGURE 1. The mechanisms of creativity are not completely mysterious. There are different parts which work together like in a gear. We try to look at some. When writing the introduction to these notes, also the task came up to visualize creativity with a picture. Of course, one could google "creativity" and paste a picture in. But that is being "anti-creative". Copying an idea is not creative.

Seminar

12.2. In a bit more than a week we have to start thinking about our first midterm, let us organize the knowledge accumulated so far. We can do that in various ways. One technique is a **mind map**. It allows on one picture to organize a vast amount of content and see connections which might otherwise be missed. In Figure (2) we started to build such a mind map. There are lots of branches still missing, even main ones. One could start also with one entry like "matrix", put it in the center then build connections to other objects definitions or results.
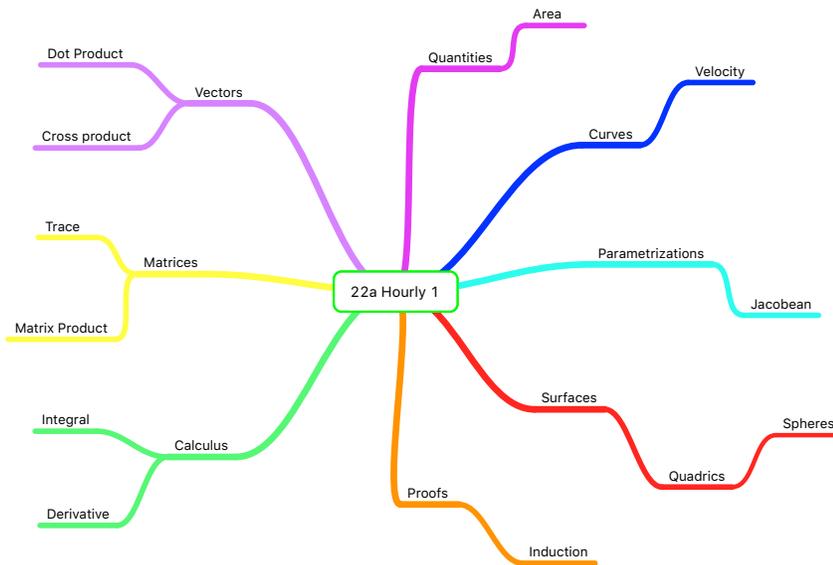
Figure 2.

12.3. What does this have to do with creativity? It turns out that in order to be creative, one has to have a fertile base of knowledge. You can not assemble new building blocks before possessing and understanding some already. In order to prove the point that knowledge is important, one can also look at computer science and especially the field artificial intelligence (AI). One of the great pioneers in AI, Marvin Minsky once wrote: "the best way to solve a problem is to know how to solve it". The modern paradigms in **machine learning** confirm that in order to train an AI entity, one has to feed in a lot of knowledge to work with. New models come then through data fitting, gradient decent methods or more sophisticated algorithms. [1]

> **Problem A:** Make a mind map of the most important facts which have appeared in the course so far. Do it on paper, a blackboard, whiteboard or using software. Figure (2) makes a start. Refine it as much as possible.

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[1]See the Ahlfors lecture talk of 9/11/2018 by Sanjeev Arora, now on Youtube

12.4. To illustrate how difficult it can be to get a new solution, try the following problem. Of course, if you know the answer or have seen it already, it can be easy. If you have never seen it, it can be very hard. It is important that you try to find the solution for at least a half an hour even if you should not be successful.

> **Problem B:** Given 6 sticks of the same length 1, arrange them so that you get 4 equilateral triangles of side length 1.

12.5. Finding proofs of theorems needs creativity. Creativity is neither "God-given" nor inherited; it can be be trained like everything else. To back this claim up, we refer to a scientist who has **demonstrated creativity** by discovering new things which nobody else has thought about before. It is the Swiss scientist **Fritz Zwicky** who taught at Caltech and wrote a book "Everybody a genius". Why does Zwicky have "street cred"? Well, he was not only extraordinarily creative, he also developed and communicated creativity techniques **that work** and have been used since both in industry and academia.
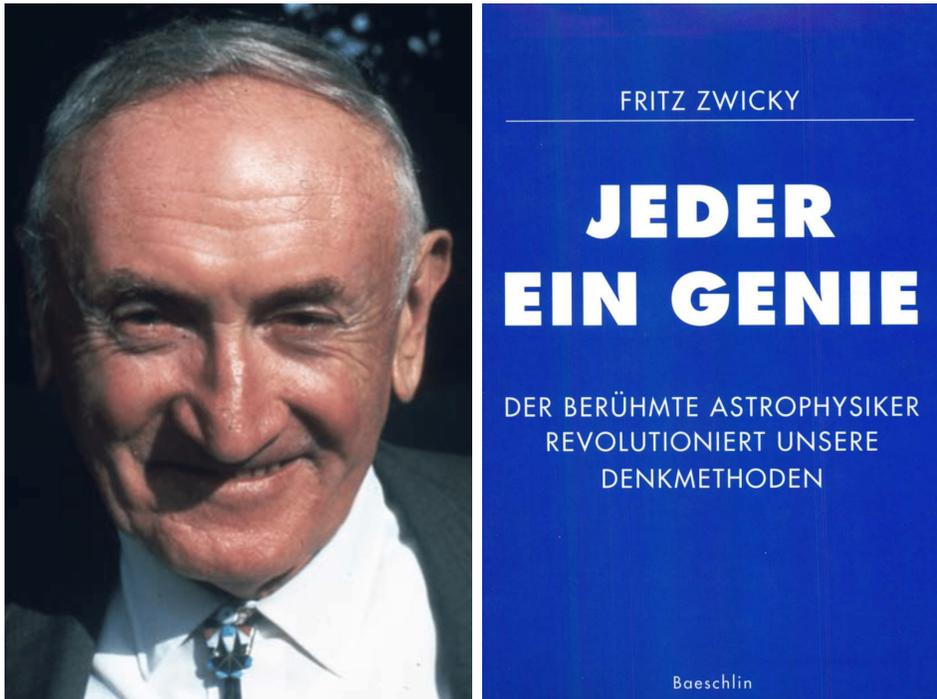
FIGURE 3. Fritz Zwicky at the International Astronomical Union meeting in Brighton, England, in 1970. Image credit: AIP Emilio Segre Visual Archives, John Irwin Slide Collection. Book: Fritz Zwicky, "Jeder ein Genie" (everybody genius), Lang and Lang, 1971.

12.6. First to the credentials: Fritz Zwicky proposed the existence of **dark matter**, **supernovas** (together with Walter Baade), **neutron stars**, **galactic cosmic rays**, **gravitational lensing** by galaxies, and **galaxy clusters**. He was also a pioneer in rocket technology. He proposed **and realized** the first shot of a human produced object to go into outer space. Each of these achievements alone would merit to be in

the list of greatest astronomers of all time. Still, Zwicky is not that well known. Why? Maybe it has to do with the fact that Zwicky used to call his colleagues "spherical bastards". Why spherical? "Because they are bastards from whatever side you looked at them!" No wonder he was not that much admired ...

12.7. One of the techniques is the **morphological box**. It is very simple. Produce a matrix in which one has one type of objects, ideas or activities on one side and another type of objects, ideas or activities. Now, just go through the matrix and look for connections. Here is such a matrix:

	Earth	Moon	Sun
shoot			
dig			
travel			

12.8. Now look what Zwicky proposed: shoot onto the moon (he actually did that with used V2 rockets which had an actual gun on top. At the end of the burn the gun was fired, the bullet would travel to space), he proposed travel by large scale digging through the earth (this is now realized by a company formed by Elon Musk) travel with the sun (the proposal was to travel to a nearby star by moving the entire solar system).

12.9. The matrix entry "dig sun" might come in when realizing Zwicky's space travel idea. We might have to target part of the sun differently to trigger asymmetric burn and so a travel. By the way there is an entire field of engineering, "macro-engineering". In 1997, I suggested in an essay (to the occasion of the 100th birthday of Zwicky) to implement Zwicky's idea by deliberate triggering of asymmetric fusion and fission in the Sun. This is mentioned in a macro-engineering book. [2]

12.10. Here is a beautiful problem assigned in a course Math 101 taught a couple of years ago taught by Sebastien Vasey. Borrowing a problem from another course does not make much of the point for creativity: but the problem is too beautiful to be missed. It is an example of an induction proof which needs some creativity. Try to solve it.

> **Problem C:** You have bathroom tiles which have three squares arranged in an L shape. Prove that you can cover a square shaped bath room floor of length and width 2^n with such tiles such that one square is left empty.

> **Problem D:** Martin Gardner wrote many books with puzzles. One of them is "The mathematical magic show" (1977). On the book cover of the German edition (1988), there is a famous puzzle: you have a cherry in a glass built by 4 matches. Move two of the four matches to get the cherry out of the glass. The glass should have the same shape as before. You are not allowed to move the cherry. Solve the cherry puzzle.

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[2]V. Badescu, R.B. Cathcart, R.D. Schuiling, Macro-Engineering, Springer, 2006

FIGURE 4. The German edition of "mathematical magic show".

In all the following question, **creativity is key**. Your object has to be original. It is ok to modify a known object. And of course, use technology so that one can admire your creation.

Problem 12.1 Be creative and generate your own parametrized curve. If you like it enough, you are allowed to name it with your name.

Problem 12.2 Be creative and generate your own parametric surface. Again, if the surface is creative enough and the surface is indeed new, you deserve to have the surface named after you.

Problem 12.3 Be creative and generate a level surface $f(x, y, z) = c$. Also here, try to get something which has never been seen.

Problem 12.4 Be creative and generate your own coordinate system $\mathbb{R}^2$.

Problem 12.5 a) Write a first hourly! b) take it! c) grade it!

Remark: According to the Apocrypha of Krantz (page 79), part a) and b) were once given as an algebraic geometry exam given here at Harvard. It is rumored that this was then used also at the Harvard philosophy department, where (and this is creative too), part c) was added. As far as we know, giving the homework assignment of writing an exam assignment is a first! Heureka! We were creative.

Unit 13: Partial differential equations

INTRODUCTION

13.1. If we can relate the changes in one quantity with changes in an other quantity, **partial differential equations** come in. One of the simplest rules is that the rate of change of a function $f(t, x)$ in time is related to the rate of change in space. Such a rule could be expressed for example as a rule $f_t(t, x) = f_x(t, x)$, where f_t is the partial derivative with respect to t and f_x is the partial derivative with respect to x. You can check that $f(t, x) = \sin(t+x)$ is an example of a function which satisfies this differential equation. You can see even that for any function g, the function $f(t, x) = g(t + x)$ satisfies $f_t = f_x$. A typical situation is to be given $f(0, x)$, the situation of "now". We then can see what $f(t, x)$ is for a **later time** t. This describes the situation in the future. As you see, the differential equation $f_t = f_x$ describes "transport". The initial situation is translated to the left. Check this out and draw for example $f(0, x) = x^2$. We see that $f(t, x) = (x + t)^2$ and especially $f(1, x) = (x + 1)^2$. The graph has moved to the left.
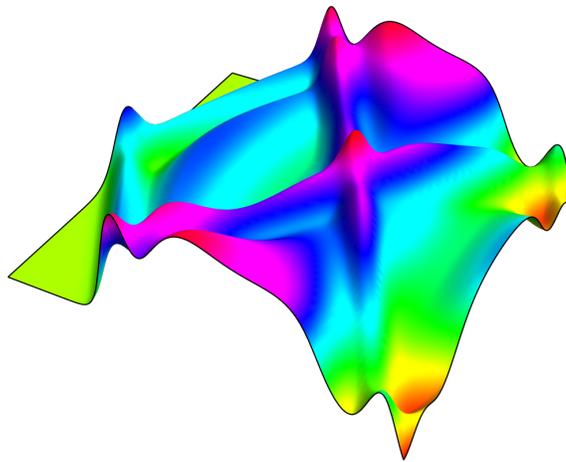

FIGURE 1. A function $f(t, x)$ satisfying a differential equation $f_{tt} - f_{xx} = \sin(u)$. This PDE is called $\mathrm{Sin} - Gordon$ equation, a nonlinear wave equation featuring **solitons**. Space is here one dimensional time goes from left to right. We see a wave going left and right, reflecting at the boundary and building up to a larger peak. A "rogue wave".

Lecture

13.2. A **partial differential equation** is a rule which combines the rates of changes of different variables. Our lives are affected by partial differential equations: the **Maxwell equations** describe electric and magnetic fields E and B. Their motion leads to the propagation of light. The **Einstein field equations** relate the metric tensor g with the mass tensor T. The **Schrödinger equation** tells how quantum particles move. Laws like the **Navier-Stokes equations** govern the motion of fluids and gases and especially the currents in the ocean or the winds in the atmosphere. Partial differential equations appear also in unexpected places like in finance, where for example, the **Black-Scholes equation** relates the prices of options in dependence of time and stock prices.

13.3. If $f(x, y)$ is a function of two variables, we can differentiate f with respect to both x or y. We just write $f_x(x, y)$ for $\partial_x f(x, y)$. For example, for $f(x, y) = x^3 y + y^2$, we have $f_x(x, y) = 3x^2 y$ and $f_y(x, y) = x^3 + 2y$. If we first differentiate with respect to x and then with respect to y, we write $f_{xy}(x, y)$. If we differentiate twice with respect to y, we write $f_{yy}(x, y)$. An equation for an unknown function f for which partial derivatives with respect to at least two different variables appear is called a **partial differential equation** PDE. If only the derivative with respect to one variable appears, one speaks of an **ordinary differential equation** ODE. An example of a PDE is $f_x^2 + f_y^2 = f_{xx} + f_{yy}$, an example of an ODE is $f'' = f^2 - f'$. It is important to realize that it is a function we are looking for, not a number. The ordinary differential equation $f' = 3f$ for example is solved by the functions $f(t) = Ce^{3t}$. If we prescribe an initial value like $f(0) = 7$, then there is a unique solution $f(t) = 7e^{3t}$. The **KdV partial differential equation** $f_t + 6ff_x + f_{xxx} = 0$ is solved by (you guessed it) $2\mathrm{sech}^2(x - 4t)$. This is one of many solutions. In that case they are called **solitons**, nonlinear waves. Korteweg-de Vries (KdV) is an icon in a mathematical field called **integrable systems** which leads to insight in ongoing research like about **rogue waves** in the ocean.

13.4. We say $f \in C^1(\mathbb{R}^2)$ if both f_x and f_y are continuous functions of two variables and $f \in C^2(\mathbb{R}^2)$ if all f_{xx}, f_{yy}, f_{xy} and f_{yx} are continuous functions. The next theorem is called the **Clairaut theorem**. It deals with the partial differential equation $f_{xy} = f_{yx}$. The proof demonstrates the **proof by contradiction**. We will look at this technique a bit more in the proof seminar.

> **Theorem:** Every $f \in C^2$ solves the Clairaut equation $f_{xy} = f_{yx}$.

13.5. Proof. We use **Fubini's theorem** which will appear later in the double integral lecture: integrate $\int_{x_0}^{x_0+h} (\int_{y_0}^{y_0+h} f_{xy}(x, y) \, dy) dx$ by applying the **fundamental theorem of calculus** twice $\int_{x_0}^{x_0+h} f_x(x, y_0 + h) - f_x(x, y_0) \, dx = f(x_0 + h, y_0 + h) - f(x_0, y_0 + h) - f(x_0 + h, y_0) + f(x_0, y_0)$. An analogous computation gives $\int_{y_0}^{y_0+h} (\int_{x_0}^{x_0+h} f_{yx}(x, y) \, dx) dy = f(x_0 + h, y_0 + h) - f(x_0, y_0 + h) - f(x_0 + h, y_0) + f(x_0, y_0)$. Fubini applied to $g(x, y) = f_{xy}(x, y)$ assures $\int_{y_0}^{y_0+h} (\int_{x_0}^{x_0+h} f_{yx}(x, y) \, dx) dy = \int_{x_0}^{x_0+h} (\int_{y_0}^{y_0+h} f_{yx}(x, y) \, dy) dx$ so that $\int \int_A f_{xy} - f_{yx} \, dy dx = 0$. $\boxed{\text{Assume}}$ there is some (x_0, y_0), where $F(x_0, y_0) = f_{xy}(x_0, y_0) - f_{yx}(x_0, y_0) = c > 0$, then also for small h, the function F is bigger than $c/2$ everywhere on $A = [x_0, x_0 + h] \times [y_0, y_0 + h]$ so that $\int \int_A F(x, y) \, dx dy \geq \text{area}(A)c/2 = h^2 c/2 > 0$ $\boxed{\text{contradicting}}$ that the integral is zero.

13.6. The statement is false for functions which are only C^1. The standard counter example is $f(x,y) = 4xy(y^2 - x^2)/(x^2 + y^2)$ which has for $y \neq 0$ the property that $f_x(0,y) = 4y$ and for $x \neq 0$ has the property that $f_y(x,0) = -4x$. You can see the comparison of $f(x,y) = 2xy = r^2 \sin(2\theta)$ and $f(x,y) = 4xy(y^2 - x^2)/(x^2 + y^2) = r^2 \sin(4\theta)$. The later function is not in C^2. The values f_{xy} and f_{yx}, changes of slopes of tangent lines, turn differently.
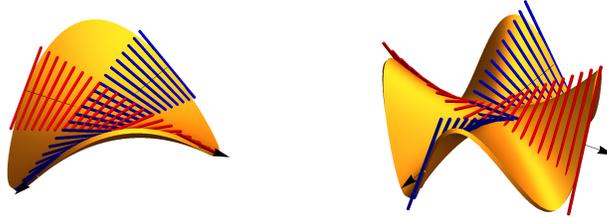
FIGURE 2. Clairaut holds for $f(x,y) = 2xy$ which is in polar coordinates $r^2 \sin(2\theta)$. It does not for the function $f(x,y) = 4xy(y^2 - x^2)/(x^2 + y^2)$ which is in polar coordinates $2r^2 \sin(2\theta)\cos(2\theta) = r^2 \sin(4\theta)$.

ILLUSTRATION

13.7. In many cases, one of the variables is **time** for which we use the letter t and keep x as the **space variable**. The differential equation $f_t(t,x) = f_x(t,x)$ is called the **transport equation**. What are the solutions if $f(0,x) = g(x)$? Here is a cool derivation: if $Df = f'$ is the derivative, [1] we can build operators like $(D + D^2 + 4D^4)f = f' + f'' + 4f''''$. The transport equation is now $f_t = Df$. Now as you know from calculus, the only solution of $f' = af$, $f(0) = b$ is be^{at}. If we boldly replace the number a with with the operator D we get $f' = Df$ and get its solution

$$e^{Dt}g(x) = (1 + Dt + D^2t^2/2! + \cdots)g(x) = g(x) + g'(x)t + g''(x)t^2/2! + \cdots.$$

By the **Taylor formula**, this is equal to $g(x+t)$. You should actually remember Taylor as $\boxed{g(x+t) = e^{Dt}g(x)}$. We have derived for $g(x) = f(0,x)$ in $C^1(\mathbb{R}^2)$:

> **Theorem:** $f_t = f_x$ is solved by $f(t,x) = g(x+t)$.

Proof. We can ignore the derivation and verify this very quickly: the function satisfies $f(0,x) = g(x)$ and $f_t(t,x) = f_x(t,x)$. QED.

13.8. Another example of a partial differential equation is the **wave equation** $f_{tt} = f_{xx}$. We can write this $(\partial_t + D)(\partial_t - D)f = 0$. One way to solve this is by looking at $(\partial_t - D)f = 0$. This means transport $f_t = f_x$ and $f(t,x) = f(x+t)$. We can also have $(\partial_t + D)f = 0$ which means $f_t = -f_x$ leading to $f(x-t)$. We see that every combination $af(x+t) + bf(x-t)$ with constants a, b is a solution. Fixing the constants a, b so that $f(x,0) = g(x)$ and $f_t(x,0) = h(x)$ gives the following **d'Alembert solution**. It requires $g, h \in C^2(\mathbb{R})$.

---

[1] We usually write df for derivative but D tells it is an operator. D also stands for Dirac.

Theorem: $f_{tt} = f_{xx}$ is solved by $f(t, x) = \frac{g(x+t)+g(x-t)}{2} + \frac{h(x+t)-h(x-t)}{2}$.

13.9. Proof. Just verify directly that this indeed is a solution and that $f(0, x) = g(x)$ and $f_t(0, x) = h(x)$. Intuitively, if we throw a stone into a narrow water way, then the waves move to both sides.

13.10. The partial differential equation $f_t = f_{xx}$ is called the **heat equation**. Its solution involves the **normal distribution**

$$N(m, s)(x) = e^{-(x-m)^2/(2s^2)}/\sqrt{2\pi s^2}$$

in probability theory. The number m is the **average** and s is the **standard deviation**.

13.11. If the initial heat $g(x) = f(0, x)$ at time $t = 0$ is continuous and zero outside a bounded interval $[a, b]$, then

Theorem: $f_t = f_{xx}$ is solved by $f(t, x) = \int_a^b g(m)N(m, \sqrt{2t})(x)\, dm$.

Proof. For every fixed m, the function $N(m, \sqrt{2t})(x)$ solves the heat equation.

```
f=PDF[NormalDistribution[m,Sqrt[2 t]],x];  Simplify[D[f,t]==D[f,{x, 2}]]
```

Every Riemann sum approximation $g(x) = (1/n)\sum_{k=1}^n g(m_k)$ of g defines a function $f_n(t, x) = (1/n)\sum_{k=1}^n g(m_k)N(m_k, \sqrt{2t})(x)$ which solves the heat equation. So does $f(t, x) = \lim_{n\to\infty} f_n(t, x)$. To check $f(0, x) = g(x)$ which need $\int_{-\infty}^\infty N(m, s)(x)\, dx = 1$ and $\int_{-\infty}^\infty h(x)N(m, s)(x)\, dx \to h(m)$ for any continuous h and $s \to 0$, proven later.

13.12. For functions of three variables $f(x, y, z)$ one can look at the partial differential equation $\Delta f(x, y, z) = f_{xx} + f_{yy} + f_{zz} = 0$. It is called the **Laplace** equation and Δ is called the **Laplace operator**. The operator appears also in one of the most important partial differential equations, the **Schrödinger equation**

$$i\hbar f_t = Hf = -\frac{\hbar^2}{2m}\Delta f + V(x)f ,$$

where $\hbar = h/(2\pi)$ is a scaled **Planck constant** and $V(x)$ is the potential depending on the position x and m is the mass. For $i\hbar f_t = Pf$ with $P = -i\hbar D$, then the solution $f(x - t)$ is forward translation. The operator P is the **momentum operator** in quantum mechanics. The Taylor formula tells that P generates translation.

Problem 13.1: Verify that for any constant b, the function
$$f(x,t) = e^{-bt}\sin(x+t)$$
satisfies the driven transport equation
$$f_t(x,t) = f_x(x,t) - bf(x,t) \ .$$
This PDE is sometimes called the **advection equation** with damping b.

Problem 13.2: We have seen in class that $f(t,x) = e^{-x^2/(4t)}/\sqrt{4\pi t}$ solves the heat equation $f_t = f_{xx}$. Verify more generally that
$$e^{-x^2/(4at)}/\sqrt{4a\pi t}$$
solves the **heat equation**
$$f_t = af_{xx} \ .$$

Problem 13.3: The **Eiconal equation** $f_x^2 + f_y^2 = 1$ is used in optics. Let $f(x,y)$ be the distance to the circle $x^2+y^2=1$. Show that it satisfies the eiconal equation. Remark: the equation can be written rewritten as $||df||^2 = 1$, where $df = \nabla f = [f_x, f_y]$ is the gradient of f which is the Jacobian matrix for the map $f : \mathbb{R}^2 \to \mathbb{R}$.

Problem 13.4: The differential equation
$$f_t = f - xf_x - x^2 f_{xx}$$
is a version of the **Black-Scholes equation**. Here $f(x,t)$ is the price of a **call option** and x is the stock price and t is time. Find a function $f(x,t)$ solving it which depends both on x and t. Hint: look first for solutions $f(x,t) = g(t)$ or $f(x,t) = h(x)$ and then for functions of the form $f(x,t) = g(t) + h(x)$.

Problem 13.5: The partial differential equation
$$f_t + ff_x = f_{xx}$$
is called **Burgers equation** and describes waves at the beach. In higher dimensions, it leads to the **Navier-Stokes** equation which is used to describe the weather. Verify that the function
$$f(t,x) = \frac{\left(\frac{1}{t}\right)^{3/2} xe^{-\frac{x^2}{4t}}}{\sqrt{\frac{1}{t}e^{-\frac{x^2}{4t}}+1}}$$
is a solution of the Burgers equation. You better use technology.

Unit 14: Keywords for First Hourly

This is a bit of a checklist. Make your own list. But here is a checklist which tries to be comprehensive. Check off the topics you know and check back with things you do not recall. You will need to have the following on your finger tips.

Theorems

- [ ] Cauchy-Schwarz $|v \cdot w| \leq |v||w|$ in general for $M(n, m)$
- [ ] Pythagoras $c^2 = a^2 + b^2$ for any inner product space
- [ ] Al Khashi $c^2 = a^2 + b^2 - 2ab\cos(\alpha)$ for any triangle
- [ ] Uniqueness of row reduction: $rref(A)$ is unique in $M(n, m)$
- [ ] The dot product formula $v \cdot w = |v||w|\cos(\alpha)$
- [ ] The cross product formula $|v \times w| = |v||w|\sin(\alpha)$
- [ ] Image of transpose $\text{im}(A^T)$ is kernel $\ker(A)$
- [ ] Cauchy-Binet formula $|v \times w|^2 = |v|^2|w|^2 - (v \cdot w)^2$
- [ ] Arc length $\int_a^b |r'(t)|\, dt$ for differentiable r
- [ ] Curvature formulas $|T'|/|r'| = |r' \times r''|/|r'|^3$
- [ ] Euler formula $e^{it} = \cos(t) + i\sin(t)$ and special case
- [ ] Distortion formula $\sqrt{\det(dr^T dr)} = |r_u \times r_v|$ for $r : \mathbb{R}^2 \to \mathbb{R}^3$

Proofs

- [ ] The use of precise definitions and notation
- [ ] Be able to argue by contradiction
- [ ] Think visually, make good pictures
- [ ] Use algebra to tackle geometric problems
- [ ] Master the method of induction
- [ ] Know the benefits and risks of intuition
- [ ] Be aware of computer assisted verification
- [ ] Believe in your creativity

Algorithms

- [ ] Find the angle between vectors or matrices
- [ ] Find the area of parallelogram
- [ ] Find the volume of parallelepiped
- [ ] Row reduce a matrix in $M(n, m)$
- [ ] Get position from velocity or acceleration
- [ ] Find the vector perpendicular to a plane

- [ ] Find the length of a curve or matrix
- [ ] Find the curvature at some point
- [ ] Compute with complex numbers
- [ ] Switch between coordinate systems
- [ ] Compute the distortion factor
- [ ] Get distances between objects

Objects

- [ ] Matrices A
- [ ] Column- and row vectors
- [ ] Parametrized curves $r(t) = [x(t), y(t), z(t)]^T$
- [ ] Parametrized surfaces $r(u, v) = [x(u, v), y(u, v), z(u, v)]^T$
- [ ] Functions $f(x, y, z)$.
- [ ] Level surfaces $f(x, y, z) = d$
- [ ] Linear manifolds $\{x | Ax = d\}$
- [ ] Quadratic manifolds $\{x | x^T Bx + Ax = d\}$
- [ ] Kernel of a linear map $\{x | Ax = 0\}$
- [ ] Image of a linear map $\{Ax | x \in \mathbb{R}^n\}$

Differentiation

- [ ] Velocity r'
- [ ] Acceleration r''
- [ ] Jerk r'''
- [ ] Free fall: $r'' = v$ given
- [ ] TNB frame, $T = r'/|r'|, N = T'/|T'|, B = T \times N$
- [ ] derivative $dr \in M(n, m)$ of a map $\mathbb{R}^m \to \mathbb{R}^n$
- [ ] Jacobian matrix dr of a map $\mathbb{R}^n \to \mathbb{R}^n$
- [ ] Distortion factor $\sqrt{\det(dr^T dr)}$
- [ ] Distortion factor for $n = m$ simplifies to $|\det(dr)|$
- [ ] Example: $r'(t) = dr$, $\sqrt{\det(dr^T dr)} = |r'|$ is speed
- [ ] Curvature $|T'|/|r'|$. In $\mathbb{R}^3$ also $|r' \times r''|/|r'|^3$

Integration

- [ ] Integrate to get arc length.
- [ ] Integrate to get position from velocity etc.
- [ ] Integration technique: substitution
- [ ] Integration technique: integration by parts
- [ ] Integration technique: partial fractions
- [ ] Integration technique: simplification

Coordinate systems

- [ ] Cartesian coordinates
- [ ] Polar coordinates
- [ ] Cylindrical coordinates
- [ ] Spherical coordinates

General coordinate change

Distortion factor $|\det(dr)| = \sqrt{\det(dr^T dr)}$

Parametrized Surfaces

Spheres

Surfaces of revolution

Graphs

Planes

Torus

Helicoid

People

Mandelbrot

Hamilton

Descartes

Cauchy

Binet

Schwarz

Euler

Heine

Cantor

Bolzano

Archimedes

Newton

Einstein

Napoleon

Geometry of Space

$v = [v_1, v_2, v_3]^T, w = [w_1, w_2, w_3]^T, v + w = [v_1 + w_1, v_2 + w_2, v_3 + w_3]^T$

dot product $v.w = v_1 w_1 + v_2 w_2 + v_3 w_3 = |v||w| \cos(\alpha)$

angle $\cos(\alpha) = (v \cdot w)/|v||w|$.

cross product $v \cdot (v \times w) = 0$, $w \cdot (v \times w) = 0$

area parallelogram $|v \times w| = |v||w| \sin(\alpha)$

triple scalar product $u \cdot (v \times w)$

volume of parallelepiped: $|u \cdot (v \times w)|$

parallel vectors $v \times w = 0$, orthogonal vectors: $v \cdot w = 0$

scalar projection $\text{comp}_w(v) = v \cdot w/|w|$

vector projection $\text{proj}_w(v) = (v \cdot w)w/|w|^2$

completion of square: $x^2 - 4x + y^2 = 1$ gives $(x-2)^2 + y^2 = 5$

unit vector = direction: vector of length 1.

Lines, Planes, Functions

parametric equation for plane $r(t, s) = p + tv + sw$ containing p

plane $A^T[x, y, z] = ax + by + cz = d$

parametric equation for line $r(t) = p + tv$ containing p

- graph $G = \{(x, y, f(x, y)) \mid (x, y) \text{ in the domain of } f\}$
- plane $ax + by + cz = d$ has normal $n = [a, b, c]^T$
- line $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ contains $v = [a, b, c]^T$
- plane through A, B, C: find normal vector $(a, b, c) = AB \times CB$

Level surfaces

- intercepts: intersections of a surface with coordinate axis
- traces: intersections of a surface with coordinate planes
- generalized traces: intersections with $\{x = c\}, \{y = c\}$ or $\{z = c\}$
- level surface $g(x, y, z) = c$: Example: graph $g(x, y, z) = z - f(x, y)$
- linear equation like $2x + 3y + 5z = 7$ defines plane
- quadric: ellipsoid,paraboloid,hyperboloid,cylinder,cone

Distance formulas

- distance $d(P, Q) = |PQ| = \sqrt{(P_1 - Q_1)^2 + (P_2 - Q_2)^2 + (P_3 - Q_3)^2}$
- distance point-plane: $d(P, \Sigma) = |(PQ) \cdot n|/|n|$
- distance point-line: $d(P, L) = |(PQ) \times u|/|u|$
- distance line-line: $d(L, M) = |(PQ) \cdot (u \times v)|/|u \times v|$
- distance parallel lines L, M: distance point $d(P, M)$ where P is in L.
- distance parallel planes: $d(P, \Sigma)$ where P is in first plane.

Functions

- graph: $z = f(x, y)$
- contour curve: $f(x, y) = c$ is a curve in the plane
- contour map: draw curves $f(x, y) = c$ for various c.
- contour surface: $f(x, y, z) = c$ in space

Curves

- plane and space curves $r(t)$
- circle: $x^2 + y^2 = r^2$, $r(t) = [r\cos t, r\sin t]^T$.
- ellipse: $(x - x_0)^2/a^2 + (y - y_0)^2/b^2 = 1$, $r(t) = [x_0 + a\cos t, y_0 + b\sin t]^T$
- velocity $r'(t)$, acceleration $r''(t)$, $|r'(t)|$ speed
- unit tangent vector $T(t) = r'(t)/|r'(t)|$
- integration: get $r(t)$ from $r'(t)$ and $r(0)$ by integration.
- integration: get $r(t)$ from acceleration $r''(t)$ as well as $r'(0)$ and $r(0)$.
- $r'(t)$ is tangent to the curve at the point $r(t)$.
- $r(t) = [f(t)\cos(t), f(t)\sin(t)]^T$ polar curve to polar graph $r = f(\theta)$.
- $\int_a^b |r'(t)|\, dt$, arc length of parametrized curve.
- $N(t) = T'(t)/|T'(t)|$ normal vector, is perpendicular to $T(t)$.
- $B(t) = T(t) \times N(t)$ bi-normal vector, is perpendicular to T and N.
- $\kappa(t) = |T'(t)|/|r'(t)|$ curvature $= |r'(t) \times r''(t)|/|r'(t)|^3$.
- $\kappa(t)$ and arc length are independent of parametrization

Coordinates

☐ Cartesian coordinates (x, y, z)
☐ polar coordinates $(x, y) = (r\cos(\theta), r\sin(\theta))$, $r \geq 0$
☐ cylindrical coordinates $(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$, $r \geq 0$
☐ spherical coordinates $(x, y, z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi))$
☐ radius: $r = x^2 + y^2$ and spherical radius $\rho = x^2 + y^2 + z^2$.
☐ radius: important relation $r = \rho\sin(\phi)$
☐ Jacobian matrix
☐ Distortion factor

Surfaces

☐ $g(r, \theta) = 0$ polar curve, especially $r = f(\theta)$, polar graphs
☐ $r = f(z, \theta)$ cylindrical surface, $r = r(z)$ surface of revolution
☐ $g(\rho, \theta, \phi) = 0$ spherical surface: example $\rho = 1$ sphere
☐ $f(x, y) = c$ level curves of $f(x, y)$
☐ plane: $ax + by + cz = d$, $r(s, t) = r_0 + sv + tw$, $[a, b, c]^T = v \times w$
☐ surface of revolution: $x^2 + y^2 = r(z)^2$, $r(\theta, z) = [r(z)\cos(\theta), r(z)\sin(\theta), z]^T$
☐ graph: $g(x, y, z) = z - f(x, y) = 0$, $r(x, y) = [x, y, f(x, y)]^T$
☐ rotated graph $g(x, y, z) = y - f(x, z) = 0$, $r(x, z) = [x, f(x, z), z]^T$
☐ ellipsoid: $r(\theta, \phi) = [a\cos\theta\sin\phi, b\sin\theta\sin\phi, c\cos\phi]^T$
☐ unit sphere: $x^2 + y^2 + z^2 = 1$, $r(u, v) = [\cos u\sin v, \sin u\sin v, \cos v]^T$

Oliver Knill, 2/22/22 last update: 2/25/22

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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total:

Unit 14: First Hourly (Practice A)

PROBLEMS

Problem 14A.1 (10 points):
The **Fibonacci numbers** are defined recursively as follows: start with $F_0 = 0, F_1 = 1$ then define $F_{n+1} = F_n + F_{n-1}$, so that $F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5$ etc. Prove that

$$F_0 + F_1 + \cdots + F_n = F_{n+2} - 1$$

for every positive integer n.

Problem 14A.2 (10 points):
Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

a) (4 points) Compute AB and $\mathrm{rref}(AB)$.
b) (4 points) Now row reduce both A and B and form $\mathrm{rref}(A)\mathrm{rref}(B)$.
c) (2 points) Is the statement $\mathrm{rref}(AB) = \mathrm{rref}(A)\mathrm{rref}(B)$ true for all A, B?

Problem 14A.3 (10 points):
a) (2 points) Parametrize the line through $(1, 1, 1)$ and $(4, 3, 1)$ in $\mathbb{R}^3$.
b) (2 points) Parametrize the ellipse $x^2/16 + y^2/25 = 1$ in $\mathbb{R}^2$.
c) (2 points) Parametrize the graph $y = x^5 + x$ in $\mathbb{R}^2$.
d) (2 points) Parametrize the circle $x^2 + (y - 2)^2 = 1, z = 4$ in $\mathbb{R}^3$.
e) (2 points) Parametrize the line $x = y = z$ in $\mathbb{R}^3$.

Problem 14A.4 (10 points):
Find the arc length of the curve

$$r(t) = [t \cos(t^2), t \sin(t^2), t^2]$$

for $0 \le t \le 2$.

Problem 14A.5 (10 points):

a) (2 points) What is the Heine-Cantor theorem?
b) (2 points) Formulate the triangle inequality.
c) (2 points) What is the Al Kashi identity?
d) (2 points) Give the name of a nowhere differentiable function.
e) (2 points) Is it true that a continuous curve $r(t)$ has a finite arc length?

Problem 14A.6 (10 points):

a) (2 points) Find $(3+i)(4+2i)$
b) (2 points) What is $e^{i3\pi/4}$?
c) (2 points) Convert from cylindrical $(r, \theta, z) = (2, \pi/2, 1)$ to Cartesian.
d) (2 points) What are the spherical coordinates of $(1, \sqrt{3}, 2)$?
e) (2 points) What surface is in spherical coordinates given as $\rho \sin(\phi) = 1$?

Problem 14A.7 (10 points):

a) (5 points) You are given $r'''(t) = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ and $r(0) = (7, 8, 9)$ and $r'(0) = (1, 0, 0)$ and $r''(0) = (0, 1, 0)$. Find $r(1)$.

b) (5 points) What is the curvature of $r(t) = [t, t+t^2, t+t^2+t^3]$ at $t = 0$?

Problem 14A.8 (10 points):

a) (5 points) Find a parametrization $r(u, v)$ of the cylinder $x^2 + z^2 = 9$.
b) (5 points) Find $r(u, v)$ for the paraboloid $y^2 + 3z^2 = x$.

Problem 14A.9 (10 points):

Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$. a) (2 points) The image of A is a plane. By using the cross product, write it as $ax + by + cz = d$.

b) (2 points) What is the first fundamental form $g = A^T A$?
c) (2 points) From a) you have $[a, b, c]^T = v \times w$. Find $\sqrt{a^2 + b^2 + c^2}$.
d) (2 points) Find the distortion factor $||A|| = \sqrt{\det(A^T A)}$ of A.
e) (2 points) What theorem was involved to see $||A|| = |v \times w|$?

Problem 14A.10 (10 points):

a) (5 points) What is the Jacobian matrix df of the map
$$f(x, y, z) = [x^2 + y^2 + z^2, x + y, -x^2]^T ?$$
b) (5 points) Find the distortion factor $\det(df)$.

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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total :

Unit 14: Hourly 1 (Practice B)

Problem 14B.1 (10 points):
Prove that $1 + 2 + 4 + 8 + \cdots + 2^n = 2^{n+1} - 1$ for every positive integer n.

Problem 14B.2 (10 points):

a) (5 points) Row reduce the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix}$.

b) (5 points) Compute the matrix product $\begin{bmatrix} 3 & 4 & 5 \end{bmatrix} A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Problem 14B.3 (10 points):
a) (2 points) Parametrize the curve $x = \sin(y)$ in $\mathbb{R}^2$.
b) (2 points) Parametrize the curve $r = \sin^2(5\theta)$ in $\mathbb{R}^2$.
c) (2 points) Parametrize the curve $y = x^5 + x, z = 4$ in $\mathbb{R}^3$.
d) (2 points) Parametrize the line $2x + y = 4$ in $\mathbb{R}^2$.
e) (2 points) Parametrize the ellipse $(x - 1)^2 + \frac{y^2}{4} = 1$ in $\mathbb{R}^2$.

Problem 14B.4 (10 points):
Find the arc length of the curve
$$r(t) = \begin{bmatrix} e^t \\ e^{-t} \\ \sqrt{2}t \end{bmatrix}$$
for $0 \leq t \leq 1$.

Problem 14B.5 (10 points):
a) (2 points) Formulate the Cauchy-Schwarz inequality.
b) (2 points) What formula gives the area of the parallelogram spanned by two vectors v and w?
c) (2 points) What formula gives the volume of a parallelepiped spanned by three vectors u, v, w?
d) (2 points) Who invented the quaternions?
e) (2 points) Assume $\text{rref}(A) = \text{rref}(B)$. Does this mean $A = B$?

Problem 14B.6 (10 points):
a) (2 points) Write the complex number $z = e^{-i\pi/2}$ in the form $z = a + ib$.
b) (2 points) Which point (x, y, z) has the cylindrical coordinates $(r, \theta, z) = (1, \pi/2, 0)$?
c) (2 points) What are the spherical coordinates (ρ, ϕ, θ) of the point $(x, y, z) = (\sqrt{2}, \sqrt{2}, -2)$?
d) (2 points) What surface is $\rho \sin^2(\phi) = \cos(\phi)$? Give the name and write it in Cartesian coordinates
e) (2 points) What surface is given in cylindrical coordinates by the equation $r \sin(\theta) = 2$?

Problem 14B.7 (10 points):

a) (5 points) You are given $r''(t) = \begin{bmatrix} 0 \\ 3 \\ t \end{bmatrix}$ and $r(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $r'(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Find $r(1)$.

b) (5 points) What is the curvature of $r(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$ at $t = 0$?

Problem 14B.8 (10 points):
a) (2 points) Find a parametrization of the cone $x^2 + y^2 = z^2$.
b) (2 points) Find a parametrization of $x^2/4 + y^2/9 + z^2/16 = 1$.
c) (2 points) Find a parametrization of the surface $x^2 - y^2 = z$.
d) (2 points) Find a parametrization of the plane $z = 2$.
e) (2 points) Find a parametrization of the cylinder $x^2 + z^2 = 1$.

Problem 14B.9 (10 points):
a) (5 points) Find the dot product $A \cdot B = \text{tr}(A^T B)$ between the two matrices

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, $$

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

b) (5 points) Find the cosine of the angle between these two matrices.

Problem 14B.10 (10 points):
a) (5 points) What is the Jacobian matrix df of the coordinate change

$$f(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2x - y + \sin(x) \\ x \end{bmatrix}.$$

b) (5 points) What is the distortion factor $\det(df)$ of the map f which by the way is called the **Chirikov map**.

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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total :

Unit 14: Hourly 1

PROBLEMS

Problem 14.1 (10 points):
Prove by induction that that for every $n \geq 1$ the formula $2 \sum_{k=0}^{n-1} 3^k = 3^n - 1$ holds.

Problem 14.2 (10 points):

a) (5 points) Row reduce the matrix $A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 4 & 4 & 4 & 4 & 4 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}$ using basic row reduction steps.

b) (5 points) For $B = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ compute either AB or BA depending on which of the two makes sense.

Problem 14.3 (10 points):
a) (2 points) Parametrize the curve $4x^2 + y^2 = 1$ in $\mathbb{R}^2$.
b) (2 points) Parametrize the curve $y - e^x = 0$ in $\mathbb{R}^2$.
c) (2 points) Parametrize the curve $x = y^3, z = 4$ in $\mathbb{R}^3$.
d) (2 points) Parametrize the line $x + y = 4, z = 2$ in $\mathbb{R}^3$.
e) (2 points) Parametrize the circle $x^2 + y^2 + z^2 = 4, z = 1$ in $\mathbb{R}^3$.

Problem 14.4 (10 points):
a) (8 points) Compute arc length of $r(t) = \begin{bmatrix} \frac{t^3}{3}, \sqrt{2}\frac{t^4}{4}, \frac{t^5}{5} \end{bmatrix}$ for $0 \leq t \leq 1$.
b) (2 points) Without doing any calculation, what is the arc length of the new parametrization $r(t^3)$ with $0 \leq t \leq 1$.

Problem 14.5 (10 points):
a) (2 points) Formulate the Al Khashi formula.
b) (2 points) We have seen a theorem of Heine- .... Fill in the second name!
c) (2 points) The linear space $\{x, Ax = 0\}$ is also called the ...... of A.
d) (2 points) Give the Euler's formula $e^{it} = \ldots \ldots$ and deduce the "most beautiful formula in math".
e) (2 points) Is $\mathrm{rref}(A) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ row reduced?

Problem 14.6 (10 points):
a) (2 points) Express $z = e^{i\pi/2} + 3e^{i\pi}$ in the form $z = a + ib$.
b) (2 points) Write $(r, \theta, z) = (2, -\pi/2, 0)$ in Cartesian coordinates.
c) (2 points) Write $(x, y, z) = (2, 2, 0)$ in spherical coordinates (ρ, ϕ, θ).
d) (2 points) Write the surface $\rho \cos(\phi) = 2$ in Cartesian coordinates.
e) (2 points) Write the surface $r \cos(\theta) = 2$ in Cartesian coordinates.

Problem 14.7 (10 points):

a) (5 points) You are given $r''(t) = \begin{bmatrix} 0 \\ 1 \\ \cos(t) \end{bmatrix}$ and $r(0) = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ and

$r'(0) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Find $r(1)$.

b) (2 points) Is there a time t such that the curve $r(t)$ ever reaches the ground $z = 0$?

c) (3 points) What is the curvature of $r(t) = \begin{bmatrix} t^2 \\ \cos(t) \\ \sin(t) \end{bmatrix}$ at $t = 0$?

Problem 14.8 (10 points):
We parametrize some surfaces. Chose the parameters on your own.
a) (2 points) Find a parametrization of the hyperboloid $x^2 + y^2 - z^2 = 1$.
b) (2 points) Find a parametrization of the cylinder $(x-1)^2/4 + y^2/9 = 1$.
c) (2 points) Find a parametrization of the surface $z = \cos(xy)$.
d) (2 points) Find a parametrization of the plane $x + y - 3z = 1$.
e) (2 points) Find a parametrization of the cylinder $x^2/9 + (y-2)^2 = 1$.

Problem 14.9 (10 points):
a) (4 points) Compute the dot product (inner product) $A \cdot B = \text{tr}(A^T B)$ of the two matrices

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 3 \end{bmatrix}.$$

b) (4 points) Now determine the cosine of the angle between A and B.
c) (2 points) Finally find the distance $|A - B|$ between A and B.

Problem 14.10 (10 points):
a) (4 points) What is the Jacobian matrix dr of the coordinate change

$$r(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 4x + y \\ y^2 \end{bmatrix} ?$$

b) (2 points) Now find the first fundamental form $g = dr^T dr$.
c) (2 points) Compute the distortion factor $|\det(dr)|$.
d) (2 points) Check in this case that $|\det(dr)| = \sqrt{\det(g)}$.

Unit 15: Contradiction and Deformation

Introduction

15.1. One of the most common fallacies which are done in logical argumentation is to reverse an implication. If A implies B, then B implies A. Right? If you are an idiot, you do stupid things. So, if you do stupid things, you are an idiot. This is not true. The implication $A \Rightarrow B$ does not imply $B \Rightarrow A$, but it implies $\neg B \implies \neg A$. This is called **contradiction**. We write $\neg A$ for the negation of A. Related to contradiction is the method of "Reductio ad absurdum". To prove a statement B from some statements A, we can assume that B is false and deduce from this that A is false.

FIGURE 1. Reductio ad absurdum. Picture by the Scottish painter John Pettie (1839-1993). This picture is featured on the Wikipedia page about Reductio ad absurdum.

15.2. Here is an example: Let A be the statement "It rains". And let B be the statement "The street is wet". Obviously A implies B. But B does not imply A. It could be that the street is wet from a rain which stopped earlier or that somebody was cleaning the street. But we can conclude: if the street is not wet, then it does not rain. The statement $A \implies B$ indeed is equivalent to $\neg B \implies \neg A$.

15.3. Geoffrey Hardy describes as follows: *"The proof is by reductio ad absurdum, and reductio ad absurdum, which Euclid loved so much, is one of a mathematician's finest weapons".* But every mathematician who has done proofs knows about the pitfalls. Here is a well formulated statement by Henry Cohn from MIT *"Unfortunately, this proof technique can really cause problems for beginners. Typically, what happens is that the proof starts off quite reasonably, and then gets lost in a maze of complexity. Somewhere in the confusion, a mistake is made, which leads to a contradiction. Then it looks like the proof is done, but unfortunately the contradiction has nothing to do with the initial assumption, and comes solely from the mistake in the middle."*

SEMINAR

15.4. We have already seen one proof technique, the **"method of induction."** Other proofs were done either by **direct computations** or by **combining already known theorems or inequalities**. Today, we look at two new and fundamentally different proof techniques. The first is the method **"by contradiction."** The second method is the **"method of deformation."** Both methods are illustrated by a theorem.

15.5. The first theorem is one of the earliest results in mathematics. It is the **Hypassus theorem** from 500 BC. It was a result which shocked the Pythagoreans so much that Hypassus got killed for its discovery. That is at least what the rumors tell.

> **Theorem:** The diagonal of a unit square has irrational length.

Proof. $\boxed{\text{Assume}}$ the statement is false and the diagonal has rational length p/q. Then by Pythagoras theorem $2 = p^2/q^2$ or $2q^2 = p^2$. By the fundamental theorem of arithmetic, the left hand side has an odd number of factors 2, the right hand side an even number. This is a $\boxed{\text{contradiction}}$. The assumption must have been wrong.

15.6.

> **Problem A:** Prove that the cube root of 2 is irrational.

15.7. Note that the proof relied on the fundamental theorem of arithmetic which assured that every integer has a unique prime factorization.

> **Problem B:** Figure (2) is a geometric proof by contradiction which does not need the fundamental theorem of arithmetic. Complete the proof.

1

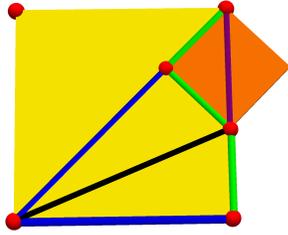[1]for more explanation, see https://www.youtube.com/watch?v=Ih16BIoR9eM

FIGURE 2. $\sqrt{2}$ is irrational. Start by assuming the side length and diagonal of the large yellow square are integers. Conclude that for the strictly smaller orange square, the side length and diagonal are integers.

15.8. Proofs by contradiction can be dangerous. A flawed proof can "$\boxed{\text{assume}}$ the contrary, mess around with arguments, make a mistake somewhere and get a $\boxed{\text{contradiction}}$. QED". Better than a proof by contradiction is a constructive proof.

15.9. Here is a non-constructive proof which is amazing:

> **Theorem:** There exist two irrational x, y such that x^y is rational.

Proof: there are two possibilities. Either $z = \sqrt{2}^{\sqrt{2}}$ is irrational or not. In the first case, we have found an example where $x = y = \sqrt{2}$. In the second case, take $x = z$ and take $y = \sqrt{2}$. Now $x^y = \sqrt{2}^2 = 2$ is rational and we have an example.

15.10. The second proof technique we see today is a **deformation argument**. To illustrate it, take a closed C^2 curve in $\mathbb{R}^2$ without self intersections. We have defined its curvature $\kappa(t)$ already. For curves in $\mathbb{R}^2$, define the **signed curvature** $K(t)$. If the curve parametrized so that $|r'(t)| = 1$ and $T(t) = [\cos(\alpha(t)), \sin(\alpha(t))]$, then $K(t) = \alpha'(t)$. Note that $\kappa(t) = |T'(t)| = |[-\sin(\alpha(t)), \cos(\alpha(t))]\alpha'(t)| = |K(t)|$. Now if we have a curve $r : [a, b] \to \mathbb{R}^2$, we can define the **total curvature** as $\int_a^b K(t)\,dt$. By the **fundamental theorem of calculus**, this total curvature is the change of the angle $\alpha(b) - \alpha(a)$. Now, if the curve is closed, the initial and final angles have to differ by a multiple of 2π. The **Hopf Umlaufsatz** tells that

> **Theorem:** The total curvature of a simple closed curve is 2π or -2π.
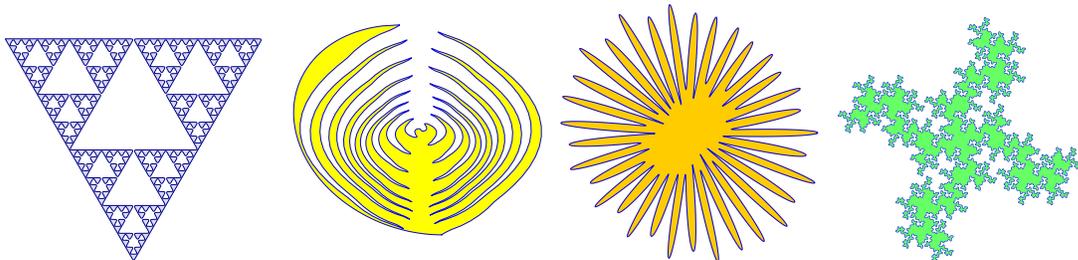

FIGURE 3. Four simple closed curves for which it is not obvious that the total curvature is 2π.

15.11.

Problem C: a) Why is the total curvature not always 2π?

b) Formulate out what happens in in Figure (4).

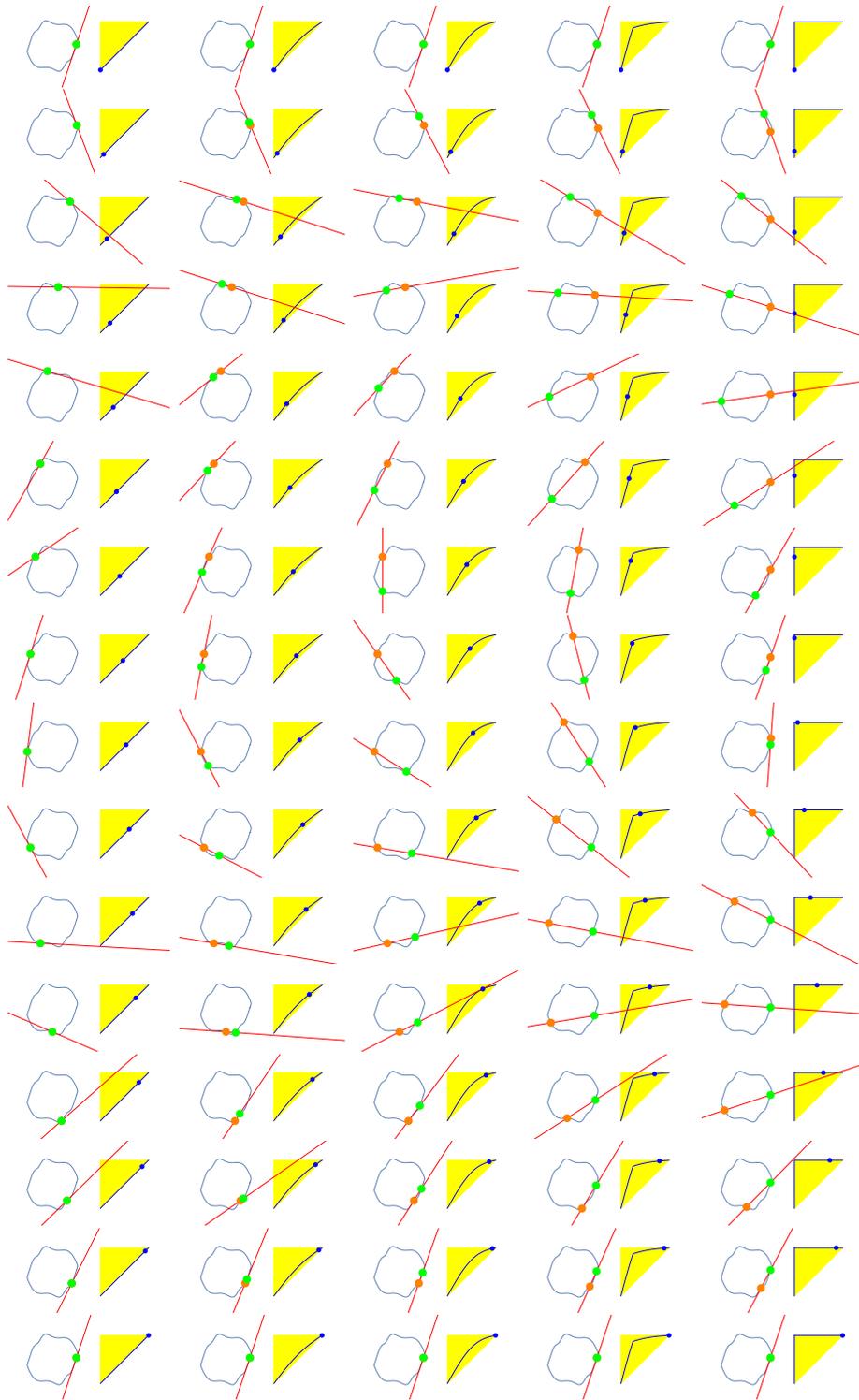FIGURE 4. Hopf's deformation proof: each picture shows the line through $r(s), r(t)$ and to the right the parameter (s, t). In the left column, where $s = t$, we deal with the tangent turning. We have to show it turns by 2π. The next columns deform the situation where the path through the parameter square is changed. In the very right column, we twice turn the segment by π, in total 2π.

Homework

Problem 15.1 Prove by contradiction that $\sqrt{12}$ is irrational.

Problem 15.2 Prove by contradiction that $\log_{10}(2)$ is irrational. $\log_{10}$ is the logarithm with respect to the base 10.

Problem 15.3 Prove by contradiction that there are infinitely many primes of the form $4k - 1$. Hint. If p_i are of the form $4k - 1$ then $4 \prod_j p_i - 1$ is again of the form $4k - 1$.

Problem 15.4 Verify the Hopf Umlaufsatz for a circle of radius 5, where $r(t) = \begin{bmatrix} 5\cos(t) \\ 5\sin(t) \end{bmatrix}$. Optional: what does the Umlaufsatz say for a triangle?
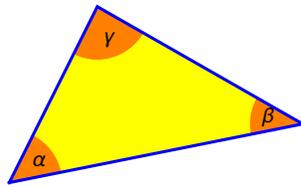
FIGURE 5. Can you adapt the Hopf Umlaufsatz for triangles?

Problem 15.5 There is a variant of proof by contradiction which is **proof by infinite descent**. It was used in proving a special case of **Fermat's Last theorem**. This special result tells that the equation $r^2 + s^4 = t^4$ has no solution with positive r, s, t. Look up and write down the proof of this theorem.

FIGURE 6. Pierre de Fermat: cropped from Foto by Didier Descouens: showing the Monument to Pierre de Fermat by Alexandre Falguière in Beaumont-de-Lomagne, Tarn-et-Garonne France.

Unit 16: Chain rule

INTRODUCTION

16.1. In calculus, we can build from basic functions more general functions. One possibility is to add functions like $f(x) + g(x) = x^2 + \sin(x)$. An other possibility is to multiply functions like $f(x)g(x) = x^2 \sin(x)$. A third possibility is to combine functions like $f \circ g(x) = f(g(x)) = \sin^2(x)$. The composition of functions is non-commutative: $f \circ g \neq g \circ f$. Indeed, we have $g \circ f(x) = \sin(x^2)$ which is completely different from $f \circ g(x) = \sin^2(x)$.

$$d(f(g(x))) = df(g(x)) \, dg(x)$$
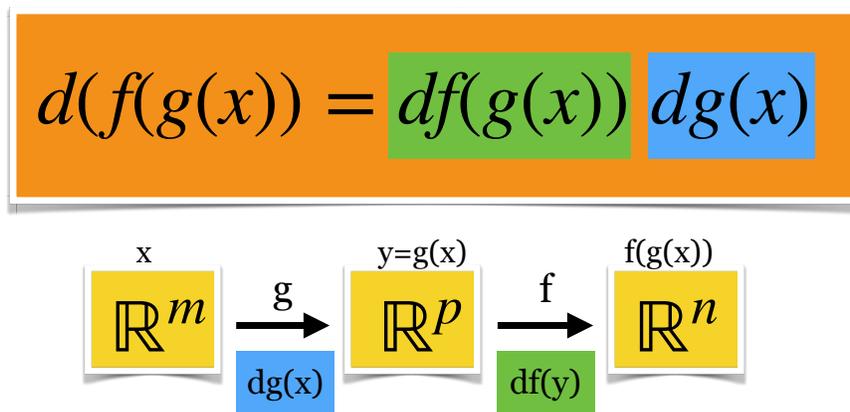
FIGURE 1. $f : \mathbb{R}^p \to \mathbb{R}^n$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ can be combined to $f(g) : \mathbb{R}^m \to \mathbb{R}^n$.

16.2. How can we express the rate of change of a composite function in terms of the basic functions it is built of? For the sum of two functions, we have the **addition rule** $(f + g)'(x) = f'(x) + g'(x)$, for multiplication we have the **product rule** $(fg)'(x) = f'(x)g(x) + f(x)g(x)$. We usually just write $(f + g)' = f' + g'$ or $(fg)' = f'g + fg'$ and do not always write the argument. As you know from single variable calculus, the derivative of the composite function is given by **chain rule**. This is $(f \circ g)' = f'(g)g'$. Written out in more details with argument, we can write

$\frac{d}{dx}f(g(x)) = \frac{d}{dx}f'(g(x))g'(x)$. We generalize this here to higher dimensions. Instead of $\frac{d}{dx}f$ we just write df. This is the Jacobean matrix we know. Now, the same rule holds as before $\boxed{df(g(x)) = df(g(x))dg(x)}$ and this is called the chain rule in higher dimensions. On the right hand side, we have the matrix product of two matrices.

16.3. Let us see why this makes sense in terms of dimensions: $g : \mathbb{R}^m \to \mathbb{R}^p$ and $f : \mathbb{R}^p \to \mathbb{R}^n$, then $dg(x) \in M(p,m)$ and $df(g(x)) \in M(n,p)$ and $df(g(x))dg(x) \in M(n,m)$ which is the same type of matrix than $d(f \circ g)$ because $f \circ g(x)$ maps $\mathbb{R}^m \to \mathbb{R}^n$ so that also $d(f \circ g)(x) \in M(n,m)$. The name **chain rule** comes because it deals with functions that are chained together.

<div align="center">LECTURE</div>

16.4. Given a differentiable function $r : \mathbb{R}^m \to \mathbb{R}^p$, its derivative at x is the Jacobian matrix $dr(x) \in M(p,m)$. If $f : \mathbb{R}^p \to \mathbb{R}^n$ is another function with $df(y) \in M(n,p)$, we can combine them and form $f \circ r(x) = f(r(x)) : \mathbb{R}^m \to \mathbb{R}^n$. The matrices $df(y) \in M(n,p)$ and $dr(x) \in M(p,m)$ combine to the matrix product $df\ dr$ at a point. This matrix is in $M(n,m)$. The **multi-variable chain rule** is:

> **Theorem:** $d(f \circ r)(x) = df(r(x))dr(x)$

16.5. For $m = n = p = 1$, the single variable calculus case, we have $df(x) = f'(x)$ and $(f \circ r)'(x) = f'(r(x))r'(x)$. In general, df is now a matrix rather than a number. By checking a single matrix entry, we reduce to the case $n = m = 1$. In that case, $f : \mathbb{R}^p \to \mathbb{R}$ is a **scalar function**. While df is a **row vector**, we define the **column vector** $\nabla f = df^T = [f_{x_1}, f_{x_2}, \ldots f_{x_p}]^T$. If $r : \mathbb{R} \to \mathbb{R}^p$ is a curve, we write $r'(t) = [x_1'(t), \cdots, x_p'(t)]^T$ instead of $dr(t)$. The symbol ∇ is addressed also as "nabla". [1] The special case $n = m = 1$ is:

> **Theorem:** $\frac{d}{dt}f(r(t)) = \nabla f(r(t)) \cdot r'(t)$.

16.6. Proof. $d/dt f(x_1(t), x_2(t), \ldots, x_p(t))$ is the limit $h \to 0$ of

$$[f(x_1(t+h), x_2(t+h), \ldots, x_p(t+h)) - f(x_1(t), x_2(t), \ldots, x_p(t))]/h =$$
$$= [f(x_1(t+h), x_2(t+h), \ldots, x_p(t+h)) - f(x_1(t), x_2(t+h), \ldots, x_p(t+h)]/h$$
$$+ [f(x_1(t), x_2(t+h), \ldots, x_p(t+h)) - f(x_1(t), x_2(t), \ldots, x_p(t+h)]/h + \cdots$$
$$+ [f(x_1(t), x_2(t), \ldots, x_p(t+h)) - f(x_1(t), x_2(t), \ldots, x_p(t))]/h$$

which is (1D chain rule) in the limit $h \to 0$ the sum $f_{x_1}(x)x_1'(t) + \cdots + f_{x_p}(x)x_p'(t)$.

16.7. Proof of the general case: Let $h = f \circ r$. The entry ij of the Jacobian matrix $dh(x)$ is $dh_{ij}(x) = \partial_{x_j}h_i(x) = \partial_{x_j}f_i(r(x))$. The case of the entry ij reduces with $t = x_j$ and $h_i = f$ to the case when $r(t)$ is a curve and $f(x)$ is a scalar function. This is the case we have proven already.

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[1]Etymology tells that the symbol is inspired by a Egyptian or Phoenician harp.

16.8. Assume a ladybug walks on a circle $r(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ and $f(x, y) = x^2 - y^2$ is the temperature at the position (x, y), then $f(r(t))$ is the rate of change of the temperature. We can write $f(r(t)) = \cos^2(t) - \sin^2(t) = \cos(2t)$. Now, $d/dt f(r(t)) = -2\sin(2t)$. The gradient of f and the velocity are $\nabla f(x, y) = \begin{bmatrix} 2x \\ -2y \end{bmatrix}, r'(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$. Now

$$\nabla f(r(t)) \cdot r'(t) = \begin{bmatrix} 2\cos(t) \\ -2\sin(t) \end{bmatrix} \cdot \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} = -4\cos(t)\sin(t) = -2\sin(2t) .$$
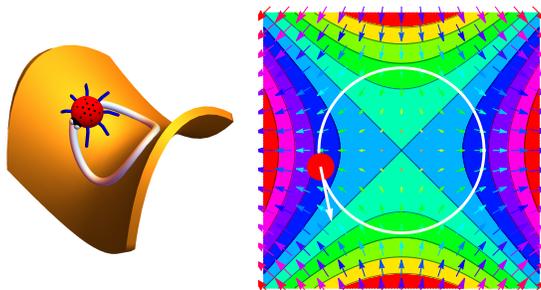

FIGURE 2. If $f(x, y)$ is a height, the rate of change $d/dt f(r(t))$ is the gain of height the bug climbs in unit time. It depends on how fast the bug walks and in which direction relative to the gradient ∇f it walks.

ILLUSTRATIONS

16.9. The case $n = m = 1$ is extremely important. The chain rule $d/dt f(r(t)) = \nabla f(r(t)) \cdot r'(t)$ tells that the rate of change of the **potential energy** $f(r(t))$ at the position $r(t)$ is the dot product of the **force** $F = \nabla f(r(t))$ at the point and the **velocity** with which we move. The right hand side is **power = force** times **velocity**. We will use this later in the fundamental theorem of line integrals.

16.10. If $f, g : \mathbb{R}^m \to \mathbb{R}^m$, then $f \circ g$ is again a map from $\mathbb{R}^m$ to $\mathbb{R}^n$. We can also **iterate** a map like $x \to f(x) \to f(f(x)) \to f(f(f(x))) \dots$. The derivative $df^n(x)$ is by the chain rule the product $df(f^{n-1}(x)) \cdots df(f(x)) df(x)$ of Jacobian matrices. The number $\lambda(x) = \limsup_{n \to \infty} (1/n) \log(|df^n(x)|)$ is called the **Lyapunov exponent** of the map f at the point x. It measures the amount of **chaos**, the "sensitive dependence on initial conditions" of f. These numbers are hard to estimate mathematically. Already for simple examples like the **Chirikov map** $f([x, y]) = [2x - y + c\sin(x), x]$, one can measure **positive entropy** $S(c)$. A conjecture of Sinai tells that that the **entropy of the map** is positive for large c. **Measurements** show that this entropy $S(c) = \int_0^{2\pi} \int_0^{2\pi} \lambda(x, y) \, dxdy/(4\pi^2)$ satisfies $S(x) \geq \log(c/2)$. The conjecture is still open. [2]

16.11. If $H(x, y)$ is a function called the **Hamiltonian** and $x'(t) = H_y(x, y), y'(t) = -H_x(x, y)$, then $d/dt H(x(t), y(t)) = 0$. This can be interpreted as **energy conservation**. We see that a Hamiltonian differential equation always preserves the energy. For the **pendulum**, $H(x, y) = y^2/2 - \cos(x)$, we have $x' = y, y' = -\sin(x)$ or $x'' = -\sin(x)$.

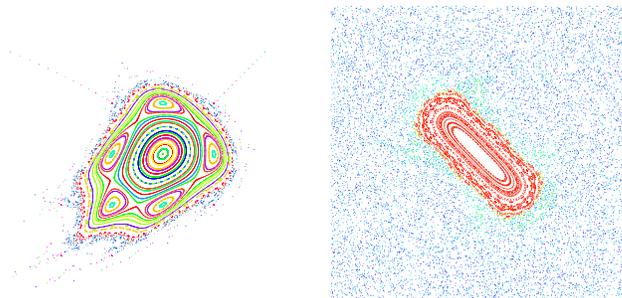[2] To generate orbits, see http://www.math.harvard.edu/~knill/technology/chirikov/.

FIGURE 3.   The map $f([x,y]) = [x^2 - x/2 - y, x]$ is a **Henon map**. We see some orbits. The map $f([x,y]) = [2x - y + 4\sin(x), x]$ on the right appeared in the first hourly. The torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ is filled with a blue "stochastic sea" containing red "stable islands".

16.12. The chain rule is useful to get derivatives of inverse functions. Like

$$1 = \frac{d}{dx}x = \frac{d}{dx}\sin(\arcsin(x)) = \cos(\arcsin(x))\arcsin'(x)$$

which then gives $\arcsin'(x) = 1/\sqrt{1 - \sin^2(\arcsin(x))} = 1/\sqrt{1 - x^2}$.

16.13. Assume $f(x,y) = x^3y + x^5y^4 - 2 - \sin(x - y) = 0$ is a curve. We can not solve for y. Still, we can assume $f(x, y(x)) = 0$. Differentiation using the chain rule gives $f_x(x, y(x)) + f_y(x, y(x))y'(x) = 0$. Therefore

$$y'(x) = -\frac{f_x(x, y(x))}{f_y(x, y(x))} .$$

In the above example, the point $(x, y) = (1, 1)$ is on the curve. Now $g_x(x, y) = 3 + 5 - 1 = 7$ and $g_y(x, y) = 1 + 4 + 1 = 6$. So, $g'(1) = -7/6$. This is called **implicit differentiation**. We could compute with it the derivative of a function which was not known.

16.14. The **implicit function theorem** assures that a differentiable implicit function $g(x)$ exists near a root (a, b) of a differentiable function $f(x, y)$.

> **Theorem:** If $f(a, b) = 0, f_y(a, b) \neq 0$ there exists $c > 0$ and a function $g \in C^1([b - c, b + c])$ with $f(x, g(x)) = 0$.

Proof. Let c be so small that for fixed $x \in [a - c, a + c]$, the function $y \in [b - c, b + c] \to h(y) = f(x, y)$ has the property $h(b-c) < 0$ and $h(b+c) > 0$ and $h'(y) \neq 0$ in $[b-c, b+c]$. The **intermediate value theorem** for h now assures a unique root $z = g(x)$ of h near b. The chain rule formula above then assures that for $a - c < x < a + c$, the differential quotient $[g(x+h) - g(x)]/h$ written down for g has a limit $-f_x(x, g(x))/f_y(x, g(x))$.

P.S. We can get the root of h by applying **Newton steps** $T(y) = y - h(y)/h'(y)$. Taylor (seen in the next class) shows the error is squared in every step. The Newton step $T(y) = y - dh(y)^{-1}h(y)$ works also in arbitrary dimensions. One can prove the implicit function theorem by just establishing that $\text{Id} - T = dh^{-1}h$ is a contraction and then use the **Banach fixed point theorem** to get a fixed point of $\text{Id} - T$ which is a root of h.
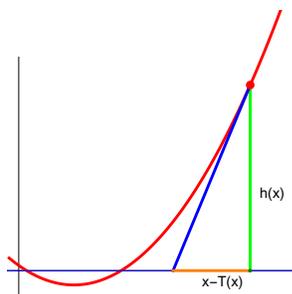
FIGURE 4. The Newton step.
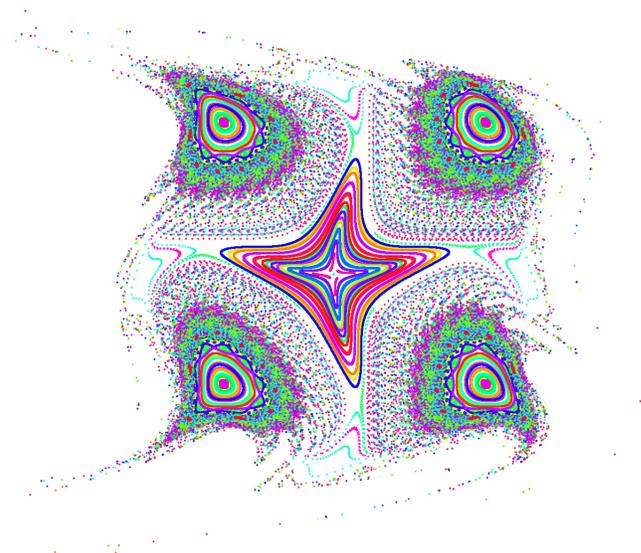
FIGURE 5. If we apply the map $f([x, y]) = [x^2 - x^4 - y, x]$ again and again and plot points, we get an **orbit**. Such simple dynamical systems are largely not understood. Which points do not escape to infinity? What is the boundary of this set. Proving that there are regions which stay bounded is hard and needs "hard implicit function theorems". The Newton method allows to get a grip on proving this, where the Newton step is applied on spaces of functions. Some of the hardest analysis which humans have invented for tackling mathematical problems come to play in this seemingly simple map $f : \mathbb{R}^2 \to \mathbb{R}^2$.

Units 16 and 17 are together taught on Wednesday. Homework is all in unit 17.

Unit 17: Taylor approximation

INTRODUCTION

17.1. According to legend [1], Richard Feynman got into the challenge to compute the cube root of 1729.03 against an Abacus computation. By using linear approximation and a bit o luck, he could get 12.002384 using paper and pencil. The actual cube root is 12.002383785691718123057. How did Feynman do it? The secret is in linear approximation. This means that we approximate a function like $f(x) = x^{1/3}$ with a linear function. The same can be done with functions of several variables. The linear approximation if of the form $L(x) = f(a) + f'(a)(x - a)$.

FIGURE 1. The Abacus scene in the movie "Infinity".

17.2. One can also do higher order approximations. The function $f(x) = e^x$ for example has the linear approximation $L(x) = 1 + x$ at $a = 0$ and the quadratic approximation $Q(x) = 1 + x + x^2/2$ at $a = 0$. To get the quadratic term, we just need to make sure that the first and second derivative at $x = a$ agree. This gives the formula $Q(x) = f(a) + f'(a)(x - a) + f''(a)(x - a)^2/2$. Indeed, you can check that $f(x)$ and $Q(x)$ have the same first derivatives and the same second derivatives at $x = a$. A

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[1]"Feynmans book "What do you care what other people think"

degree n approximation is then the **polynomial**

$$P_n(x) = \sum_{k=0}^{n} f^{(k)}(a) \frac{(x-a)^k}{k!} \ .$$

For the function e^x for example, we have the m'th order approximation

$$e^x = 1 + x + x^2/2! + x^3/3! + ... + x^n/n! \ .$$

17.3. The same can be done in higher dimensions. Everything is the same. We just have to use the derivative df rather than the usual derivative f'. We look here only at linear and quadratic approximation of functions $\mathbb{R}^n \to \mathbb{R}$ The linear approximation is then

$$L(x) = f(a) + \nabla f(a)(x-a)$$

where $\nabla f(a) = df(a) = [f_{x_1}(a), \cdots, f_{x_n}(a)]$ is the Jacobian matrix, which ii a row vector. Now, since we can see $df(x) : \mathbb{R}^n \to \mathbb{R}^n$ the second derivative is a matrix $d^2 f(x) = H(x)$. It is called the Hessian. It encodes all the second derivatives $H_{ij}(x) = f_{x_i x_j}$.

LECTURE

17.4. Given a function $f : \mathbb{R}^m \to \mathbb{R}^n$, its derivative $df(x)$ is the Jacobian matrix. For every $x \in \mathbb{R}^m$, we can use the matrix $df(x)$ and a vector $v \in \mathbb{R}^m$ to get $D_v f(x) = df(x)v \in \mathbb{R}^m$. For fixed v, this defines a map $x \in \mathbb{R}^m \to df(x)v \in \mathbb{R}^n$, like the original f. Because D_v is a map on $\mathcal{X} = \{$ all functions from $\mathbb{R}^m \to \mathbb{R}^n \}$, one calls it an **operator**. The **Taylor formula** $f(x+t) = e^{Dt} f(x)$ holds in arbitrary dimensions:

Theorem: $f(x+tv) = e^{D_v t} f = f(x) + \frac{D_v t f(x)}{1!} + \frac{D_v^2 t^2 f(x)}{2!} + ...$

17.5. Proof. It is the single variable Taylor on the line $x+tv$. The directional derivative $D_v f$ is there the usual derivative as $\lim_{t \to 0}[f(x+tv) - f(x)]/t = D_v f(x)$. Technically, we need the sum to converge as well: like functions built from polynomials, $\sin, \cos, \exp$.

17.6. The Taylor formula can be written down using successive derivatives $df, d^2 f, d^3 f$ also, which are then called **tensors**. In the scalar case $n = 1$, the first derivative $df(x)$ leads to the gradient $\nabla f(x)$, the second derivative $d^2 f(x)$ to the **Hessian matrix** $H(x)$ which is a bilinear form acting on pairs of vectors. The third derivative $d^3 f(x)$ then acts on triples of vectors etc. One can still write as in one dimension

Theorem: $f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + \cdots$

if we write $f^{(k)} = d^k f$. For a polynomial, this just means that we first write down the constant, then all linear terms then all quadratic terms, then all cubic terms etc.

17.7. Assume $f : \mathbb{R}^m \to \mathbb{R}$ and stop the Taylor series after the first step. We get

$$L(x_0 + v) = f(x_0) + \nabla f(x_0) \cdot v \ .$$

It is custom to write this with $x = x_0 + v, v = x - x_0$ as

$$L(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0)$$

This function is called the **linearization** of f. The kernel of $L - f(x_0)$ is a linear manifold approximating the surface $\{x \mid f(x) - f(x_0) = 0\}$. If $f : \mathbb{R}^m \to \mathbb{R}^n$, then the just said can be applied to every component f_i of f, with $1 \le i \le n$. One can not stress enough the importance of this linearization. [2]

17.8. If we stop the Taylor series after two steps, we get the function $Q(x + v) = f(x) + df(x) \cdot v + v \cdot d^2 f(x) \cdot v/2$. The matrix $H(x) = d^2 f(x)$ is called the **Hessian matrix** at the point x. It is also here custom to eliminate v by writing $x = x_0 + v$.

$$Q(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0) + (x - x_0) \cdot H(x_0)(x - x_0)/2$$

is called the **quadratic approximation** of f. The kernel of $Q - f(x_0)$ is the **quadratic manifold** $Q(x) - f(x_0) = x \cdot Bx + Ax = 0$, where $A = df$ and $B = d^2 f/2$. It approximates the surface $\{x \mid f(x) - f(x_0) = 0\}$ even better than the linear one. If $|x - x_0|$ is of the order ϵ, then $|f(x) - L(x)|$ is of the order ϵ^2 and $|f(x) - Q(x)|$ is of the order ϵ^3. This follows from the exact **Taylor with remainder formula**. [3]
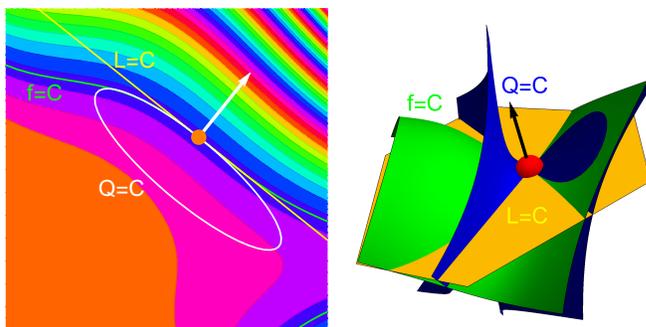
FIGURE 2. The manifolds $f(x, y) = C, L(x, y) = C$ and $Q(x, y) = C$ for $C = f(x_0, y_0)$ pass through the point (x_0, y_0). To the right, we see the situation for $f(x, y, z) = C$. We see the best linear approximation and quadratic approximation. The gradient is perpendicular.

17.9. To get the **tangent plane** to a surface $f(x) = C$ one can just look at the linear manifold $L(x) = C$. However, there is a better method:

The tangent plane to a surface $f(x, y, z) = C$ at (x_0, y_0, z_0) is $ax + by + cz = d$, where $[a, b, c]^T = \nabla f(x_0, y_0, z_0)$ and $d = ax_0 + by_0 + cz_0$.

---

[2] Again: the linearization idea is utmost important because it brings in linear algebra.

[3] If $f \in C^{n+1}$, $f(x+t) = \sum_{k=0}^{n} f^{(k)}(x) t^k / k! + \int_0^t (t-s)^n f^{(n+1)}(x+s) ds / n!$ (prove this by induction!)

17.10. This follows from the **fundamental theorem of gradients**:

> **Theorem:** The gradient $\nabla f(x_0)$ of $f : \mathbb{R}^m \to \mathbb{R}$ is perpendicular to the surface $S = \{f(x) = f(x_0) = C\}$ at x_0.

Proof. Let $r(t)$ be a curve on S with $r(0) = x_0$. The chain rule assures $d/dt f(r(t)) = \nabla f(r(t)) \cdot r'(t)$. But because $f(r(t)) = c$ is constant, this is zero assuring $r'(t)$ being perpendicular to the gradient. As this works for any curve, we are done.

<div align="center">EXAMPLES</div>

17.11. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given as $f(x, y) = x^3 y^2 + x + y^3$. What is the quadratic approximation at $(x_0, y_0) = (1, 1)$? We have $df(1,1) = [4, 5]$ and

$$\nabla f(1, 1) = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, H(1, 1) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 8 \end{bmatrix}.$$

The linearization is $L(x, y) = 4(x - 1) + 5(y - 1) + 3$. The quadratic approximation is $Q(x, y) = 3 + 4(x - 1) + 5(y - 1) + 6(x - 1)^2/2 + 12(x - 1)(y - 1)/2 + 8(y - 1)^2/2$. This is the situation displayed to the left in Figure (2). For $v = [7, 2]^T$, the directional derivative $D_v f(1, 1) = \nabla f(1, 1) \cdot v = [4, 5]^T \cdot [7, 2] = 38$. The Taylor expansion given at the beginning is a finite series because f was a polynomial: $f([1, 1] + t[7, 2]) = f(1 + 7t, 1 + 2t) = 3 + 38t + 247t^2 + 1023t^3 + 1960t^4 + 1372t^5$.

17.12. For $f(x, y, z) = -x^4 + x^2 + y^2 + z^2$, the gradient and Hessian are

$$\nabla f(1, 1, 1) = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, H(1, 1, 1) = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} -10 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

The linearization is $L(x, y, z) = 2 - 2(x - 1) + 2(y - 1) + 2(z - 1)$. The quadratic approximation

$$Q(x, y, z) = 2 - 2(x - 1) + 2(y - 1) + 2(z - 1) + (-10(x - 1)^2 + 2(y - 1)^2 + 2(z - 1)^2)/2$$

is the situation displayed to the right in Figure (2).

17.13. What is the tangent plane to the surface $f(x, y, z) = 1/10$ for $f(x, y, z) =$

$$10z^2 - x^2 - y^2 + 100x^4 - 200x^6 + 100x^8 - 200x^2y^2 + 200x^4y^2 + 100y^4 = 1/10$$

at the point $(x, y, z) = (0, 0, 1/10)$? The gradient is $\nabla f(0, 0, 1/10) = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$. The tangent plane equation is $2z = d$, where the constant d is obtained by plugging in the point. We end up with $2z = 2/10$. The linearization is $L(x, y, z) = 1/20 + 2(z - 1/10)$.
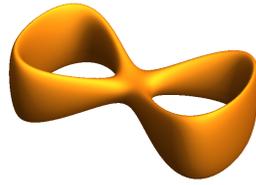
FIGURE 3.

HOMEWORK

Problem 16-17.1: Let $r(t) = [3t + \cos(t), t + 4\sin(t)]^T$ be a curve and $f([x,y]^T) = [x^3 + y, x + 2y + y^3]^T$ be a coordinate change.
a) Compute $v = r'(0)$ at $t = 0$, then $df(x,y)$ and $A = df(r(0))$ and $df(r(0))r'(0) = Av$.
b) Compute $R(t) = f(r(t))$ first, then find $w = R'(0)$. It should agree with a).

Problem 16-17.2: a) The surface

$$f(x,y,z) = x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 4 + 1/4 + 1/9$$

is an ellipsoid. Compute $z_x(x,y)$ at the point $(x,y,z) = (2,1,1)$ using the implicit differentiation rule. (Use the formula).
b) Apply the Newton step 3 times starting with $x = 2$ to solve the equation $x^2 - 2 = 0$.

Problem 16-17.3: Evaluate without technology the cube root of 1002 using quadratic approximation. Especially look how close you are to the real value.

Problem 16-17.4: a) Find the tangent plane to the surface $f(x,y,z) = \sqrt{xyz} = 60$ at $(x,y,z) = (100, 36, 1)$. b) Estimate $\sqrt{100.1 \cdot 36.1 \cdot 0.999}$ using linear approximation (compute $L(x,y,z)$ rather than $f(x,y,z)$.)

Problem 16-17-5: Find the quadratic approximation $Q(x,y)$ of $f(x,y) = x^3 + x^2 y + x^2 + y^2 - 2x + 3xy$ at the point $(1,2)$ by computing the gradient vector $\nabla f(1,2)$ and the Hessian matrix $H(1,2)$. The vector $\nabla f(1,2)$ is a 1×2 matrix (row vector) and the Hessian matrix $H(1,2)$ is a 2×2 matrix.

Unit 18: Number Magic

INTRODUCTION

18.1. One of the most famous open problems in mathematics is the **Goldbach conjecture**:

Every even integer larger than 2 is a sum of two primes.

Let $g(n)$ denote the function which tells in how many ways we can write n as a sum of two primes. For example $g(5) = 2, g(6) = 1$ because $5 = 2 + 3 = 3 + 2$ and $g(6) = 1$ because $6 + 3 + 3$.
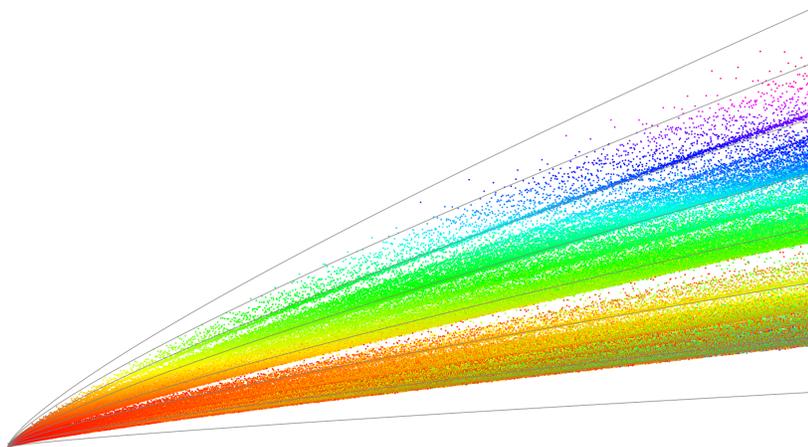

FIGURE 1. The Goldbach comet and the suspected lower and upper bounds which are of the form $n/\log(n)^2$, $Cn/\log(n)^2$.

18.2. Here is Mathematica code which allows to plot the comet, the graph of the function g.

```
n=100000;  f=Sum[If[PrimeQ[a],x^a,0],{a,n}];  g = Expand[f*f];
G=CoefficientList[g,x];  ListPlot[Table[G[[2k-1]],{k,3,n/2}]]
```

18.3. Why is this remarkable? It shows that computing the numbers $f(n)$ could be done nicely using **calculus** by defining a function f. Using **Taylor's theorem** we can compute the entries $g(n)$. The Goldbach conjecture is equivalent to

$$g^{2n}(0) \text{ is nonzero for all } n \geq 1.$$

The only thing we would really need is to get a grip on the function f. Unfortunately, nobody has seen how to write down the function f in terms of known functions. But it is not completely hopeless that there should not exist a modification $f(x) = \sum_{p \text{ prime}} a_p x^p$ with positive a_p such that $f(x)$ is expressible using known functions. Also then, if $g(x) = f(x)^2$ had positive even derivatives, Goldbach would follow.

<center>SEMINAR</center>

18.4. In this seminar, we see how calculus can help to compute things effectively and also hope to get insight into topics which are of more number theoretical nature. To find the cube root of 10 for example, we have

$$10^{1/3} \sim 8^{1/3} + \frac{2}{3 \cdot 8^{2/3}} = 2 + \frac{2}{12} = 2.1666\ldots.$$

The actual value is 2.15443. We can also use linearization to find exact roots

18.5.

> **Problem A:** Find $(1030301)^{1/3}$ using linear approximation at $x = 1000000$.
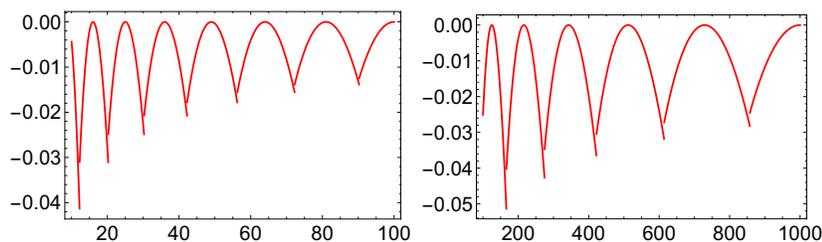
FIGURE 2. The error of the linear approximation when computing square roots and cube roots is in the 5 percent range.

18.6. We could not mention the **Newton method** to find roots in class. It is a simple but effective iterative method. We can also do that to find roots. In order to find the cube root of 9 for example, we start with a first approximation like 2, then introduce the function $f(x) = x^3 - 9$ for which we aim to find the root, then apply the **Newton step**

$$T(x) = x - \frac{f(x)}{f'(x)}.$$

We have $f'(x) = 3x^2$ and so $T(x) = x - (x^3 - 9)/(3x^2)$. This gives $T(2) = 25/12 = 2.08333$. Already quite close to $9^{1/3} = 2.08008$.

18.7. There is an interesting story here when applying the Newton method in the complex plane. The function $f(x) = x^3 - 9$ has exactly 3 roots in the complex plane. They are $9^{1/3}, 9^{1/3}e^{i2\pi/3}$ and $9^{1/3}e^{i4\pi/3}$. Check that these three numbers satisfy $f(x) = 0$! Investigating the Newton method in the complex actually predated the Mandelbrot story. One can wonder what happens if you apply the Newton method with a given initial condition. The solution will end up in one of the three roots, but which ones? When drawing this, we see the **Newton fractal**. here is how you can plot the Newton fractal. [1]

```
f[z_]:=z^3-1;T[z_]:=z-f[z]/f'[z];T[z_]:=N[1/(3z^2)+(2z)/3];
g[z_]:=Arg[FixedPoint[T,z,20]];
ContourPlot[g[x+I y],{x,-4,4},{y,-4,4},PlotPoints->200]
```
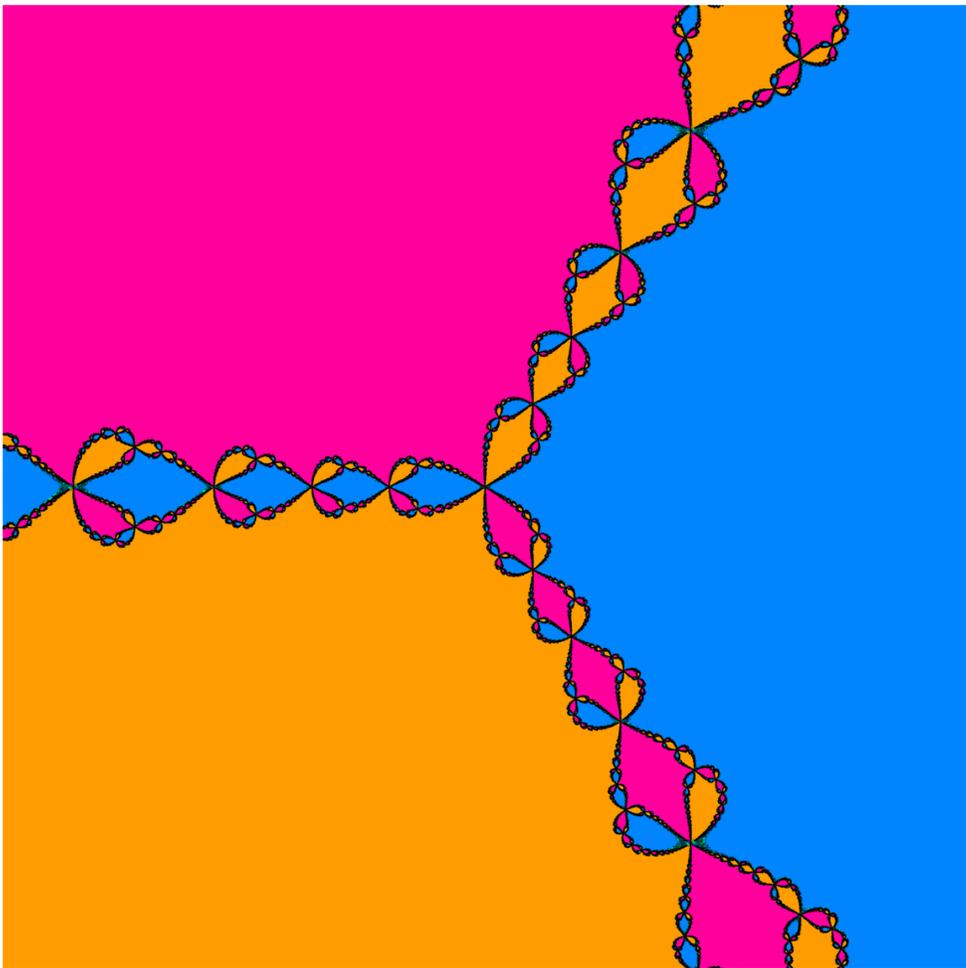

FIGURE 3. The Newton Fractal

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[1]T is define a second time because we do not want to differentiate f symbolically in each evaluation of T and N[] forces floating point arithmetic.

18.8. In the exam you have proven $1 + 3 + 3^2 + \cdots + 3^{n-1} = (3^n - 1)/2$. This is a special case of the **geometric series formula**

$$1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}.$$

Of course, we could also prove this formula by induction. Better do it directly:

> **Problem B:** Verify the geometric series formula by multiplying with $1 - a$.

18.9. These were all finite sums but seeing the pattern allows us to **take a limit** and compute the infinite series:

> **Problem C:** For which a is $1 + a + a^2 + a^3 + ... = \frac{1}{1-a}$ valid?

18.10. The Taylor series of a nice function is $f(x) = \sum_{k=0}^{\infty} f^{(k)}(0) x^k$. Having just looked at C) can answer the trick question:

> **Problem D:** What is the Taylor series of $f(x) = \frac{1}{(1-x)}$ at $x_0 = 0$?

18.11. How can you get from the last exercise the following identity?

> **Problem E:** $-\log(1 - x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$.

18.12. Now lets see what happens at $x = -1$.

> **Problem F:** Use E to see what happens for $x = -1$.

18.13. How come that great number theorists like Leonard Euler or Godfrey Hardy were also masters in calculus? The reason is that many results of number theoretic nature have intimate relations with calculus. Lets look at the following problem:

> **Problem G:** What is the value of the Leibniz series
> $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots .$$

18.14. Hint: compute first the Taylor series of $f(x) = \arctan(x)$ using the Taylor series of $1/(1 + x^2)$ (the later is a geometric series), then evaluate f at $x = 1$.

18.15.

$$\mathrm{Li}_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s} = x + \frac{x^2}{2^s} + \frac{x^3}{3^s} + \cdots$$

is called the **poly logarithm function**. For $s = 0$ it is Problem D, for $s = 1$ it is problem E. While in calculus, we might be more interested in the function as a function of x, **number theorists** are more interested in the function as a function of s and s is complex. In the case $x = 1$, the function $\mathrm{Li}_s(x)$ is the **Riemann zeta function** $\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$.

18.16. The **Euler golden key** relates ζ with primes:

Theorem: $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - \frac{1}{p^s})^{-1}$.

18.17.

Problem I: Verify the Euler golden key identity.

18.18. First verify (maybe look at Problem C) that for a single prime p

$$\frac{1}{1 - \frac{1}{p^s}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots$$

which is the sum over all $\frac{1}{n^s}$, where n has only prime factors p. Then look at the product of these for two primes p, q and see that this is the sum over all $\frac{1}{n^s}$ where n has only prime factors p and q.

18.19. Lets come back to the topic of the introduction. Remember that the **Goldbach conjecture** tells that every even number larger than 2 is the sum of two primes. What is the relation with calculus? Define $g(x) = (f(x))^2$ with

$$f(x) = \sum_p \frac{x^p}{p!} = \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots.$$

For the following, try to verify this carefully by showing **both directions**. If a statement A, B are **equivalent** then this means that we have to show two things. We have to verify that $A \implies B$ and $B \implies A$.

Problem J: Goldbach is equivalent to $g^{(n)}(x) > 0$ for all even $n > 2$.

HOMEWORK

Problem 18.1 The weak Goldbach conjecture claims that every integer larger or equal than 6 is a sum of three primes. Check this for $n = 6, 7, 8, 9, 10, 11, \ldots$. The theorem is proven since 2015 (will appear in the Annals of mathematics). Use a computer to draw a picture of the **weak Goldbach comet**.

Problem 18.2 The function f defined by $f(x) = e^{-1/x}$ for $x > 0$ and 0 for $x \leq 0$ is smooth and that all derivatives at 0 are zero. Check $f'(0), f''(0), f'''(0) = 0$. Conclude that there are smooth functions for which the Taylor expansion does not work. Check then that is $b(x) = f(r^2 - |x|^2)$ a "bump function" (see figure 3). First define what a "bump function" is.
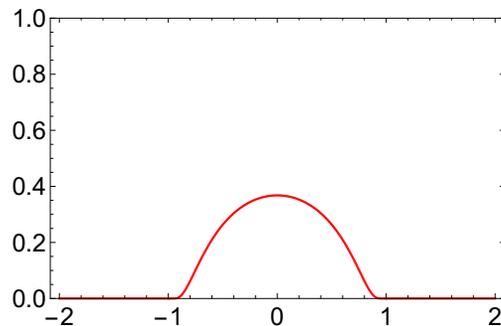
FIGURE 4. The function $f(x) = e^{-1/x}$ allows to define a **smooth bump function** $b(x)$ which is zero outside a ball of radius r.

Problem 18.3 The series
$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$
a long history. Research it a bit. Especially: What is the value of $\zeta(2)$. Who found this problem first? What is the name of the problem? Now look at $\zeta(3)$. Is there like for $\zeta(2)$ an explicit formula? Does one know whether $\zeta(3)$ is rational or not?

Problem 18.4 By looking it up, give an explanation why it makes sense that
$$\zeta(-1) = 1 + 2 + 3 + 4 + \cdots$$
can be assigned a finite value. You can also look up its value $-1/12$ with Mathematica $Zeta[-1]$. How is such a finite value possible? In your explanation, we just want to know which field of mathematics is involved and what the idea is to define $\zeta(s)$ also for $s = -1$, a point where the sum diverges. Finally, what are the values $\zeta(-2), \zeta(-3), \zeta(-4), \zeta(-5), ...\zeta(-9)$. The last one is
$$\zeta(-9) = 1^9 + 2^9 + 3^9 + 4^9 + ...$$

Problem 18.5 You can practice computing square roots of numbers between 1 and 100 by linear approximation in your head. For example, if somebody asks you to compute $\sqrt{20}$ you would immediately tell $4 + 4/(2 \cdot 4) = 4.5$. The actual result is 4.472... You you could also get $5 - 5/(2 \cdot 5) = 4.5$. Find another non-square integer in 1 to 100 for which these two estimates agree. (There are a couple of them).

Unit 19: Extrema

INTRODUCTION

19.1. Learning is an optimization process with the goal to increase knowledge, skills and creative power. This applies both for education as well as for machine learning. In order to track the learning process, we need a function which measures progress. An old fashioned metric is the GPA averaging some grades in an educational system, an other or IQ scores measured by doing tests. An other metric example in a research setting is a social network score like the number of citations or the h-index. For a car driving autonomously it could be the $f(x) = 100/(1 + N(x))$ where $N(x)$ is the number of accidents produced using the parameter configuration x in a fixed period.

From D. Perkins: The Eureka Effect, the Art and Logic of Breakthrough Thinking , 2000

Search in a Homing Space: 1. Clueless regions. 2. Large clued regions leading to the target.

Search in a Klondike Space: 1. A large space wtih few solutions (a wilderness trap). 2. Regions with no clues pointing direction (plateau traps). 3. A barrier isolates the solution (creating a canyon trap). 4. An area of high promise but no solution (an oasis trap).

FIGURE 1. The Klondike picture of David Perkins illustrates the search for solutions in a higher dimensional landscape defined by a height function f. Calculus suggests to follow the gradient ∇f as this leads to **local maxima**. To find global maxima (which could be breakthrough ideas), we have to search harder and make a list of all maxima. The process is named after Klondike region in Canada which became infamous during the Gold rush.

19.2. Once the frame work and the function f is fixed, the question is how to increase f most effectively. This simplistic picture is quite effective both for human intelligence or artificial intelligence. For many functions which have been considered (winning in chess games, computational power, data retention, feature detection, driving cars or flying planes) machines progressed rapidly. There is hardly anybody who seriously doubts that humans eventually will lose the battle for any function f which can be considered. There are still domains where machines have not taken over. Examples are art or writing scientific papers. [1]

19.3. Once a machine knows the function f, it can quit comfortably determine from a position x in which direction to change to increase f most rapidly. The direction of **fastest increase** is the direction of the gradient ∇f of f. In calculus, we look at situations, where the position consists of a few variables only. Single variable calculus deals with the situation of one variable. We look here at the situation with n variables but will mostly work with 2 variables as this already gives the main idea. The principle is that we have reached an optimum where no change any more can increase the function f. This means mathematically that the derivative df of f is zero. We call such points "critical points".

19.4. Let us first look at the rate of change of a function along a direction v. Take a curve $r(t) = x + tv$ where v is a unit vector. By the chain rule, the rate of change at x is given by $f(r(t)) = \nabla f(r(t)) \cdot r'(0) = \nabla f(x) \cdot v$. We know for the dot product that this is equal to $|\nabla f(x)||v| \cos(\alpha) = |\nabla f(x)| \cos(\alpha)$. This is maximized for $\cos(\alpha) = 1$ which means that v points into the same direction than ∇f. So, The gradient points into the direction of maximal increase. This is important to remember. If you are in a landscape given by the height $f(x)$ you have to go into the direction of $\nabla f(x)/|\nabla f(x)|$ in order to increase most. Of course, this does not make sense if $\nabla f(x) = 0$ but that is the situation where you are at a maximum, and where you can not increase f any more.

LECTURE

19.5. All functions are assumed here to be in C^2, meaning that they are two times continuously differentiable. It all starts with an observation going back to Pierre de Fermat:

> **Theorem:** If x_0 is a maximum of $f : \mathbb{R}^m \to \mathbb{R}$, then $\nabla f(x_0) = 0$.

Proof. We prove this by contradiction. Assume $\nabla f(x_0) \neq 0$, define the vector $v = \nabla f(x_0)$ and look at $g(t) = f(x_0 + tv)$, which is a function of one variable. By the chain rule, it satisfies $g'(0) = \nabla f(x_0 + 0v) \cdot v = |\nabla f|^2 > 0$. This means that $f(x_0 + tv) > 0$ for small $t > 0$. The point x_0 can not have been maximal. This is a **contradiction**. QED.

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[1]There could be resistance: humans might decide not to cite scientific breakthroughs by machines. On the other hand, who would not want to learn a "theory of everything" even if it is discovered by a machine?

19.6. A point x with $\nabla f(x) = 0$ is called a **critical point** of f. By the Taylor formula, we have at a critical point x_0 the quadratic approximation $Q(x) = f(x_0) + (x - x_0)^T H(x_0)(x - x_0)/2$, where $H(x_0)$ is the **Hessian matrix**

$$H(x_0) = \begin{bmatrix} f_{x_1 x_1} & f_{x_1 x_2} & \cdots & f_{x_1 x_m} \\ f_{x_2 x_1} & f_{x_2 x_2} & \cdots & f_{x_2 x_m} \\ \cdots & \cdots & \cdots & \cdots \\ f_{x_m x_1} & f_{x_m x_2} & \cdots & f_{x_m x_m} \end{bmatrix}.$$

19.7. As in one dimension, having a critical point does not assure that a point is a local maximum or minimum. The second derivative test in single variable calculus assures that if $f'(x_0) = 0$, $f''(x_0) > 0$, we have a local minimum and if $f'(x_0) = 0$, $f''(x_0) < 0$, we have a local maximum. If $f''(x_0) = 0$, we can not say anything without looking at higher derivatives.

19.8. A matrix A is called **positive definite** if $v \cdot Av > 0$ for all vectors $v \neq 0$. It is called **negative definite** if $v \cdot Av < 0$ for all vectors $v \neq 0$. A diagonal matrix with positive diagonal entries is positive definite. In the following statements, we assume x_0 is a critical point.

19.9. We say x_0 is a **local maximum** of f if there exists $r > 0$ such that $f(x) \leq f(x_0)$ for all $|x - x_0| < r$. We say, it is a **local minimum** of f if $f(x) \geq f(x_0)$ for all $|x - x_0| < r$. How can we check whether a point is a local maximum or minimum?

> **Theorem:** Assume $\nabla f(x_0) = 0$. If $H(x_0)$ is positive definite, then x_0 is a local minimum. If $H(x_0)$ is negative definite, then x_0 is a local maximum.

19.10. Proof: as $\nabla f(x_0) = 0$, the quadratic approximation at x_0 is $Q(x) = f(x_0) + H(x_0)v \cdot v/2 > f(x_0)$ for small non-zero $v = x - x_0$ and Hessian H. The analogue statement for the minimum can be deduced by replacing f with $-f$.

19.11. Let us look at the case, where $f(x, y)$ is a function of two variables such that $f_x(x_0, y_0) = 0$ and $g_x(x_0, y_0) = 0$. The Hessian matrix is

$$H(x_0, y_0) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}.$$

In this two dimensional case, we can classify the critical points if the determinant $D = \det(H) = f_{xx}f_{yy} - f_{xy}^2$ of H is non-zero. The number D is also called the **discriminant** at a critical point.
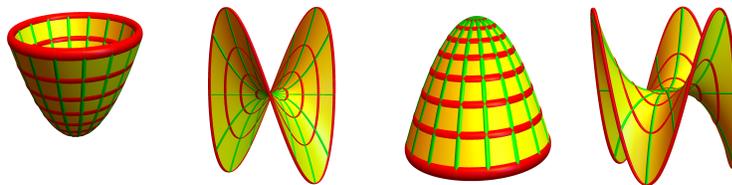

FIGURE 2. $f = x^2 + y^2$ gives a minimum, $f = -x^2 - y^2$ a maximum and $f = x^2 - y^2$ a saddle. The case $f = x^2 y - yx^2$ is not Morse.

19.12. We say (x_0, y_0) is a **Morse point**, if (x_0, y_0) is a critical point and the determinant is non-zero. A C^2 function is a **Morse function** if every critical point is Morse. Examples of Morse functions are $f(x, y) = x^2 + y^2$, $f(x, y) = -x^2 - y^2$ and $f(x, y) = x^2 - y^2$. The last case is called a **hyperbolic saddle**. In general, a critical point is a hyperbolic saddle if $D \neq 0$ and if it is neither a maximum nor a minimum. Here is the **second derivative test** in dimension 2:

> **Theorem:** Assume $f \in C^2$ has a critical point (x_0, y_0) with $D \neq 0$.
> If $D > 0$ and $f_{xx} > 0$ then (x_0, y_0) is a local minimum.
> If $D > 0$ and $f_{xx} < 0$ then (x_0, y_0) is a local maximum.
> If $D < 0$ then (x_0, y_0) is a hyperbolic saddle.

19.13. Proof. After translation $(x, y) \rightarrow (x - x_0, y - y_0)$ and replacing f with $f - f(x_0, y_0)$, we have $(x_0, y_0) = (0, 0)$ and $f(0, 0) = 0$. At the critical point, the quadratic approximation is now
$$Q(x, y) = ax^2 + 2bxy + cy^2 .$$
This can be rewritten as $a(x + \frac{b}{a}y)^2 + (c - \frac{b^2}{a})y^2 = a(A^2 + DB^2)$ with $A = (x + \frac{b}{a}y)$, $B = b^2/a^2$ and discriminant D. If $a = f_{xx} > 0$ and $D > 0$ then $c - b^2/a > 0$ and the function has positive values for all $(x, y) \neq (0, 0)$. The point $(0, 0)$ is then a minimum. If $a = f_{xx} < 0$ and $D > 0$, then $c - b^2/a < 0$ and the function has negative values for all $(x, y) \neq (0, 0)$ and the point (x, y) is a local maximum. If $D < 0$, then f takes both negative and positive values near $(0, 0)$. QED

19.14. One can ask, why f_{xx} and not f_{yy} is chosen. It does not matter, because if $D > 0$, then both f_{xx} and f_{yy} need to be non-zero and have the same sign. Instead of f_{xx}, one could also have pick the more natural **trace** $\text{tr}(H)$. It is invariant under coordinate changes similarly as the determinant D. The discriminant D happens also to be the **Gauss curvature** of the surface at the point.

19.15. In higher dimensions, the situation is described by the **Morse lemma**. It tells that near a critical point there is a coordinate change ϕ such that $g(x) = f(\phi(x))$ is a quadratic function $f(x) = B(x - x_0) \cdot (x - x_0)$ where B is diagonal with entries $+1$ or -1. Critical point can then be given a **Morse index**, the number of entries -1 in B. The Morse lemma is actually a theorem (theorems are more important than lemmata=helper theorems)

> **Theorem:** Near a Morse critical point x_0 of a C^2 function f, there is a coordinate change $\phi : \mathbb{R}^m \rightarrow \mathbb{R}^m$ such that $g(x) = f(\phi(x)) - f(x_0)$ is
> $$g(x) = -x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_m^2 .$$

19.16. Proof. We use induction with respect to m. **(i) Induction foundation.** For $m = 1$, the result tells that for a Morse critical point, the function looks like $y = x^2$ or $y = -x^2$. First show that if $f(0) = f'(0) = 0, f''(0) \neq 0$, then $f(x) = x^2 h(x)$ or $f(x) = -x^2 h(x)$ for some positive C^2 function h. Proof. By a linear coordinate change we assume $x_0 = 0$ and $f(0) = 0$. There exists then $g(x)$ such that $f(x) = xg(x)$: it is $g(x) = f(x)/x$ for $x \neq 0$ and is in the limit $x \rightarrow 0$ the value of $\lim_{x \to 0}(f(x) - f(0))/x = f'(0)$. By the product rule, $f'(x) = g(x) + xg'(x)$ with $g(0) = 0$. Because

$f'(0) = g(0) = 0$ can define $f(x)/x^2$ for $x \neq 0$ and take the limit $x \to 0$, because by applying Hôpital twice, the limit is $f''(0)$. The coordinate change is now given by a function $y = \phi(x)$ satisfying $g(x, y) = y\sqrt{h(y)} = x$. Implicit differentiation gives $g_y(0, 0) = \sqrt{h(y)} \neq 0$ so that by the implicit function theorem $y(x)$ exists.

(ii) Induction step $m \to m+1$: we first note that Taylor for C^2 with remainder term implies that $f(x_1, \ldots, x_n) = \sum_{i,j} x_i x_j h_{ij}(x_1, \ldots, x_n)$ with some continuous functions h_{ij}. Furthermore, the function value $h_{ij}(0) = f_{x_i x_j}(0) = H_{ij}(0)$ are the coordinates of the Hessian. Apply first a rotation so that $h_{11} \neq 0$. Now look at x_1 and keep the other coordinates constant. As in (i), find a coordinate change ϕ such that $f(\phi(x)) = \pm x_1^2 + g(x_2, \ldots, x_m)$, where g inherits the properties of but is of one dimension less. By induction assumption, there is a second coordinate change such that $g(\psi(x)) = x_2^2 - \cdots - x_l^2 + x_{l+1}^2 + \cdots + x_m^2$. Combining ϕ and ψ produces the Morse normal form.

<center>EXAMPLES</center>

19.17. Q: Classify the critical points of $f(x, y) = x^3 - 3x - y^3 - 3y$. A: As $\nabla f(x, y) = [3x^2 - 3, -3y^2 + 3]^T$, the critical points are $(1, 1), (-1, 1), (1, -1)$ and $(-1, -1)$. We compute $H(x, y) = \begin{bmatrix} 2x & 0 \\ 0 & -2y \end{bmatrix}$. For $(1, 1)$ and $(-1, -1)$ we have $D = -4$ and so saddle points. For $(-1, 1)$, we have $D = 4$, $f_{xx} = -2$, a local max. For $(1, -1)$ where $D = 4$, $f_{xx} = 2$ we have a local min.

<center>HOMEWORK</center>

Problem 19.1: a) Classify the critical points of the function
$$f(x, y) = x^2 + y^3 - xy .$$
(Maxima, minima or saddle points).
b) Now do the same for
$$f(x, y, z) = x^2 + y^3 - xy + z^2$$
and find the Morse index at each critical point.

**Problem 19.2:** Find all critical points of the **3D area 51** function
$$f(x, y, z) = x^{51} - 51x + y^{51} - 51y + z^{51} - 51z .$$
Compute the Hessian $H = d^2 f$ at each critical point and determine the maxima (all eigenvalues are negative) and minima (all eigenvalues are positive) P.S. Area 51 is an old hat. But 3D Area 51 is still highly classified and rumored to be near the dark side of the moon.

Problem 19.3: Where on the parametrized surface $r(u, v) = [u^2, v^3, uv]$ is the temperature $T(x, y, z) = 12x + y - 12z$ minimal. Classify all the critical points of the function $f(u, v) = T(r(u, v))$. [If you have found the function $f(u, v)$, you can replace u, v again with x, y if you like to work with a function $f(x, y)$.]

Problem 19.4: Find all the critical points of the function $f(x, y, z) = (x - 1)^2 - y^2 + xz^2$. In each of the cases, find the Hessian matrix. Also here compute the eigenvalues. These are numbers λ such that $Hv = \lambda v$ for some non-zero vector. One can find them by looking for the roots of the characteristic polynomial $\chi_H(\lambda) = \det(L - \lambda)$. You can calculate them on a computer. Find in each case the eigenvalues.

Problem 19.5: a) Find a function $f(x, y)$ with 3 maxima and 3 saddle points and one minimum.
b) You see below a contour map of a function of two variables. How many critical points are there? Is the function a Morse function?
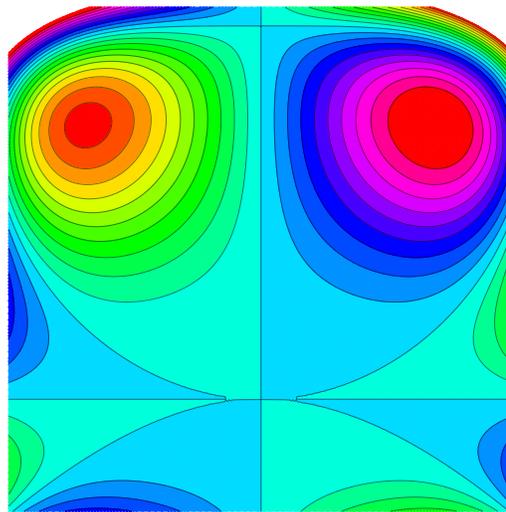
Unit 20: Constraints

20.1. There is rarely a "free lunch". If we want to maximize a quantity, we often have to work with constraints. Obstacles might prevent us to change the parameters arbitrarily. The gradient can still be used as a guiding principle. While we can not achieve ∇f to be zero, we can look for points where the gradient is perpendicular to the constraint. This gives us an optimal point under the confinement. If you hike on a path in the mountains, you often reach a local maximum without being on top of the mountain. What happens at such points x is that $\nabla f(x)$ is perpendicular to the curve meaning that $\nabla f(x)$ is parallel to $\nabla g(x)$.
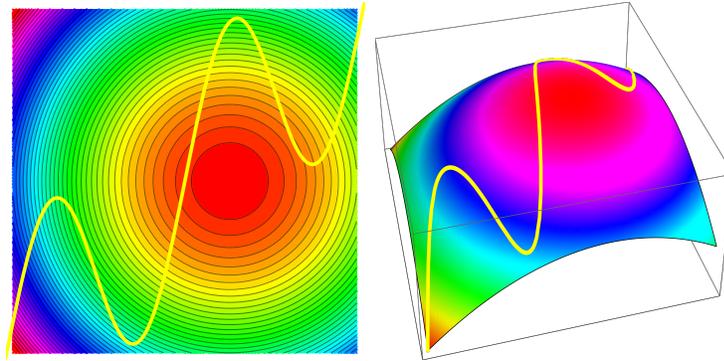

FIGURE 1. The situation, where a function $f(x, y)$ is optimized along a curve $g(x, y) = c$ is a frame-work which can be tackled with Lagrange. The condition of being maximal means that the gradient of f is perpendicular to the curve. This means that the gradients of f and g are parallel. $\nabla f = \lambda \nabla g$.

20.2. The method of Lagrange is much more general. We can work with arbitrary many constraints and still use the same principle. The gradient of $f : \mathbb{R}^n \to \mathbb{R}$ is then perpendicular to the constraint surface which means that is a linear combination of the gradients of all the m constraints: these are n equations $\nabla f = \sum_{j=1}^{m} \lambda_j \nabla g_j$ because the vectors have n components. Together with the m equations $g_j = c_j$ we have $n + m$ equations for $n + n$ variables $x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m$.

LECTURE

20.3. If we want to maximize a function $f : \mathbb{R}^m \to \mathbb{R}$ on the constraint $S = \{x \in \mathbb{R}^m \mid g(x) = c\}$, then both the gradients of f and g matter. We call two vectors v, w **parallel** if $v = \lambda w$ or $w = \lambda v$ for some real λ. The zero vector is parallel to everything. Here is a variant of Fermat:

> **Theorem:** If x_0 is a maximum of f under the constraint $g = c$, then $\nabla f(x_0)$ and $\nabla g(x_0)$ are parallel.

20.4. Proof by contradiction: assume $\nabla f(x_0)$ and $\nabla g(x_0)$ are not parallel and x_0 is a local maximum. Let T be the tangent plane to $S = \{g = c\}$ at x_0. Because $\nabla f(x_0)$ is not perpendicular to T we can project it onto T to get a non-zero vector v in T which is not perpendicular to ∇f. Actually the angle between ∇f and v is acute so that $\cos(\alpha) > 0$. Take a curve $r(t)$ in S with $r(0) = x_0$ and $r'(0) = v$. We have $d/dt f(r(0)) = \nabla f(r(0)) \cdot r'(0) = |\nabla f(x_0)||v|\cos(\alpha) > 0$. By linear approximation, we know that $f(r(t)) > f(r(0))$ for small enough $t > 0$. This is a contradiction to the fact that f was maximal at $x_0 = r(0)$ on S.
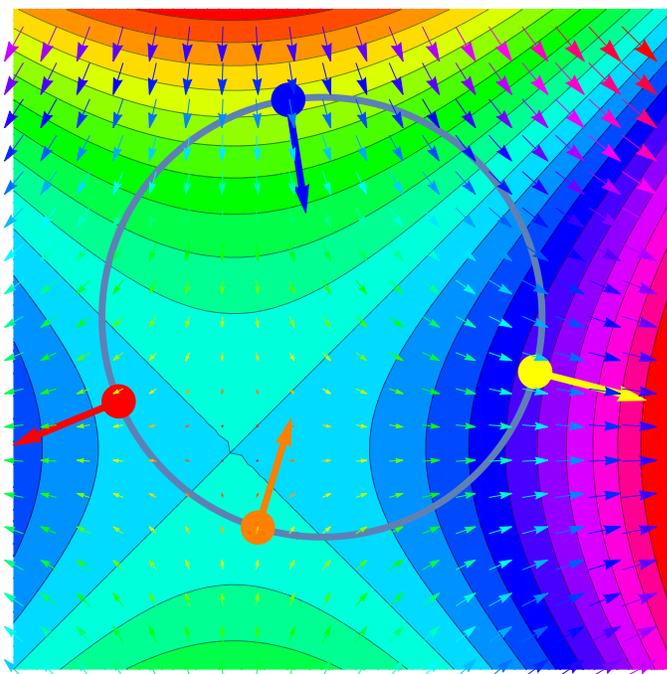

FIGURE 2.   A Lagrange problem

20.5. This immediately implies: (distinguish $\nabla g \neq 0$ and $\nabla g = 0$)

> **Theorem:** For a maximum of f on $S = \{g = c\}$ either the Lagrange equations $\nabla f(x_0) = \lambda \nabla g(x_0), g = c$ hold, or then $\nabla g(x_0) = 0, g = c$.

20.6. For functions $f(x, y), g(x, y)$ of two variables, this means we have to solve a system with three equations and three unknowns:

$$\begin{aligned} f_x(x_0, y_0) &= \lambda g_x(x_0, y_0) \\ f_y(x_0, y_0) &= \lambda g_y(x_0, y_0) \\ g(x, y) &= c \end{aligned}$$

20.7. To find a maximum, solve the Lagrange equations and add a list of critical points of g on the constraint. Then pick a point where f is maximal among all points. We don't bother with a second derivative test. But here is a possible statement:

$$\frac{d^2}{dt^2} D_{tv} D_{tv} f(x_0)|_{t=0} < 0$$

for all v perpendicular to $\nabla g(x_0)$, then x_0 is a local maximum.

20.8. Of course, the case of maxima and minima are analog. If f has a maximum on $g = c$, then $-f$ has a minimum at $g = c$. We can have a maximum of f under a smooth constraint $S = \{g = c\}$ without that the Lagrange equations are satisfied. An example is $f(x, y) = x$ and $g(x, y) = x^3 - y^2$ shown in Figure (3).
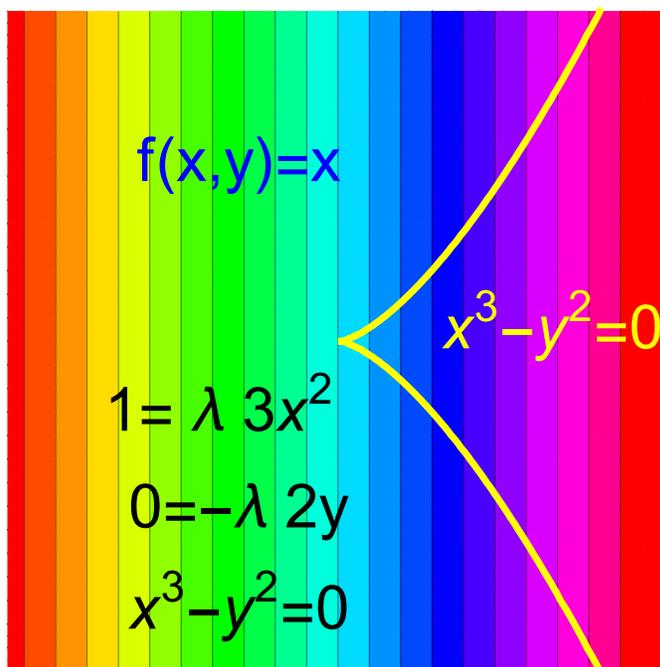
FIGURE 3. An example of a function, where the Lagrange equations do not give the minimum, here $(0, 0)$. It is a case, where $\nabla g = 0$.

20.9. The method of Lagrange can maximize functions f under several constraints. Lets show this in the case of a function $f(x, y, z)$ of three variables and two constraints $g(x, y, z) = c$ and $h(x, y, z) = d$. The analogue of the Fermat principle is that at a maximum of f, the gradient of f is in the plane spanned by ∇g and ∇h. This leads to the **Lagrange equations** for 5 unknowns x, y, z, λ, μ.

$$\begin{aligned}
f_x(x_0, y_0, z_0) &= \lambda g_x(x_0, y_0, z_0) + \mu h_x(x_0, y_0, z_0) \\
f_y(x_0, y_0, z_0) &= \lambda g_y(x_0, y_0, z_0) + \mu h_y(x_0, y_0, z_0) \\
f_z(x_0, y_0, z_0) &= \lambda g_z(x_0, y_0, z_0) + \mu h_z(x_0, y_0, z_0) \\
g(x, y, z) &= c \\
h(x, y, z) &= d
\end{aligned}$$

20.10. For example, if $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = x^2 + y^2 = 1$, $h(x, y, z) = x + y + z = 4$, then we find points on the ellipse $g = 1, h = 4$ with minimal or maximal distance to 0.
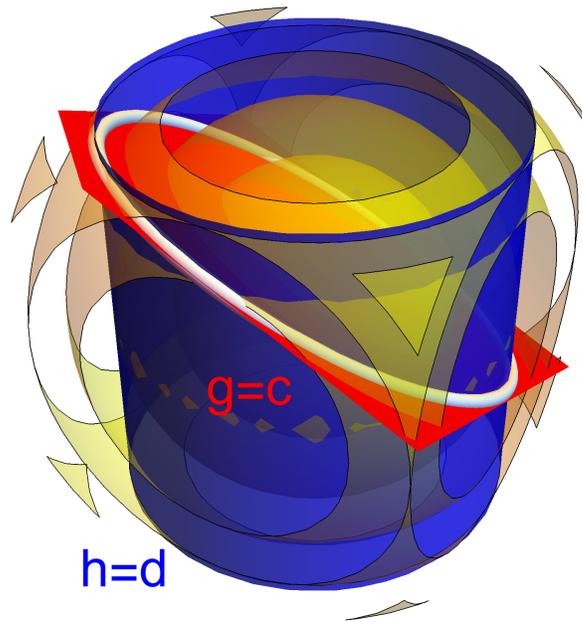

FIGURE 4. We see a situation where we try to maximize a function f under two constraints. In this case the intersection $g = c, h = d$ is an ellipse.

EXAMPLES

20.11. Problem: Minimize $f(x, y) = x^2 + 2y^2$ under the constraint $g(x, y) = x + y^2 = 1$. **Solution:** The Lagrange equations are $2x = \lambda, 4y = \lambda 2y$. If $y = 0$ then $x = 1$. If $y \neq 0$ we can divide the second equation by y and get $2x = \lambda, 4 = \lambda 2$ again showing $x = 1$. The point $x = 1, y = 0$ is the only solution.

20.12. Problem: Which cylindrical soda can of height h and radius r has minimal surface A for fixed volume V? **Solution:** We have $V(r, h) = h\pi r^2 = 1$ and $A(r, h) = 2\pi r h + 2\pi r^2$. With $x = h\pi, y = r$, you need to optimize $f(x, y) = 2xy + 2\pi y^2$ under the constrained $g(x, y) = xy^2 = 1$. We will do that in class.

20.13. Problem: If $0 \leq p_k \leq 1$ is the probability that a dice shows k, then we have $g(p) = p_1 + p_2 + \cdots + p_6 = 1$. This vector p is called a **probability distribution**. The **Shannon entropy** of p is defined as

$$S(p) = -\sum_{i=1}^{6} p_i \log(p_i) = -p_1 \log(p_1) - p_2 \log(p_2) - \cdots - p_6 \log(p_6) \ .$$

Find the distribution p which maximizes entropy S. **Solution:** $\nabla f = (-1 - \log(p_1), \ldots, -1 - \log(p_n))$, $\nabla g = (1, \ldots, 1)$. The Lagrange equations are $-1 - \log(p_i) = \lambda, p_1 + \cdots + p_6 = 1$, from which we get $p_i = e^{-(\lambda+1)}$. The last equation $1 = \sum_i \exp(-(\lambda + 1)) = 6 \exp(-(\lambda + 1))$ fixes $\lambda = -\log(1/6) - 1$ so that $p_1 = p_2 = \cdots = p_6 = 1/6$. It is the fair dice that has maximal entropy. Maximal entropy means **least information content**.

20.14. Assume that the probability that a physical or chemical system is in a state k is p_k and that the energy of the state k is E_k. Nature minimizes the **free energy**

$$F(p_1, \ldots, p_n) = -\sum_i [p_i \log(p_i) - E_i p_i]$$

if the energies E_i are fixed. The probability distribution p_i satisfying $\sum_i p_i = 1$ minimizing the free energy is called a **Gibbs distribution**. Find this distribution in general if E_i are given. **Solution:** $\nabla f = (-1 - \log(p_1) - E_1, \ldots, -1 - \log(p_n) - E_n)$, $\nabla g = (1, \ldots, 1)$. The Lagrange equation are $\log(p_i) = -1 - \lambda - E_i$, or $p_i = \exp(-E_i)C$, where $C = \exp(-1 - \lambda)$. The constraint $p_1 + \cdots + p_n = 1$ gives $C(\sum_i \exp(-E_i)) = 1$ so that $C = 1/(\sum_i e^{-E_i})$. The **Gibbs solution** is $p_k = \exp(-E_k)/\sum_i \exp(-E_i)$. [1]

20.15. If f is a quadratic function on $\mathbb{R}^m$ and g is linear that is $f(x) = Bx \cdot x/2$ with $B \in M(m, m)$ and if the constraint $g(x) = Ax = c$ is linear $A \in M(1, m)$, then $\nabla f(x) = Bx$ and $\nabla g(x) = A^T$. Lets call $b = A^T \in M(m, 1) \sim \mathbb{R}^m$. The Lagrange equations are then $Bx = \lambda b, Ax = c$. We see in general that for quadratic f and linear g, we end up with a **linear system of equations.**

20.16. Related to the previous remark is the following observation. It is often possible to reduce the Lagrange problem to a problem without constraint. This is a point of view often taken in economics. Let us look at it in dimension 2, where we extremize $f(x, y)$ under the constraint $g(x, y) = 0$. Define $F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$. The Lagrange equations for f, g are now equivalent to $\nabla F(x, y, \lambda) = 0$ in three dimensions.

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[1] This example is from Rufus Bowen, Lecture Notes in Math, 470, 1978

Homework

Problem 20.1: Find the cylindrical basket which is open on the top has has the largest volume for fixed area π. If x is the radius and y is the height, we have to maximize $f(x, y) = \pi x^2 y$ under the constraint $g(x, y) = 2\pi xy + \pi x^2 = \pi$. Use the method of Lagrange multipliers.

Problem 20.2: Given a $n \times n$ symmetric matrix B, we look at the function $f(x) = x \cdot Bx$. and look at extrema of f under the constraint that $g(x) = x \cdot x = 1$. This leads to an equation

$$Bx = \lambda x .$$

A solution x is called an **eigenvector**. The Lagrange constant λ is an **eigenvalue**. Find the solutions to $Bx = \lambda x, |x| = 1$ if B is a 2×2 matrix, where $f(x, y) = ax^2 + (b + c)xy + dy^2$ and $g(x, y) = x^2 + y^2$. Then solve the problem with $a = 4, b = 1, c = 1, d = 4$.

Problem 20.3: Which pyramid of height h over a square $[-a, a] \times [-a, a]$ with surface area is $4a\sqrt{h^2 + a^2} + 4a^2 = 4$ has maximal volume $V(h, a) = 4ha^2/3$? By using new variables (x, y) and multiplying V with a constant, we get to the equivalent problem to maximize $f(x, y) = yx^2$ over the constraint $g(x, y) = x\sqrt{y^2 + x^2} + x^2 = 1$. Use the later variables.

Problem 20.4: Motivated by the Disney movie "Tangled", we want to build a hot air balloon with a cuboid mesh of dimension x, y, z which together with the top and bottom fortifications uses wires of total length $g(x, y, z) = 6x + 6y + 4z = 32$. Find the balloon with maximal volume $f(x, y, z) = xyz$.

Problem 20.5: A **solid bullet** made of a half sphere and a cylinder has the volume $V = 2\pi r^3/3 + \pi r^2 h$ and surface area $A = 2\pi r^2 + 2\pi rh + \pi r^2$. Doctor Manhattan designs a bullet with fixed volume and minimal area. With $g = 3V/\pi = 1$ and $f = A/\pi$ he therefore minimizes $f(h, r) = 3r^2 + 2rh$ under the constraint $g(h, r) = 2r^3 + 3r^2 h = 1$. Use the Lagrange method to find a local minimum of f under the constraint $g = 1$.

Appendix: Data illustration: Cobb Douglas

20.17. The mathematician and economist **Charles W. Cobb** at Amherst college and the economist and politician **Paul H. Douglas** who was also teaching at Amherst, found in 1928 empirically a formula $F(K, L) = L^\alpha K^\beta$ which fits the **total production** F of an economic system as a function of the **capital investment** K and the **labor** L. The two authors used logarithms variables and assumed linearity to find α, β. Below are the data normalized so that the date for year 1899 has the value 100.

$Year$	K	L	P
1899	100	100	100
1900	107	105	101
1901	114	110	112
1902	122	118	122
1903	131	123	124
1904	138	116	122
1905	149	125	143
1906	163	133	152
1907	176	138	151
1908	185	121	126
1909	198	140	155
1910	208	144	159
1911	216	145	153
1912	226	152	177
1913	236	154	184
1914	244	149	169
1915	266	154	189
1916	298	182	225
1917	335	196	227
1918	366	200	223
1919	387	193	218
1920	407	193	231
1921	417	147	179
1922	431	161	240
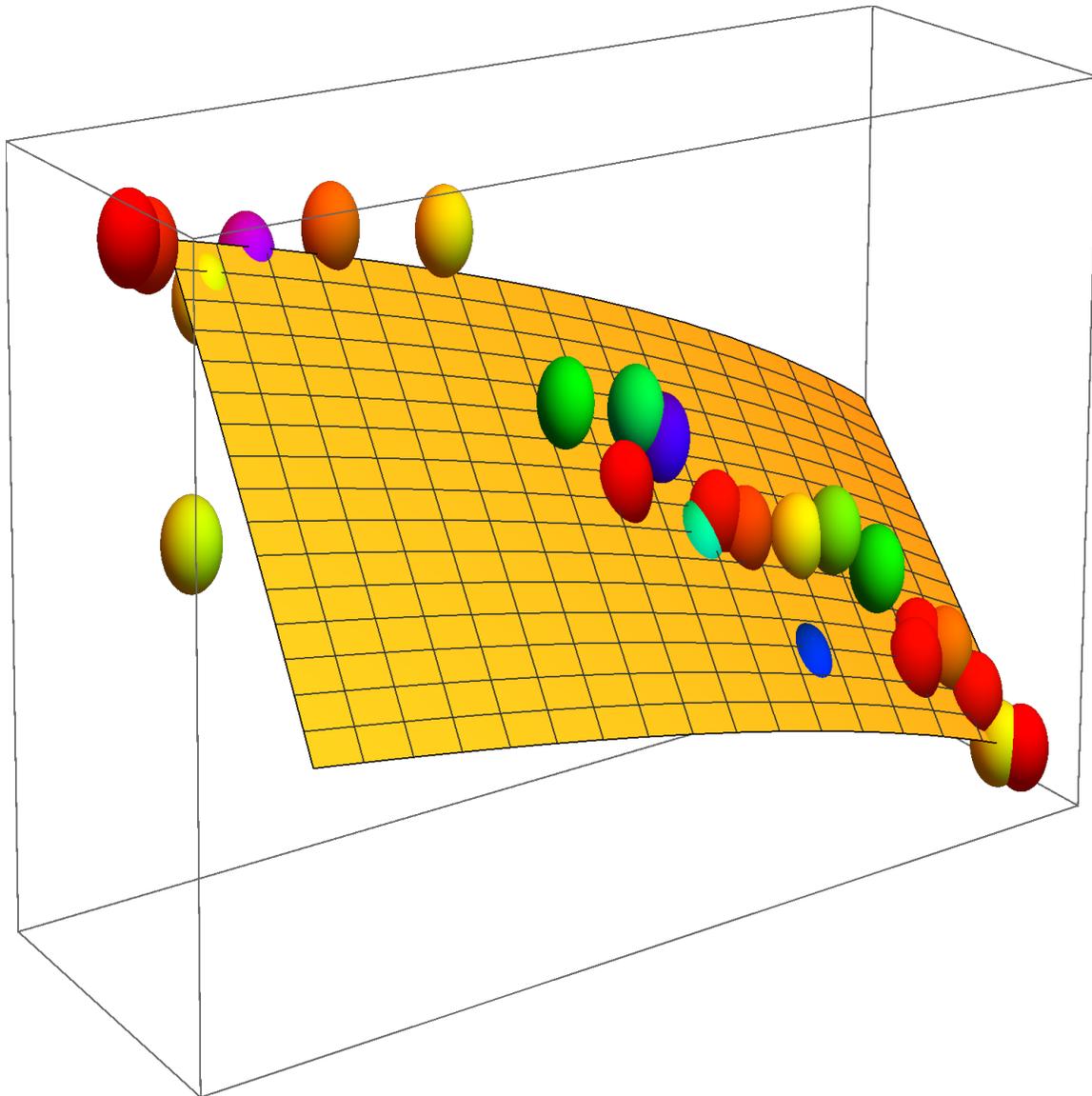
The graph of $F(L, K) = L^{3/4}K^{1/4}$ fits pretty well that data set. You can see in the data that there is an out-layer.

20.18. Assume that the labor and capital investment are bound by the additional constraint $G(L, K) = L^{3/4} + K^{1/4} = 50$. (This function G is unrelated to the function $F(L, K)$ as we are in a Lagrange problem.) Where is the production P maximal under this constraint? Plot the two functions $F(L, K)$ and $G(L, K)$.

Unit 21: Island mathematics

INTRODUCTION

21.1. We are on **spectacle island**, near the Boston Harbor and decide to understand the function $f(x, y)$ which gives the height at a position (x, y). The island has once been an "out of sight, out of mind" place occupied by a horse rendering plant (morphing horses to leather and fertilizers) and (related) a grease reclamation facility and a garbage disposal area. Barf! In the 1960ies, there came a renewal, to clean up the island and reclaim and reshape the island into a public park. It has now 5 miles of trails and fine beaches. The island has become an example of how to recreate a sustainable open space.

FIGURE 1. Spectacle island in Boston has two local hills and a saddle point of the height function $f(x, y)$. We assume that the function is 0 at the beaches. Can one say something general about the maxima, the minima and saddle points of an island? In this case we have two maxima and one saddle point.

21.2. When we were looking for maxima and minima this week, we were not concerned so much how many critical points there are or which combinations of critical points can occur. It turns out this is a very exciting topic. An other theme we can explore while vising an island is to look at the relation between its **area** and its **circumference** (the

length of the beaches). If the area is fixed, how short or how long can the total beach area become? It turns out that this is an infinite dimensional Lagrange problem but we can explore this also in finite dimensions when looking at polygonal island. We use the topic here to explore a bit some related areas to calculus.
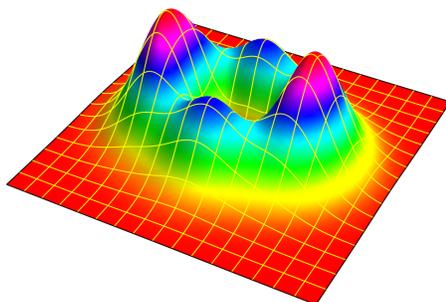

FIGURE 2.   An island with 4 mountain peaks, one sink and 4 mountain passes. The island theorem assuring that $peaks + sinks - passes = 1$ is satisfied.

MAXIMIZING AREA

21.3. An "island" a region in the plane $\mathbb{R}^2$ which is bound by a simple closed curve C which is continuous everywhere and differentiable everywhere except at a finite set of points. We allow for exceptions for continuity to allow simple polygons. Which island does have the maximal area if the length of the boundary is fixed? This is called the **isoperimetric problem**. If we look at the problem restricted to polygons with a fixed number n of vertices, we have a nice finite dimensional Lagrange problem.

21.4. Let us look at a **triangular island** $T(x, y)$ with vertices $(-1, 0), (1, 0), (x, y)$.

> **Problem A:** Assume the circumference $g(x, y) = 1 + \sqrt{(x+1)^2 + y^2} + \sqrt{(x-1)^2 + y^2}$ of the triangle is 3. What is the maximal area $f(x, y) = y/2$ we can get? Set up the Lagrange equations and solve them.

21.5. Here is a related problem from good old **Euclidean geometry**. If you should not know, look up "string method pins".

> **Problem B:** What points (x, y) in the plane satisfy $g(x, y) = 3$. This means, which points of the plane have the property that the sum of the distances to the two points is constant?

21.6. Solving the problem to find the n-gon with maximal area is a messy Lagrange problem. It can be done by a computer but there is a more elegant way:

21.7. You are on a treasure island G and have two locations A, C in G. You need to get from A to C but want to reach a point on the beach. It turns out that the solution is the **billiard law of reflection at the boundary**. Look at a triangle ABC, replace the curve with the tangent curve L at B. Reflect C at L to get a point C'. The path is shortest if the path ABC has the same length than the path connecting A with C' directly.
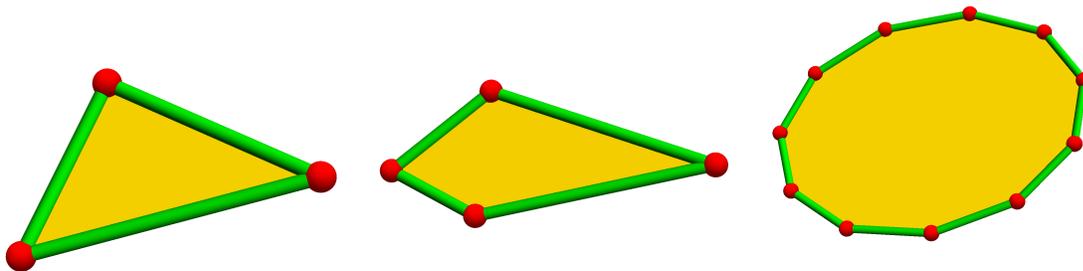
FIGURE 3. What polygon with fixed circumference has maximal area?

22. MOUNTAINS, SINKS AND MOUNTAIN PASSES

22.1. The next time you are cast away on an island, count the number m of mountain peaks, the number s of sinks and the number p of mountain passes. Make some experiments. You notice the following rule which is known as a special case of the Poincaré-Hopf theorem:

Theorem: maxima $+$ minima $-$ saddles $= 1$.

22.2. If you want to challenge yourself, see whether you can prove the island theorem by deformation. (This is probably too hard. Just enjoy the struggle!)

22.3.

FIGURE 4.   First an island with 2 mountain peaks and with 1 mountain pass. Then an island with 3 mountain peaks and 2 mountain passes. We see maxima + minima − saddles = 1.

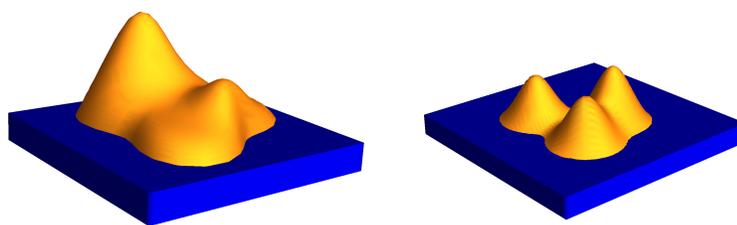FIGURE 5.   The Atafu atoll. Picture by NASA Johnson Space Center, 2009.
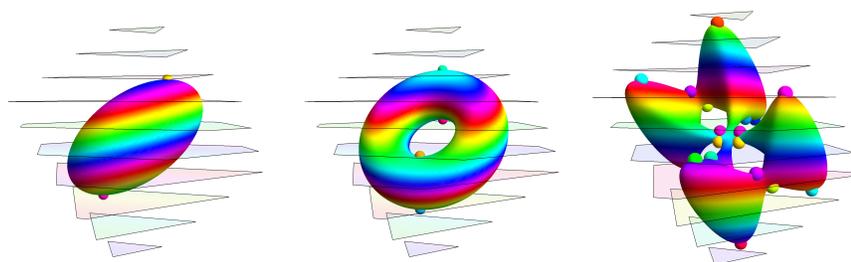
FIGURE 6.   If we place a surface $S : g = c$ in space and look at the restriction of a function $f(x, y, z)$ on S, we solve a Lagrange problem. In a Morse situation, the numbers maxima + minima − saddles add up to a number which only depends on the number of holes.

22.4. Let us look at the one-dimensional case, where we prove things easier. Assume the island is the interval $[a, b]$. Let f be a smooth function on $[a, b]$ which has the property that f is zero for $x \geq b$ and for $x \leq a$. We look at critical points of f in the interior (a, b) which are Morse, (meaning $f''(x) \neq 0$ at critical points), so that we only have only local maxima and minima as critical points. Let m be the number of maxima and s the number of minima (sinks). In order to prevent the island to be flooded, we also assume that the function f is positive for $x > a$, close to a and $x < b$ close to b.

Theorem: maxima − minima = 1.

Problem H: Verify that there is an odd number of critical points for a Morse function f which has as a support a finite interval $[a, b]$.

FIGURE 7. One-dimensional islands.

Problem I: Use a deformation argument to show that if there are $2k + 1$ critical points, we can reduce them to $2k - 1$ by merging a pair of neighboring maxima and minima

HOMEWORK

21.1 Assume $f : \mathbb{R} \to \mathbb{R}$ is a single variable Morse function on a circle. What is the relation between the number m of maxima on $[0, 2\pi)$ and the number of minima on $[0, 2\pi)$. Prove your statement. Hint: verify that for a Morse function, it is not possible that two maxima are adjacent.

21.2 If we look at maxima, minima and saddle points for a Morse function $f(x, y)$ defined on a sphere, find the island number maxima + minima − saddles there.

21.3 If we look at maxima, minima and saddle points for a function $f(x, y)$ defined on a doughnut. By looking at examples, find the island number maxima + minima − saddles there.

21.4 If we look at maxima, minima and saddle points on a brezel with two holes. Remember that you have constructed such a shape. By looking at examples, what is the island number maxima + minima − saddles there? The simplest situation is if the function is linear in $\mathbb{R}^3$.

21.5 Describe how to build a concrete function $f(x, y)$ of two variables which has two maxima and no minimum and no saddle point. (This is not possible on an island by the island theorem. It is possible however in the plane but such a function is not easy to describe. Discuss strategies with each other.)

Unit 22: Double integrals

INTRODUCTION

22.1. When integrating a continuous function $f(x, y)$ over a two-dimensional domain $R \subset \mathbb{R}^2$, we can use Riemann sums again like in one dimensions and get $\iint_R f(x, y) \, dA$. A special case of a continuous function is the function $f(x, y) = 1$. If we integrate $\iint_R 1 \, dA$ we get the **area**. Unlike in one dimensions, where a domain is just an interval, we can have much more interesting regions in two dimensions.
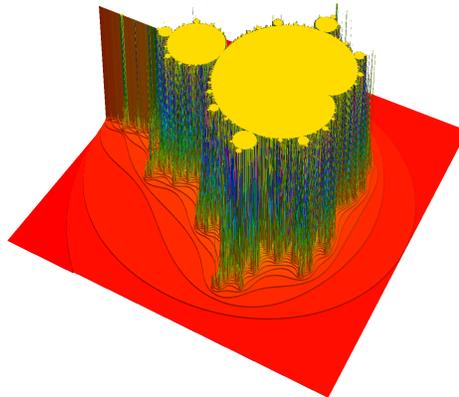
FIGURE 1. What is the area of the Mandelbrot set? The integral $\iint_R f(x, y) \, dA$ has an interpretation of the volume under the graph of f. If the height is constant 1, then the volume is the area $\iint_R 1 \, dA = |R|$ of the region R. For the Mandelbrot set we measure an area slightly above 1.5.

22.2. Integration in two dimensions is a good prototype. Knowing this **multi-dimensional** situation, will allow also to understand how to integrate in 3 or more dimensions. We will learn next week how to compute the area of a surface. But dimensional integrals also matter in higher dimensions: if we integrate a so called 2-form F over a two-dimensional surface, we get double integrals. An example of a 2-form is the **electro-magnetic field** also known as "light". String theorists work in higher dimensional spaces. The surface traced out by a moving string is a 2-dimensional surface called a

"world-sheet". Its surface area is called the **Nambu-Goto action** which plays the role of the length in classical mechanics. This is a double integral.

22.3. Like particle move on shortest paths called geodesics, strings move on paths in which the surface area is minimized. Not everybody has jumped onto the string theory wagon however and the theory lead to a dead end. We do not know yet. In any case, in the quest of understanding the basic building blocks of space and time and matter is exciting. We live in an interesting time where highly successive theories like the standard model (SM), quantum mechanics (QM) or general relativity (GR) match measurements with enormous precision. There are also other interesting theories which lack experimental verifications. Without doubt however, calculus and integration theory in particular will play an important role also in the future, whatever lies ahead.

<center>LECTURE</center>

22.4. Given a bounded region R in $\mathbb{R}^2$ and a continuous function $f(x, y) : R \to \mathbb{R}$, define the Riemann integral $I = \iint_R f(x, y) \, dA$ as the $n \to \infty$ limit of

$$I_n = \sum_{(i/n, j/n) \in R} f(\frac{i}{n}, \frac{j}{n})\frac{1}{n^2} \ .$$

The bounded region R is a defined as closed subset of $\mathbb{R}^2$ bound by finitely many differentiable curves $R = \{g_1 \leq c_1, \ldots g_k \leq c_k\}$. As already in one dimension, the definition is designed to be independent of an orientation chosen on R. We are integrating like summing up a spread sheet. Just add up all entries. To justify that the limit exists, we again can use the Heine-Cantor theorem which tells that f is continuous on R if and only if it is uniformly continuous. This means there are numbers $M_n \to 0$ such that if $|(x_1, y_1) - (x_2, y_2)| \leq 1/n$, then $|f(x_1, y_1) - f(x_2, y_2)| \leq M_n$.

> **Theorem:** For continuous f on a bounded region R, $\iint_R f \, dx dy$ exists.

22.5. Proof. In each cube $Q_{ij} = \{i/n \leq x \leq (i+1)/n, j/n \leq y \leq (j+1)/n\} \cap R$ define $a_{ij} = \min_{(x,y) \in Q_{ij}} f(x, y)$ and $b_{ij} = \max_{(x,y) \in Q_{ij}} f(x, y)$. Because the boundary was assumed to be given by a collection of curves which have finite total arc length L, the number of cubes Q_{ij} which intersect the boundary C is bounded by $4Ln$ (a curve of length 1 can maximally touch 4 squares). Define also $F = \max_{(x,y) \in R} |f(x, y)|$. We have with $K_n = 4LF/n$:

$$A_n - K_n \leq I_n \leq B_n + K_n \ ,$$

where $A_n = \sum_{i,j} a_{ij}/n^2$ and $B_n = \sum_{i,j} b_{ij}/n^2$ and K_n takes care of cubes Q_{ij} which intersect the boundary of R and so only contribute partially. Let I be the limsup of I_n. We have $B_n - A_n \leq M_n n^2/n^2 = M_n \to 0$ and $K_n \to 0$ as well so that $||I_n - I| \leq M_n + K_n \to 0$.

22.6. We rarely evaluate integrals using Riemann sums. Fortunately it is possible to reduce a double integral to single integrals. One can do that for **basic regions** which consist of two type of regions **"bottom to top" regions** $R = \{(x, y), \ a \leq x \leq b, c(x) \leq y \leq d(x)\}$ or **"left to right" regions** $R = \{(x, y), \ a(y) \leq x \leq b(y), c \leq y \leq d\}$. By cutting a general region into smaller pieces like intersecting with sufficiently small cubes $Q_{i,j}$ defined above, we can write any region as a union of such basic regions:

for large enough n, any $Q_{ij} \cap R$ us a basic region. Now we can define the integral in the first case as $\int_a^b [\int_{c(x)}^{d(x)} f(x,y)\, dy] dx$ and in the second case as $\int_c^d [\int_{a(y)}^{b(y)} f(x,y)\, dx] dy$. Is this the same? This is answered with Fubini, which we have already used. Let R be a rectangle $R = \{(x,y) \mid a \le x \le b, c \le y \le d\}$. Here is the **Fubini theorem**:
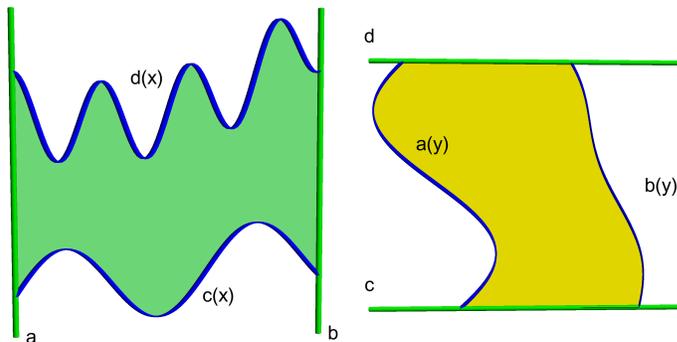

FIGURE 2.   "Bottom to top" and "left to right" regions.

Theorem: $\iint_R f(x,y)\, dA = \int_a^b [\int_c^d f(x,y)\, dy] dx = \int_c^d [\int_a^b f(x,y)\, dx] dy.$

22.7. Proof: first make a coordinate change to get $R = [0,1] \times [0,1]$, then cover R with n^2 cubes Q_{ij} of side length $1/n$. We have for every y a uniformly continuous function $x \to f(x,y)$ and for every x a uniformly continuous function $y \to f(x,y)$ and the constants M_n work for all: there is $M_n \to 0$ so that if $|x_1 - x_2| < 1/n$ and $|y_1 - y_2| < 1/n$, then $|f(x_1, y_1) - f(x_2, y_2)| \le M_n$. Now use the notation $A \sim_c B$ if $|A - B| \le c$ and get $\iint_R f(x,y) dA \sim_{M_n} \frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{n} \sum_{j=0}^{n-1} f(i/n, j/n) \sim_{2M_n} \frac{1}{n} \sum_{i=0}^{n-1} \int_0^1 f(i/n, y)\, dy \sim_{3M_n} \int_0^1 [\int_0^1 f(x,y)\, dy]\, dx$. Similarly, we can show $\iint_R f(x,y) dA \sim_{3M_n} \int_0^1 [\int_0^1 f(x,y)\, dx]\, dy$.

22.8. Without continuity, Fubini is false: the standard example is illustrated in Figure (3):

$$\frac{-\pi}{4} = \int_0^1 \int_0^1 \frac{(x^2 - y^2)}{(x^2 + y^2)^2}\, dy dx \ne \int_0^1 \int_0^1 \frac{(x^2 - y^2)}{(x^2 + y^2)^2}\, dx dy = \frac{\pi}{4}\ .$$

Proof. $\int (x^2 - y^2)/(x^2 + y^2)^2\, dx = -x/(x^2 + y^2)$, $\int (x^2 - y^2)/(x^2 + y^2)^2\, dy = y/(x^2 + y^2)$. so that $\int_0^1 (x^2 - y^2)/(x^2 + y^2)^2\, dx = -1/(1 + y^2)$ and $\int_0^1 (x^2 - y^2)/(x^2 + y^2)^2\, dy = 1/(1 + x^2)$.

22.9. Integrals in higher dimensions are defined in the same way. We will cover the three dimensional case in particular later. Lets just add the definition for now. Given a m dimensional region R in $\mathbb{R}^m$ and a continuous $f : \mathbb{R}^m \to \mathbb{R}$, using the **multi-index notation** $x = (x_1, \ldots, x_m)$, $dx = dx_1 dx_2 \cdots dx_m$ and $i/n = (i_1/n, i_2/n, \ldots, i_m/n)$ define

$$\int_R f(x) dx = \lim_{n \to \infty} \frac{1}{n^m} \sum_{\frac{i}{n} \in R} f(\frac{i}{n})\ .$$

A **region** is now a set $R = \{x \in \mathbb{R}^m \mid g_1(x) \le c_1, \ldots, g_k(x) \le c_k\}$ where g_k are smooth functions. It is called **bounded** if there exists $\rho > 0$ such that $R \subset \{|x| \le \rho\}$.
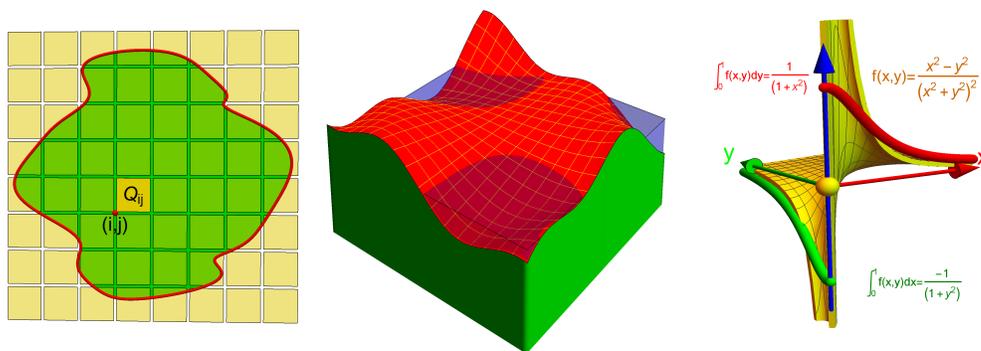
FIGURE 3. Integrating over a region via a Riemann integral. A double integral is a signed volume. Parts where $f < 0$ is negative volume. Fubini can fail, even if the two conditional integrals exist.

EXAMPLES

22.10. If $f(x, y) = 1$, then $\iint_R f(x, y) \, dxdy$ is the **area** of R. For example, if $\iint_{x^2+y^2 \leq 9} 8 \, dxdy = 8 \iint_{x^2+y^2 \leq 9} 1 \, dxdy = 8\text{Area}(R) = 72\pi$.

22.11. We know from single variable calculus that $\int_a^b f(x) \, dx$ is the **signed area** under the curve of f. For $f(x) \geq 0$, where it is the area, we can write this as $\int_a^b \int_0^{f(x)} 1 \, dydx$. Note that as we have defined the integrals, the equivalence would be wrong if $f(x)$ is negative somewhere. It is the double integral which is the correct notion of area. **Example:** The area of the region bounded by the curve $y = 1/(1 + x^2)$, the curve $y = 0$ and the curve $x = -1$ and $x = 1$ is $\int_{-1}^1 \int_0^{1/(1+x^2)} \, dydx = \arctan(x)|_{-1}^1 = \pi/2$.
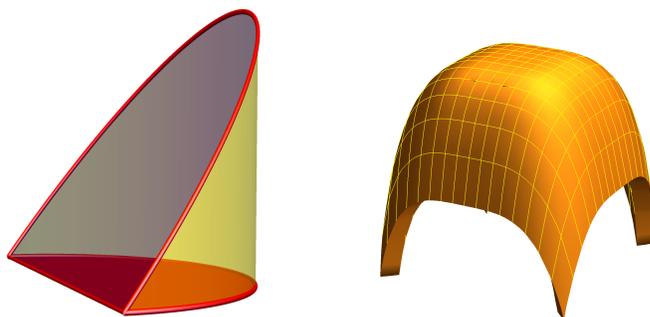
FIGURE 4.

22.12. The integral $\iint_R f(x, y) \, dxdy$ can be interpreted as the signed volume under the graph of f above the region R. Find the volume of the region bound by $z = 4 - 2x^4 - 2y^4$ and $z = 4 - 2x^2 - 2y^2$ and $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. Solution: $\int_0^1 \int_0^1 (4 - 2x^4 - 2y^4) - (4 - 2x^2 - 2y^2) \, dxdy = (4/15)^2$.

22.13. Problem. Find the area of a disc of radius a. Solution:

$$\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 1 \, dydx = \int_{-a}^a 2\sqrt{a^2 - x^2} \, dx \; .$$

Use **trig substitution** $x = a\sin(u)$, $dx = a\cos(u)$, to get

$$\int_{-\pi/2}^{\pi/2} 2\sqrt{a^2 - a^2\sin^2(u)}a\cos(u)\,du = \int_{-\pi/2}^{\pi/2} 2a^2\cos^2(u)\,du \ .$$

Using a double angle formula, this gives $a^2 \int_{-\pi/2}^{\pi/2} 2\frac{(1+\cos(2u))}{2}\,du = a^2\pi$. We will next time compute this much more effectively.

22.14. Problem. Let R be the triangle $\{1 \geq x \geq 0, 0 \leq y \leq x\}$. Evaluate $\int\int_R e^{-x^2}\,dxdy$. **Solution.** We can not evaluate the integral directly because e^{-x^2} has no anti-derivative given in terms of elementary functions. But we can write the integral as $\int_0^1 [\int_0^x e^{-x^2}\,dy]\,dx$

$$= \int_0^1 xe^{-x^2}\,dx = -\frac{e^{-x^2}}{2}\Big|_0^1 = \frac{(1 - e^{-1})}{2} \ .$$

HOMEWORK

Problem 22.1: Calculate the iterated integral $\int_0^1 \int_x^{2-x}(x^3 - y)\,dydx$ in two ways, once as a "left to right" and once as a "bottom to top" integral.

Problem 22.2: Find the integral

$$\int_0^1 \int_{\sqrt{y}}^{y^2} \frac{3x^7}{\sqrt{x - x^2}}\,dx\,dy \ .$$

Problem 22.3: a) Compute the area of the elliptical region bound by the ellipse $x^2/4^2 + y^2/9^2 = 1$ using trig substitution.
b) Now do this in general for an ellipse $x^2/a^2 + y^2/b^2 = 1$.
(It is the "hardest problem in geometry", according to the comedy-drama "Rushmore", a movie from 1998).

Problem 22.4: Find the integral

$$\int_0^{\pi^2} \int_{\sqrt{y}}^{\pi} \frac{\sin(x)}{x^2}\,dxdy \ .$$

Problem 22.5: Find the volume of the hoof solid $x^2 + y^2 \leq 1, 0 \leq z \leq x$. The hoof solid was considered by Archimedes already.

Appendix: Data illustration: Monte Carlo

22.15. Often, when we deal with real data, we do not have analytic expressions for the region or function we want to integrate. The Riemann integral has its limitations. In other branches of mathematics like in **probability theory**, a better integral is needed. Its definition is close to the Riemann integral which we have given as the limit $\int_{(x_k,y_l)\in R} f(x_k,y_l)\,\frac{1}{n^2}$, where $x_k = k/n, y_l = l/n$. The Lebesgue integral replaces the regularly spaced (x_k,y_l) grid with random points (x_k,y_l) and uses the same formula.

22.16. How do we find the area of Mandelbrot set

$$M = \{c = a + ib \in \mathbb{C} \in \mathbb{R}^2 \mid T_c(0)^n \text{ stays bounded }\},$$

where $T_c(z) = z^+c$. In real coordinates, this is the map $T_c(x,y) = (x^2 - y^2 + a, 2xy + b)$.
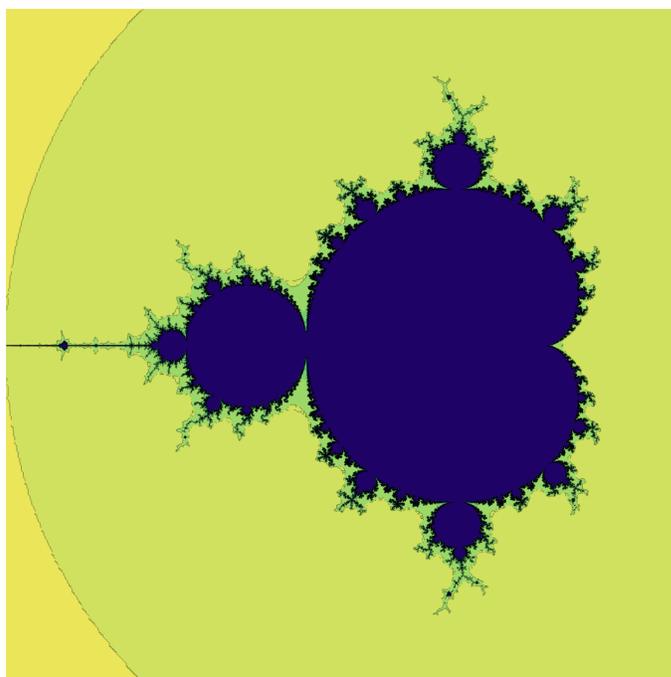
22.17. What is the area of the Mandelbrot set? We know it is contained in the rectangle $x \in [-2,1]$ and $y \in [-3/2, 3/2]$. We now just randomly shoot into this rectangle and see whether we are in the Mandelbrot set or not after 1000 iterations. Here is some Mathematica code which allows you to compute things. When we ran it, it gave a value of about 1.515.... More accurate measurements reported hint for a slightly smaller value like 1.506.... Others have given bounds $[1.50311, 1.5613027]$.

```
M=Compile[{x,y},Module[{z=x+I y,k=0},
    While[Abs[z]<2.&&k<1000,z=N[z^2+x+I y];++k];Floor[k/1000]]];
9*Sum[M[-2+3 Random[],-1.5+3 Random[]],{1000000}]/1000000
```

Unit 23: Substitution

Introduction

23.1. We have introduced a general notion of derivative dr of a function from r : $\mathbb{R}^m \to \mathbb{R}^n$. The determinant $|\det(\sqrt{dr^T dr}|$ was called the distortion factor. In the case of a map from $\mathbb{R}^n \to \mathbb{R}^n$ of the same dimension, the distortion factor is simply $|\det(dr)|$ because the now square matrix dr^T has the same determinant than dr and the determinant is multiplicative. The **first fundamental form** $g = dr^T dr$ is also called the **metric tensor**. In general relativity it plays an important role. Before we start, let us say that instead of using $r : R \to S$ as a coordinate change, we will use $\Phi : R \to S$, the reason being that r will be used in polar coordinates.
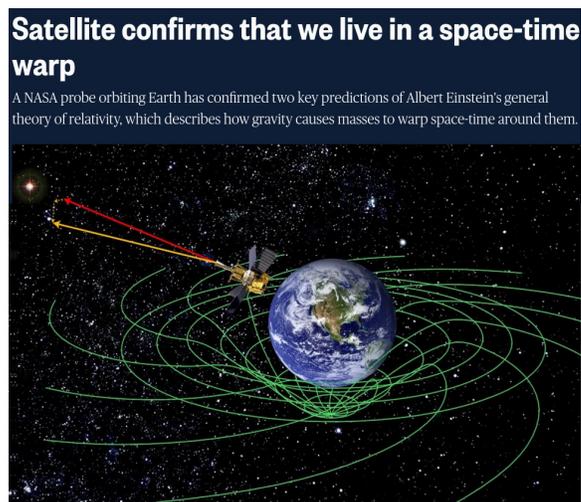
Figure 1. Coordinate changes even make it to main stream news. Here is a NBCnews page from 2011 reporting about "space-time warp. **Gravity probe B** (active between 2004 and 2010) carried two gyroscopes pointing to a star. The gyroscopes experienced small spin rotation changes matching the predictions of general relativity.

23.2. It describes a space in which distances are warped: it is matter in space that produces a coordinate change which changes the metric. How this happens is described by a complicated partial differential equation, the Einstein equations. We look here again at the distortion factor. The reason is that when we do integration in other

coordinates, the distortion factor comes in. We will learn here how to integrate in polar coordinates or integrate in spherical coordinates.

LECTURE

23.3. If $\Phi : R \to S$, $\begin{bmatrix} u \\ v \end{bmatrix} \to \begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix}$ is a **coordinate change**, then the **distortion factor** was defined as $|d\Phi| = |\det(d\Phi)|$, where

$$d\Phi(u,v) = \begin{bmatrix} \partial_u x(u,v) & \partial_v x(u,v) \\ \partial_u y(u,v) & \partial_v y(u,v) \end{bmatrix} .$$

The **change of variable theorem** is the same in all dimensions. In the following proof, we assume that Φ is C^2. Because of Heine-Cantor, we know there exists $M_n \to 0$ with $|\frac{d^2}{dt^2} \Phi(u_0 + tv, v_0 + tw)| \le M_n$ for $\sqrt{v^2 + w^2} \le 1/n$ and all $(u_0, v_0) \in R$. [1]

Theorem: $\iint_R f(\Phi(u,v)) \, |d\Phi(u,v)| du dv = \iint_S f(x,y) \, dx dy.$

23.4. Proof. Cover S with cubes Q_{ij} as in the last lecture. Then

$$\iint_S f(x,y) \, dx dy = \sum_{Q_{ij}} \iint_{Q_{ij} \cap S} f(x,y) \, dx dy \sim \sum_{i,j} f(\frac{i}{n}, \frac{j}{n}) \frac{1}{n^2} .$$

The transformed squares $\Phi(Q_{ij})$ are close to the **parallelograms** $d\Phi(Q_{ij})$ which have area $|d\Phi(i/n, j/n)|/n^2$. Now make a quadratic Taylor expansion $\Phi(x,y) = \Phi(x_0, y_0) + d\Phi(x_0, y_0)(x - x_0, y - y_0) + d^2\Phi(x_0, y_0)(x - x_0, y - y_0)^2/2$ at $(x_0, y_0) = (i/n, j/n)$, where $|d^2\Phi(x_0, y_0)(x - x_0, y - y_0)^2| \le M_n$. Let $F = \max_{(x,y) \in R}(|f(x,y)|)$. Applying in every direction, Taylor with remainder, we see

$$| \int_{\Phi(Q_{ij}) \cap S} f(x,y) \, dx dy - f(\Phi(\frac{i}{n}, \frac{j}{n})) |d\Phi(\frac{i}{n}, \frac{j}{n})| \frac{1}{n^2} | \le \frac{M_n F}{n^2} .$$

As the number of squares hitting R is bound by $An^2 + 4Ln$ where A is the area of R and L is the length of the boundary of R, the sum of the non-linear errors is therefore bound by $(An^2 + 4Ln)M_n F/n^2$ which goes to zero for $n \to \infty$. QED.

23.5. Here is an example: If $\Phi : R = [0,1] \times [0, 2\pi] \to S = \{x^2 + y^2 \le 1\}$ is given by $\Phi(r, \theta) = [r\cos(\theta), r\sin(\theta)]^T$. Then $d\Phi(r, \theta) = r$. If $f(x,y) = x^2 + y^2 = r^2$, then $\iint_R r^2 \, r dr \, d\theta = \iint_S x^2 + y^2 \, dx dy$. The first integral is $2\pi/4$.

23.6. Let $\Phi : [0,1] \times [0,1] \to [0,1] \times [0,1]$ be given as $\Phi(x,y) = (y,x)$. Now $\det(d\Phi) = -1$ and $|d\Phi| = 1$. While we usually could ignore talking about orientation, it is evident here that the integrals considered so far, we do not care about the orientation of the space. If the change of coordinates switches the orientation, the resulting integral does not change.

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[1] For the C^1 case, see J. Schwartz, Mathematical Monthly 61, 1954, or P.D. Lax, Monthly 108, 2001

23.7. The chain rule assures that combining two coordinate changes Φ, Ψ, gives a new coordinate change with $d(\Psi \circ \Phi)(x) = d\Psi(\Phi(x))d\Phi(x)$. For example if $\Psi(x, y) = [ax, by]^T$ and $\Phi(r, \theta) = [r\cos(\theta), r\sin(\theta)]^T$ changes into polar coordinates, then $\Psi(\Phi(r, \theta)) = [ar\cos(\theta), br\sin(\theta)]^T$. Now the image of $R = [0, 1] \times [0, 2\pi]$ is the ellipse $S = \{x^2/a^2 + y^2/b^2 \leq 1\}$ and the **area of the ellipse** is $A = \iint_R abr\, drd\theta$ because $\det(d\Phi) = r$ and $\det(d\Psi) = ab$. The result is $\int_0^1 \int_0^{2\pi} abr\, d\theta dr = \pi ab$.

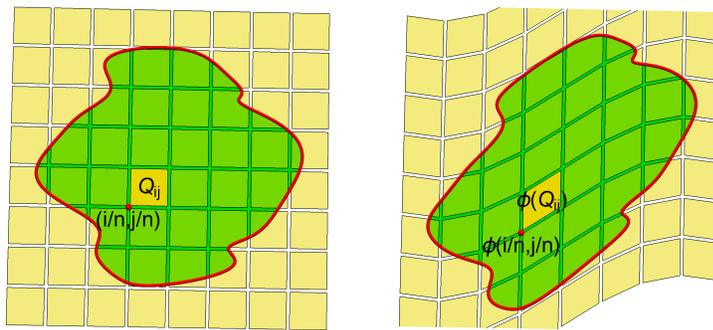FIGURE 2.   Coordinate change.

23.8. Preview: We will next week look at more general cases like $r : R \subset \mathbb{R}^2 \to \mathbb{R}^3$ of a parametrized surface, where the distortion factor is $|dr| = \sqrt{\det(dr^T dr)} = |r_u \times r_v|$ and the surface area is $\iint_R |r_u \times r_v|\, dudv = \iint_S 1\, dA$.

23.9. The theorem generalizes **substitution** $\int_c^d f(\Phi(x))|\Phi'(x)|\, dx = \int_a^b f(x)\, dx$ if $\Phi(c) = a$ and $\Phi(d) = b$. We usually insist that Φ is monotonically increasing and write $u = \Phi(x), du = \Phi'(x)dx$ to get computations like in $\int_0^{\sqrt{\pi/2}} \sin(x^2)2xdx = \int_0^{\pi/2} \sin(u)\, du$, where $\Phi(x) = x^2$. As a hack, one can extend the formula to the case when Φ can decrease in which case the $[a, b]$ interval becomes the negative $[b, a]$ interval with $a < b$. Example: Let $\Phi(x) = 2 - 2x$ which has $\Phi' = -2$, then $\int_{1/2}^1 (2-2x)^2|(-2)|dx = \int_0^1 x^2 dx$. In single variable calculus, one can also work with the negative sign case and compute $\int_1^{1/2}(2 - 2x)^2(-2)dx$ which works if $\int_1^{1/2} = -\int_{1/2}^1$ but this is **not compatible** with the defined Riemann integral: we use **"spread-sheet" summation** and do not distinguish whether we add up the function values from left to right or from right to left.

23.10. We can again look at the Fubini counter example $\iint_{x^2+y^2\leq 1}(x^2 - y^2)/(x^2 + y^2)^2\, dxdy = \int_0^1 \int_0^{2\pi} \cos(2\theta)/r\, d\theta dr = 0$. We can not change the order of integration as we can not integrate $\int_0^1 1/r\, dr$. The trouble also continues in the new coordinate system and it is even more dramatic.

23.11. If $\Phi : x \to Ax$ and $\Psi : x \to Bx$ are two linear coordinate changes then $\Psi \circ \Phi = BA$ is the matrix product and the chain rule tells $|d(\Psi \circ \Phi)| = |\det(AB)|$ which agrees with the product $|d\Psi||d\Phi| = |\det(A)||\det(B)|$. We can do the verification

of the **Cauchy-Binet** formula $\det(AB) = \det(A)\det(B)$ directly. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, then $AB = \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}$ and you can check the determinant formula.

23.12. Here is a famous open problem about coordinate changes. It is called the **Jacobian conjecture**. It deals with polynomial coordinate changes, where $x(u, v)$ and $y(u, v)$ are **polynomials** in u, v.

> **Conjecture:** If Φ is polynomial and $|d\Phi|$ is constant different from zero, then Φ has a polynomial inverse.

One knows that if the conjecture is false, then there exists a counter example with integer polynomials and Jacobian determinant 1. The conjecture is open since at least 1939. An example of a coordinate transformation with determinant 1 and integer polynomials are **Hénon maps** from lecture 16. If $\Phi([u, v]^T) = [x, y]^T = [u^2 - u^4 - v, u]^T$, then $\Phi^{-1}([x, y]^T) = [y, y^2 - y^4 - x]^T$.

EXAMPLES

23.13. Problem: What is the area of the image $S = \Phi(R)$ if $\Phi([u, v]) = [u^2 - v^2 + 1, 2uv + 2]^T$ and $R = \{1 \le u \le 3, 0 \le v \le 1\}$. (This is $\Phi(z) = z^2 + c$ with $c = 1 + 2i$ in the complex). We have $d\Phi(u, v) = \begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix}$ and $|d\Phi(u, v)| = 4u^2 + 4v^2$. We see from the change of variables formula that the area is $\int_0^1 \int_1^3 4u^2 + 4v^2 \, dudv = 112/3$.

23.14. Problem: What is the **moment of inertia** $\iint_R x^2 + y^2 \, dxdy$, where R is the polar region given in polar coordinates as $r \le 2 + \sin(3\theta)$. Solution: using the polar coordinate change of variables Φ with $|d\Phi| = r$, we get $\int_0^{2\pi} \int_0^{2+\sin(3\theta)} r^2 \, r \, drd\theta = \int_0^{2\pi} (2 + \sin(3\theta))^4/4 \, d\theta$. We explain in class how to get the answer $227\pi/4$ quickly.

23.15. Problem: Here is a famous problem. It is so popular, that it even made it to Hollywood: compute $\iint_{\mathbb{R}^2} e^{-x^2 - y^2} \, dxdy$. Solution: this problem looks difficult at first as we can not integrate with respect to x or y. The function e^{-x^2} has no elementary anti-derivative. This **improper integral** is doable in polar coordinates as it is $\int_0^{2\pi} \int_0^\infty e^{-r^2} r \, dr \, d\theta = \pi$. It is the inner part $\int_0^\infty e^{-r^2} r \, dr$ which is an improper integral. One deals with this by approximation. For every finite L we have $\int_0^L e^{-r^2} r \, dr = -e^{-r^2}/2|_0^L = 1/2 - e^{-L^2}/2$ This converges nicely to $1/2$ for $L \to \infty$. It follows (and that is the punch line) that $\int_{-\infty}^\infty e^{-x^2} \, dx = \sqrt{\pi}$.

Problem 23.1: Given a disk $R = \{x^2 + y^2 \leq 1\}$, we can make this into a probability space and define the **expectation** of a function f as $\mathrm{E}[f] = \iint_R f \, dxdy/\pi$. The expectation of the random variables $f(x, y) = x^n$ are examples of **moments**. Find $\mathrm{E}[x], \mathrm{E}[x^2], \mathrm{E}[x^3]$ and $E[x^4]$.

Problem 23.2: What is the volume of the solid bound by $z = f(x, y) = x^2 + y^2$ and $z = g(x, y) = 8 - x^2 - y^2$? You can write this as a double integral $\iint_R g(x, y) - f(x, y) \, dxdy$ over a suitable region.

Problem 23.3: The fidget spinner is so "2017" now. What is hot now is the **math 22 spinner** with 23 bearings! What is the moment of inertia $\int \int_G x^2 + y^2 \, dxdy$ of the **math 22 fidget spinner region** G given in polar coordinates as $1/2 \leq r \leq 2 + \cos(22\theta)$. To keep our bearings, we do not count the bearings.

Problem 23.4: Biologist **Piet Gielis** once patented polar regions in order to use them to describe biological shapes like cells, leaves, starfish or butterflies. Don't worry about violating patent laws when finding the area of the following butterfly $r(t) \leq |8 - \sin(t) + 2\sin(3t) + 2\sin(5t) - \sin(7t) + 3\cos(2t) - 2\cos(4t)|$. (It can produce butterflies in your stomach but there are some tricks to do that fast. Relax with the Math 22 fidget spinner for example!)
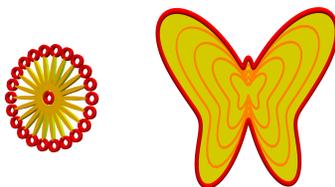
FIGURE 3. The math 22 spinner and the butterfly.

Problem 23.5: a) Prove the Jacobian conjecture for linear maps $\Phi(x) = Ax$, where A is a 2×2 matrix.
b) Find a linear coordinate change $\Phi(x, y)$ for which the Jacobian determinant is 1. It should be non-trivial in the sense, that we don't just want a diagonal matrix $d\Phi$.
c) Find a counter example of the Jacobian conjecture for cubic polynomials (just kidding). Find an example for the Jacobian conjecture where both polynomials are not linear!

Unit 24: How to solve: Literature Samples

Seminar

24.1. In this seminar, we look a bit around in the literature and collect problem solving strategies. We have seen already a few methods:

> **Already seen principles**
> 1. Induction (Theorem on unique row reduced echelon form)
> 2. Contradiction (Clairaut theorem)
> 3. Deformation (Hopf Umlaufsatz)
> 4. Invariant (Morse indices on island)

24.2. We will introduce a few more principles and tips and take the opportunity to introduce a bit the literature. We look at 4 books:
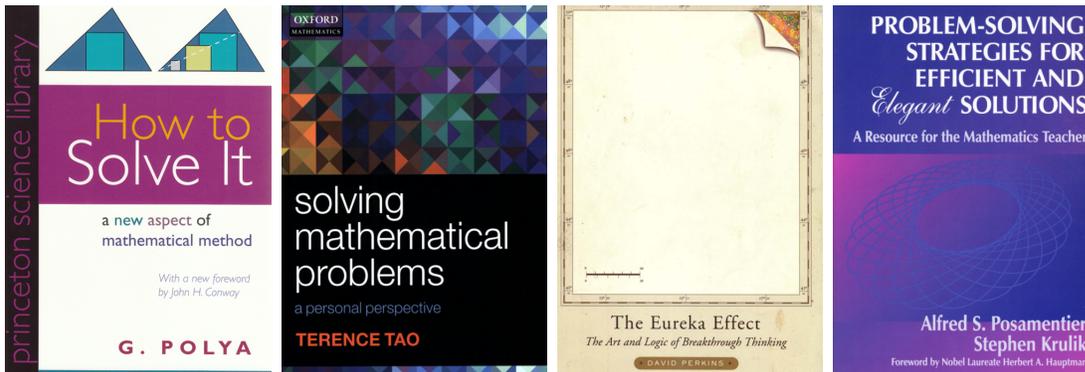
FIGURE 1. 4 Superstar books: Polya: How to Solve it. Tao: Solving Mathematical Problems, Perkins: The Eureka effect, Posamentier-Krulik: Problem solving strategies.

24.3. The mother of all problem solving books is Polya's "How to solve it" which was published in 1945. If you read and absorb this book, you immediately get measurably stronger in math. Still after more than 70 years, it is the best. Here are the now famous **Polya principles**:

> **Polya principles**
> 1. **Understand** the problem: unknowns, data, draw figure.
> 2. Devise a **plan**: similar or related problem?
> 3. **Carry out** the plan: check each step.
> 4. **Examine** the solution: can other problems be solved as such?

24.4. This sounds a bit like "open the door, step through the door, close the door" advise to "how to exit the house". But it is amazing to see the power in a method. Why is it powerful? Because if one sees a harder problem the first time, one is totally lost. (Proof: if not, then the problem was easy ....) Where do we start? This is where it is good already to have a guide telling you: well, just first start to understand the problem.

24.5. Here is an example of a problem in geometry which is mentioned in Polya's book. The problem is featured even on the cover of some later editions of the book.

> **Problem A:** Inscribe a square Q in a triangle T so that two vertices of Q on the base of T and the other sides of T each contains a vertex of Q.

24.6. An here is another problem from Polya, slightly reformulated. Work out also this problem using the Polya principles:

> **Problem B:** Water is flowing with a constant rate of one cubic meter per second into a conical vessel $x^2 + y^2 = z^2, z \geq 0$. At which rate is the water level rising if the water depth is z meters?

24.7. The second best book in our collection is "Solving mathematical problems" by Terrence Tao. Why? Like Polya, also Tao has proven new important theorems (many as a single author) and so got some street cred. Here are some problems from his book:

> **Problem C:** An integer n has the same last digit than n^5.

> **Problem D:** If k is a positive odd number, then $1^k + 2^k + ... + n^k$ is divisible by $n + 1$.

24.8. Tao calls the following identity "his favourite algebraic identity". We have done the case of the sum of the first n squares in a practice exam.

> **Problem E:** $1^3 + 2^3 + ... + n^3 = (1 + 2 + 3 + .. + n)^2$.

24.9. Tao does not give a formal list of strategies, but explains in an example on page 4 the following principles. We paraphrase here these "deformation principles":

> **Tao's deformation principles**
> a. Consider special, extreme or degenerate cases.
> b. Solve a simplified version of the problem
> c. Formulate a conjecture
> d. Derive intermediate steps which would get it.
> e. Reformulate, especially try contraposition.
> f. Examine solutions of similar problems
> g. Generalize the problem

24.10. The book of Perkins analyses skillfully the mechanisms of break through ideas. It distills the following mechanism for break through ideas. It captures it pretty well, since problems which are solved quickly rarely cover new ground.

> **Perkins**
> 1. Long search. 99 percent perspiration. Work for years or decades.
> 2. Little apparent progress. Many failures.
> 3. A precipitating event. Maybe external circumstances.
> 4. A cognitive snap. Usually in a flash. Eureka!
> 5. Transformation. Flesh it out. Consequences.

24.11. The following exercise is from Perkin's book. Try to solve it yourself and also keep track on how you pursue the task to solve the problem.

> **Problem F:** Someone brings an old coin to a museum director and offers it for sale. The coin is stamped 540 B.C.E. Instead of considering the purchase, the museum director calls the police. Why?

24.12. If this was too easy (experiments show that some people can answer it very quickly. For others it takes longer), try this one, also from Perkins:

> **Problem G:** You are driving a jeep through the Sahara desert. You encounter someone lying face down in the sand, dead. There are no tracks anywhere around. There has been no wind for days to destroy tracks. You look into the pack on the person's back. What do you find?

24.13. The book of Posamentier and Krulik is more intended for the teacher and less for the research mathematician. It goes through the following principles

> **Posamentier-Krulik**
> 1. Reason logically   2. Recognize patterns
> 3. Work backwards   4. Adopt different view
> 5. Consider extreme cases   6. Solve simpler problems
> 7. Organize data   8. Make a picture
> 9. Account all possibilities   10. Experiment, guess and test

24.14. Here is a strategy which often occurs: "make it more general". In the book "Posamentier-Krulik: Problem-Solving-Strategies in mathematics" for example is the problem:

> **Problem H:** We have a 5×5 seating arrangement of students. The teacher wants every student to change place and move to a seat to the left, right, front or left. It it possible? Solve this problem by looking first at smaller classrooms like 2×2 or 3×3 or 2×3. In which cases is it possible?

24.15. Once you have an idea, prove the statement.

HOMEWORK

> **24.1** A nursery rhyme is the riddle "As I was going to St. Ives, I met a man with seven wives, Each wife had seven sacks, Each sack had seven cats, Each cat had seven kits: Kits, cats, sacks, and wives, How many were there going to St. Ives?" Pretend not to know the answer, solve the riddle and follow the Polya principle. The rhyme was inspired by one of the oldest problems texts in math, the Rhind Papyrus. But it was a more serious question which translates: "how many kits came from St Ives"?

> **24.2** (Tao) The perpendicular bisectors in a triangle meet in a point.

> **24.3** (Tao). Find all triangles for which the length have an arithmetic progression $a, a + d, a + 2d$.

> **24.4** Here are a few children riddles. We hope you don't know all of them (if you know the answer there is little benefit). Keep a log of how you search for an answer: a) I'm tall when I'm young and I'm short when I'm old. What am I? b) What gets wetter and wetter the more it dries? c) What can run but can't walk? d) What is full of holes and still holds water?

> **24.5** In the **15 puzzle** (invented in 1874 by **Noyes Palmer Chapman**) each the numbers $1 - 15$ are arranged in a 4×4 grid. There is one hole 0 left. The task is to reorder a scrambled puzzle so that all numbers are in order and 0 at the very bottom right. The player can switch 0 with a neighboring piece. **Sam Loyd** suggested to start with stone 14 and 15 switched. and offered 1000 dollars for a solution. Prove that one can not win the prize.

Unit 25: Solids

INTRODUCTION

25.1. 1-dimensional objects are curves and 2-dimensional objects are regions or surfaces. In dimension 3, we deal with **solids**. The simplest solids imaginable are the cube or the spherical ball. Solids in three dimensional space are usually drawn by plotting their boundary surfaces. A solid polyhedron for example is bound by planes. The first figure shows the solid bound by hyperboloids. It is quite a challenge to compute its volume. [1]
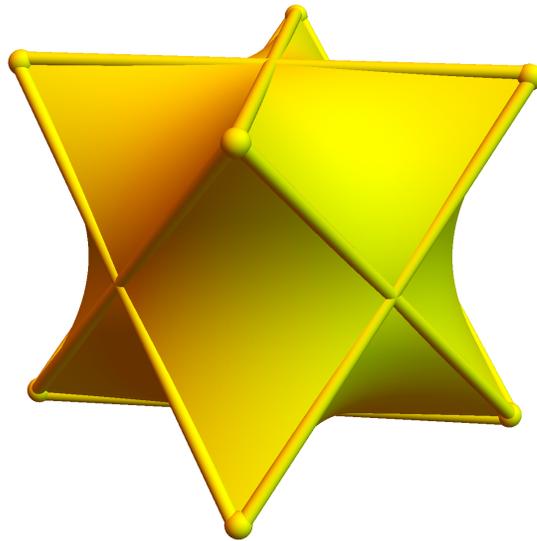
FIGURE 1. The "Archimedes revenge problem" asks to prove that $E : x^2+y^2-z^2 \leq 1, y^2+z^2-x^2 \leq 1, z^2+x^2-y^2 \leq 1$ has $\mathrm{Vol}(E) = \log(256)$.

25.2. While curves C have **length** and regions S have **area**, three dimensional solids E have **volume**. We will in the next lecture look at surface area $\int \int_S 1 \ dS$. In this lecture we look at volume $\int \int \int_E 1 \ dV$

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[1]Archimedes Revenge, first appeared in Math S21a exam, Harvard Summer School, 2017

LECTURE

25.3. A **basic solid** R in $\mathbb{R}^n$ is a bounded region enclosed by finitely many surfaces $g_i(x_1, \cdots, x_n) = c_i$. A **solid** is a finite union of such basic solids. We focus here mostly on $n = 3$. A 3D integral $I = \iiint_R f(x, y, z)\ dxdydz$ is defined in the same way as a limit of a Riemann sum I_n which for a given integer n is defined as

$$I_n = \frac{1}{n^3} \sum_{(i/n, j/n, k/n) \in R} f(\frac{i}{n}, \frac{j}{n}, \frac{k}{n})\ .$$

The convergence is proven in the same way. The boundary contribution can be neglected in the limit $n \to \infty$. If $\Phi : R \to E$ is a parametrization of the solid, then

Theorem: $\iiint_R f(u, v, w)|d\Phi(u, v, w)|dudvdw = \iiint_E f(x, y, z)\ dxdydz$
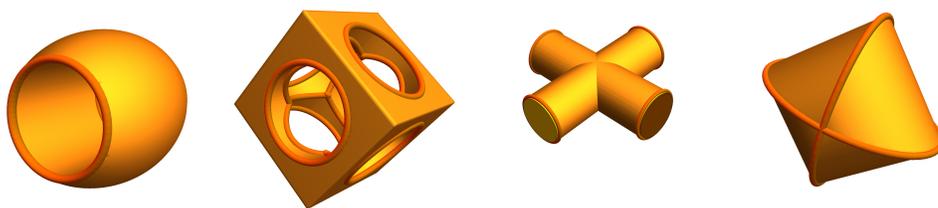

FIGURE 2. Solids in $\mathbb{R}^3$ are sets which are unions of solids bound by smooth surfaces. The second solid appears in homework 25.3, the last in 25.2

25.4. If $f(x, y, z)$ is constant 1, then $\iiint_E f(x, y, z)\ dxdydz$ is the **volume** of the solid E. For a cone $x^2 + y^2 \leq z^2, 0 \leq z \leq 1$, we can write $\iiint 1\ dzdxdy = \iint_R 1 - \sqrt{x^2 + y^2}\ dxdy$, where R is the unit disc. Its volume is $\pi - 2\pi/3 = \pi/3$. For the unit sphere $x^2 + y^2 + z^2 \leq 1$ for example, we can write $\iiint_E 1\ dzdxdy = \iint_R 2\sqrt{1 - x^2 - y^2}\ dxdy$, where R is the unit disc $x^2 + y^2 \leq 1$. In polar coordinates, we get $\int_0^{2\pi} \int_0^1 2\sqrt{1 - r^2}r\ drd\theta = 4\pi/3$. We can also use spherical coordinates $\Phi([\rho, \phi, \theta]) = [\rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi)]$, where $|d\Phi| = \rho^2 \sin(\phi)$. The volume is $\int_0^{2\pi} \int_0^\pi \int_0^1 \rho^2 \sin(\phi)\ d\rho d\phi d\theta = 4\pi/3$.

25.5. There are two basic strategies to compute the integral: the first is to slice the region up along a line like the z-axis then form $\int_a^b \iint_{R(z)} f(x, y, z)dxdydz$. To get the volume of a cone for example, integrate $\int_0^1 [\iint_{R(z)} 1dxdy]dz$. The inner double integral is the area of the slice which is πz^2. The last integral gives $\pi/3$. A second reduction is to see the solid sandwiched between two graphs of a function on a region R, then form $\iint_R [\int_{g(x,y)}^{h(x,y)} f(x, y, z)\ dz]dxdy$. In the cone case, we have for R the disc of radius 1. The lower function is $g(x, y) = \sqrt{x^2 + y^2}$ the upper function is 1. We get $\iint_R [1 - \sqrt{x^2 + y^2}]\ dxdy$, a double integral which best can be computed using polar coordinates: $\int_0^{2\pi} \int_0^1 (1 - r)rdrd\theta = 2\pi(1/2 - 1/3) = \pi/3$. Burgers and fries!
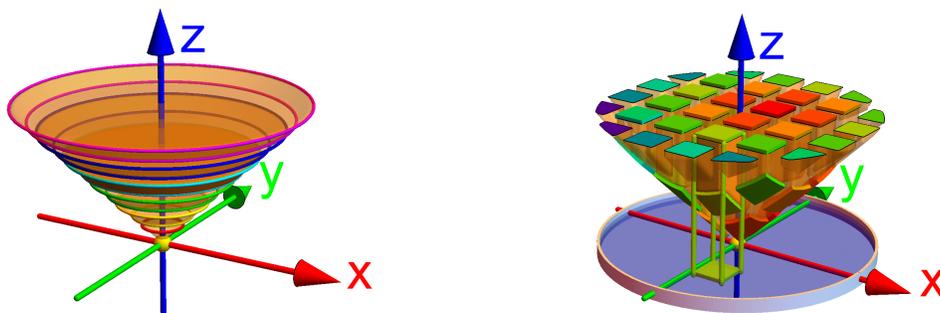
FIGURE 3. The "burger and fries methods" to compute triple integral. The first reduces to a single integral, the second to a double integral.

25.6. We have seen in the theorem the coordinate change formula if $\Phi : R \to E$ is given. For **spherical coordinates** $\Phi([\rho, \phi, \theta]) = [\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)]$, we have $\boxed{|d\phi| = \rho^2 \sin(\phi)}$. For **cylindrical coordinates**, the situation is the same as for polar coordinates. The map $\Phi([r, \theta, z]) = [r\cos(\theta), r\sin(\theta), z]$ produces $\boxed{|d\Phi| = r}$.

25.7. Let us find the integral $\iiint_E 1 \, dxdydz$, where $E = \{x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1\}$ is a **solid ellipsoid**. The most comfortable way is to introduce another coordinate change $\Psi([x, y, z]) \to [ax, by, cz]$ which maps the solid sphere S to to the solid ellipsoid E. Then take the spherical coordinate map $\phi : R \to S$, where $R = \{(\rho, \phi, \theta) \mid 0 \leq \rho \leq 1, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$. Now $\Psi \circ \Phi : R \to E$ is a coordinate change which maps R to the ellipsoid. By the chain rule, the distortion factor is $|d\Psi||d\Phi| = abc\rho^2\sin(\phi)$. The integral is $abc(1/3)(2\pi)\int_0^\pi \sin(\phi) \, d\phi = \boxed{(4\pi/3)(abc)}$.

25.8. In order to compute the volume of a **solid torus**, we can introduce a special coordinate system $\Phi([r, \psi, \theta]) = [(b + ar\cos(\psi))\cos(\theta), (b + ar\cos(\psi))\sin(\theta), a\sin(\psi)]$. The solid torus E is then the image of the cuboid $\{(r, \psi, \theta) \mid 0 \leq r \leq 1, 0 \leq \psi \leq 2\pi, 0 \leq \theta \leq 2\pi\}$. The determinant is $|d\Phi| = a^2\cos^2(s)(b + ar\cos(s))$. Integration over the cuboid gives the volume $(2\pi b)(\pi a^2)$.

<div align="center">EXAMPLES</div>

25.9. To find $\iiint_E f \, dV$ for $E = \{0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$ and $f(x, y, z) = 24x^2y^3z$, set up the integral $\int_0^1$ $\int_0^1$ $\int_0^1 24x^2y^3z \, dz$ dy dx. Start with the core $\int_0^1 24x^2y^3z \, dz = 12x^3y^3$, then integrate the middle layer, $\int_0^1 12x^3y^3 \, dy = 3x^2$ and finally handle the outer layer: $\int_0^1 3x^2dx = 1$.

25.10. To find the **moment of inertia** $I = \iiint_E x^2 + y^2 \, dV$ of a sphere $E = \{x^2 + y^2 + z^2 \leq L^2\}$, we use **spherical coordinates**. We know that $x^2 + y^2 = \rho^2 \sin^2(\phi)$ and the distortion factor is $\rho^2 \sin(\phi)$. We have therefore

$$I = \int_0^{2\pi} \int_0^\pi \int_0^L \rho^2 \sin^2(\phi) \rho^2 \sin(\phi) \, d\rho d\phi d\theta = 8\pi L^5/15 \; .$$

We will see some details in class. If we rotate the sphere around the z-axis with angular velocity ω, then $I\omega^2/2$ is the **kinetic energy** of that sphere. **Example:** the moment of inertia of the earth is $8 \cdot 10^{37} \text{kgm}^2$. With an angular velocity of $\omega = 2\pi/\text{day} = 2\pi/(86400s)$, this rotational kinetic energy is $8 \cdot 10^{37} \text{kgm}^2/(7464960000s^2) \sim 10^{29} J \sim 2.5 \cdot 10^{24} \text{kcal}$.

25.11. Problem: Find the volume E of the intersection of $x^2 + y^2 \leq 1$, $x^2 + z^2 \leq 1$ and $y^2 + z^2 \leq 1$. **Solution:** look at 1/16'th of the body given in cylindrical coordinates $0 \leq \theta \leq \pi/4, r \leq 1, z > 0$. The roof is $z = \sqrt{1 - x^2}$ because above the "one eighth disc" R only the cylinder $x^2 + z^2 = 1$ matters. The polar integration problem

$$16 \int_0^{\pi/4} \int_0^1 \sqrt{1 - r^2 \cos^2(\theta)} r \, dr d\theta$$

has an inner r-integral of $(16/3)(1 - \sin(\theta)^3)/\cos^2(\theta)$. Integrating this over θ can be done by integrating $f(x) = (1 - \sin(x)^3) \sec^2(x)$ by parts (using $\tan'(x) = \sec^2(x)$) leading to the anti-derivative $-\cos(x) + \sec(x) + \tan(x)$ of f. The result is $16 - 8\sqrt{2}$.

25.12. Problem: A **pencil** E, a hexagonal cylinder of radius 1 above the xy-plane is cut by a sharpener below the cone $z = 10 - r$. What is its volume? Solution: we consider one sixth of the pen where the base is the polar region $0 \leq \theta \leq 2\pi/6$ and $r(\theta) \leq \sqrt{3}/(\sqrt{3}\cos(\theta) + \sin(\theta))$. The pen's back is $z = 0$ and the sharpened part is $z = 10 - r$.

$$\int_0^{\pi/3} \int_0^{\sqrt{3}/(\sqrt{3}\cos(t)+\sin(t))} \int_0^{10-r} 1 \, r \, dz dr d\theta \; .$$

The integral can be computed and is a bit messy $(29 - 3\text{arctanh}(2 - \sqrt{3}))/(3\sqrt{3})$. [2]
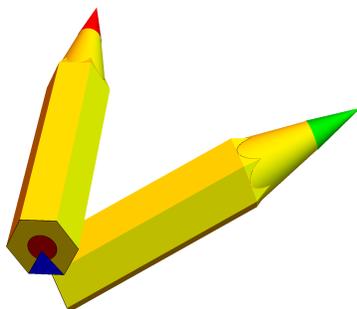
FIGURE 4.   The pen problem

The homework is combined in Unit 26.

Unit 26: Surface area

INTRODUCTION

26.1. We have looked at maps $r : R \to S$ in the context of coordinate changes and also in full generality, in the case when R is a subset of $\mathbb{R}^m$ and S is a subset of $\mathbb{R}^n$. We have learned that the Jacobian matrix dr allows to quantify the distortion $\sqrt{\det(dr^T dr)}$. If R is a subset of $\mathbb{R}^2$, then r describes a 2-dimensional surface. We usually write a point in R as (u, v) but other variables can be used. If $n = 3$, that is if we deal with a surface in three dimensional space, then the distortion factor is $|r_u \times r_v|$ and the surface area is the double integral $\iint_R |r_u \times r_v| \, dudv$. This topic is therefore a great opportunity to practice more double integrals.

FIGURE 1. A circle moving in space time produces a two dimensional surface. The surface area of this surface is of interest in physics. The surface area is the **Nambu-Goto action**.

Lecture

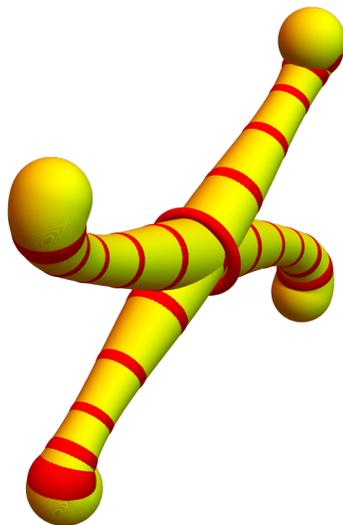**26.2.** A map $r : R \subset \mathbb{R}^2 \to \mathbb{R}^3$ has an image $r(R) = S$ which is a **parametrized surface**. What is its surface area? We have seen that the distortion factor is now $|dr| = \sqrt{\det(g)} = |r_u \times r_v|$, where $g = dr^T dr$ was the **first fundamental form** of the surface. Of course, it is more convenient to use $|r_u \times r_v|$, which is the same as $|dr|$.

> **Theorem:** The surface area $\iint_S dS$ of S is $\iint_R |r_u \times r_v| \, du dv$.

26.3. More generally if $f : R \to \mathbb{R}$ is a function which describes something like a **density** then $\iint_R f(r(u,v)) \, |r_u \times r_v| \, du dv$ is an integral which is abbreviated as $\iint_S f \, dS$ and called a **scalar surface integral**. For example, if f is a density on the surface then this $\iint_S f dS$ is the mass. Again, we have to stress that in this integral, the orientation of the surface is irrelevant. The distortion factor $|dr|$ is always non-negative. It is better to think of $\iint_S f dS$ as a **weighted surface area** generalizing area $\iint_S dS$. [1]

26.4. Here is the most general change of integration formula for maps $r : \mathbb{R}^m \to \mathbb{R}^n$, with **distortion factor** $|dr| = \sqrt{\det(dr^T dr)}$. The formula holds for $m > n$ too, det is then a pseudo determinant. If $S = r(R)$ is the image of a solid R under a C^2 map r and $f : \mathbb{R}^n \to \mathbb{R}$ is a function, then the **mother of all substitution formulas** is

> **Theorem:** $\iint_R f(r(u))|dr(u)| \, du = \iint_S f(u) \, du$.

26.5. The proof is the same as seen in the two-dimensional change of variable situation. Just because n is used for the target space $\mathbb{R}^n$, we use the basic size $1/N$. We chop up the region into parts $R \cap Q$ with cubes Q of size $1/N$ and estimate the difference $Vol(dr(Q))$ and $Vol(r(Q))$ by CM_N/N^2 leading to an overall difference bounded by FCM_N/N^2, where F is the maximal value of f on R and M_n is the **Heine-Cantor function** modulus of continuity of f. Adding everything up gives an error $FC\mathrm{Vol}(R)M_N + 2^n\mathrm{Vol}(\delta R)F/N \to 0$, where δR is the boundary of R. There is one new thing: we have to see why $\sqrt{\det(A^T A)}$ is the volume of the parallelepiped spanned by the column vectors of the Jacobian matrix $A = dr$. We will talk about determinants in detail later but if A is in row reduced echelon form then $A^T A$ is the identity matrix and the determinant is 1, agreeing with the volume. Now notice that if a column of A is scaled by λ producing a new matrix B, then $\det(B^T A) = \lambda\det(A^T A)$ and $\det(B^T B) = \lambda^2\det(A^T A)$. If two columns of A are swapped leading to a new matrix B, then $\det(B^T A) = -\det(A^T A)$ and $\det(B^T B) = \det(A^T A)$. If a column of A is added to another column, then this does change $\det(B^T B)$. The only row reduction step which affects the $|dr|$ is the scaling. But that is completely in sync what happens with the volume. QED.

26.6. The last theorem covers everything we have seen and we ever need to know when integrating scalar functions over manifolds. In the special case $n = m$ it leads to:

> **Theorem:** $\iint_R |dr(u)| \, du = \mathbf{Vol}(S)$.

---

[1] Unfortunately, scalar integrals are often placed close to the integration of differential forms (like volume forms). The later are of **different nature** and use an integration theory in which spaces come with orientation. So far, if we replace $r(u, v)$ with $r(v, u)$ gives the same result (like area or mass).

26.7. Here are the important small dimensional examples:

If $m = 1, n = 3$, then $\int_a^b |r'(t)|\, dt$ is the **arc length** of the curve $C = r(I)$.

If $m = 2, n = 2$, then $\iint_R |dr|\, dudv$ is the **area** of the region $S = r(R)$.

If $m = 2, n = 3$, then $\iint_R |r_u \times r_v|\, dudv$ is the **surface area** of $S = r(R)$.

If $m = 3, n = 3$, then $\iiint_R |dr|\, dudvdw$ is the **volume** of the solid $S = r(R)$.

EXAMPLES

26.8. In all the examples of surface area computations, we take a parametrization $r(u, v) : R \to S$, then use use that the distortion factor is $\sqrt{\det(dr^T dr)} = |r_u \times r_v|$.
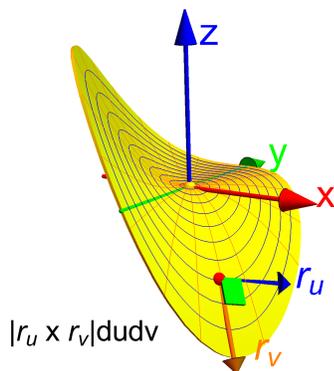

FIGURE 2.   The distortion factors $|dr| = |g| = \sqrt{\det(g)} = \sqrt{\det(dr^T dr)}$ appear in general. For $m = 2, n = 3$ we get surface area $\iint_R |r_u \times r_v|\, dudv$.

26.9. Problem: find the surface area of a sphere $x^2 + y^2 + z^2 = L^2$. **Solution:** Parametrize the surface $r([\theta, \phi]) = [L\sin(\phi)\cos(\theta), L\sin(\phi)\sin(\theta), L\cos(\phi)]$. The distortion factor is $L^2 \sin(\phi)$. The surface area is $4\pi L^2$.

26.10. Problem: find the surface area of surface of revolution given in cylindrical coordinates as $z = g(\theta), a \le z \le b$. **Solution:** Parametrize the surface $r([\theta, z]) = [g(z)\cos(\theta), g(z)\sin(\theta), z]$. The distortion factor is $g(z)\sqrt{1 + g'(z)^2}$.

26.11. As an example, we can look at the surface of revolution $x^2 + y^2 = 1/z^2, |z|^2 > 1$. The volume of the solid enclosed by the surface is π. The surface area is infinite.

26.12. Problem: find the surface area of the graph of a function $z = f(x, y), (x, y) \in R$. **Solution:** Parametrize the surface as $r([x, y]) = [x, y, f(x, y)]$. The distortion factor is $|r_x \times r_y| = \sqrt{1 + f_x^2 + f_y^2}$.

26.13. Problem: what is the surface area of the intersection of $x^2 + z^2 \le 1, 6x + 3y + 9z = 12$. **Solution:** The surface is a plane but also a graph over $R = \{x^2 + z^2 \le 1\}$ in the xz-plane. The simplest parametrization is $r([x, z]) = [x, (12 - 6x - 9z)/3, z] = [x, 4 - 2x - 3z, z]$. It gives $|r_x \times r_z| = |[-2, -1, -3]| = \sqrt{14}$. The surface area is $\iint_R \sqrt{14}dxdy = \sqrt{14}\text{Area}(R) = \sqrt{14}\pi$.

26.14. The following **hyperspherical coordinates** parametrize the 3-dimensional sphere $x^2 + y^2 + z^2 + w^2 = 1$ in $\mathbb{R}^4$.

$$r([\phi, \psi, \theta]) = [\cos(\phi), \sin(\phi)\cos(\psi), \sin(\phi)\sin(\psi)\cos(\theta), \sin(\phi)\sin(\psi)\sin(\theta)] \,,$$

with $\theta \in [0, 2\pi], \phi \in [0, \pi], \psi \in [0, \pi]$. The distortion factor is $\sqrt{\det(dr^T dr)} = \sqrt{\sin^4(\phi)\sin^2(\psi)}$ so that the surface area of the **hypersphere** is

$2\pi \int_0^\pi \int_0^\pi \sin^2(\phi)\sin(\psi)d\phi d\psi = \boxed{2\pi^2}$.
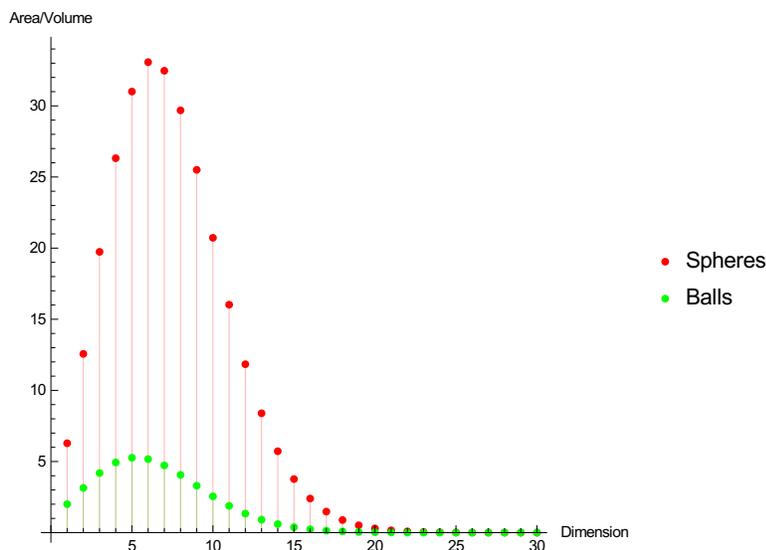

FIGURE 3. The volume and surface area of k dimensional spheres

26.15. In dimension n what is the volume $|B_n|$ of the n-dimensional **unit ball** B_n in $\mathbb{R}^n$ and the **volume** $|S_n|$ of the n-dimensional **unit sphere** S_n in $\mathbb{R}^{n+1}$? It starts with $|B_0| = 1$, as B_0 is a point and $|S_0| = 2$, as S_0 consists of two points. The n-ball of radius ρ has the volume $|B_n|\rho^n$ and the n-sphere of radius ρ has the volume $|S_n|\rho^n$. Because $|B_{n+1}| = \int_0^1 |S_n|\rho^n \, d\rho$, we have $|B_{n+1}| = |S_n|/(n+1)$. Because S_n can be written as a union of products $(n-2)$-spheres with S_1 leading to $|S_n| = 2\pi \int_0^{\pi/2} |S_{n-2}|\cos(\phi) \, d\phi = 2\pi|B_{n-1}|$. We know now all: just start with $|B_0| = 1, |S_0| = 2, |B_1| = 2, |S_1| = 2\pi$ and

> **Theorem:** $|B_n| = \frac{2\pi}{n}|B_{n-2}|$, $|S_n| = \frac{2\pi}{n-1}|S_{n-2}|$.

The 5-ball has maximal volume 5.26379... among all unit balls. The 6-sphere has maximal surface area 33.0734... among all unit spheres. The volume of the 30-ball is only 0.00002.... The surface area of the 30-sphere for example is only 0.0003. Compare with a **n-unit cube** of volume 1 and a boundary surface area $2n$. High dimensional spheres and balls are tiny!

26.16. If S is a cylinder $x^2 + y^2 = 1, 0 < z < 1$, triangulated with each triangle smaller than $1/n \to 0$, does the area converge to the surface area $A(S)$? No! A counter example is the **Schwarz lantern** from 1880. The cylinder is cut into m slices and n points are marked on the rim of each slice to get triangles like $A = (1, 0, 0), B =$

$(\cos(4\pi/n), \sin(4\pi/n, 0)), C = (\cos(2\pi/n), \sin(2\pi/n), 1/m)$ of area $\sin(2\pi/n)(1/m)\sqrt{2 + 3m^2 - 4m^2\cos(2\pi/n) + m^2\cos(4\pi/n)}/\sqrt{2}$. The nm triangles have area $\sim \sqrt{2 + 8m^2\pi^4/n^4}/\sqrt{2}$. For $m = n^3$, the triangulated area diverges.
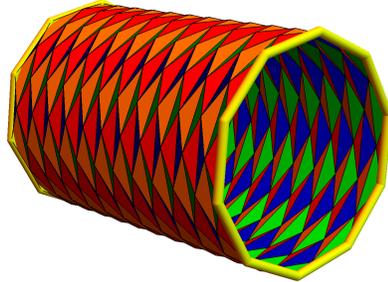
FIGURE 4.   The Schwarz lantern.

26.17. The three dimensional sphere is $x^2 + y^2 + z^2 + w^2 = 1$ in $\mathbb{R}^4$. The **Hopf parametrization** is $r : R \subset \mathbb{R}^3 \to S \subset \mathbb{R}^4$ is

$$r([\phi, \theta_1, \theta_2]) = [\cos(\phi)\cos(\theta_1), \cos(\phi)\sin(\theta_1), \sin(\phi)\cos(\theta_2), \sin(\phi)\sin(\theta_2)] \ .]$$

We compute $|dr| = \sqrt{\det(dr^T dr)} = \cos(\phi)\sin(\phi) = \sin(2\phi)/2$. If we fix ϕ, we see a two dimensional torus. Their union with $\phi \in [0, \pi/2]$ is the **Hopf fibration**. We can now compute the volume of the three dimensional sphere:

$$\int_0^{\pi/2} \int_0^{2\pi} \int_0^{2\pi} \sin(2\phi)/2 \ d\phi d\theta_1 d\theta_2 = 2\pi^2 \ .$$
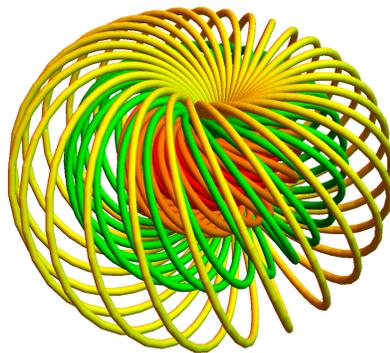
FIGURE 5.   The Hopf fibration of the 3-sphere.

HOMEWORK

Problem 25-26.1: Find the moment of inertia $\iiint_E x^2 + y^2 \, dV$, where $E = \{x^2 + y^2 \leq z^2, |z|^2 \leq 1$ is the double cone.

Problem 25-26.2: Evaluate the triple integral

$$\iiint_E xy \, dV \, ,$$

where E is bounded by the parabolic cylinders $y = 3x^2$ and $x = 3y^2$ and the planes $z = 0$ and $z = x + y$.

Problem 25-26.3: We have seen the problem in the movie "Gifted" to compute the improper integral of e^{-x^2}. Here is another approach: verify

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2+z^2)} \, dx \, dy \, dz = (\sqrt{\pi})^3.$$

Use this as in the "Gifted" computation to find $\int_{-\infty}^{\infty} e^{-x^2} \, dx$. You can do that without knowing that the later is $\sqrt{\pi}$.

Problem 25-26.4: Find the surface area of the **Einstein-Rosen bridge** $r(u, v) = [3v^3, v^9 \cos(u), v^9 \sin(u)]^T$, where $0 \leq u \leq 2\pi$ and $-1 \leq v \leq 1$. Tunnels connecting different parts of space-time appear frequently in science fiction.
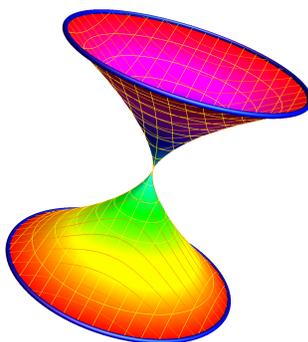
FIGURE 6. A "wormhole".

Problem 25-26.5: Find the area of the surface given by the **helicoid** $r(u, v) = [u \cos(v), u \sin(v), v]^T$ with $0 \leq u \leq 1, \ 0 \leq v \leq \pi$.

Unit 27: A visit by Archimedes

Introduction

27.1. Drum rolls .... Here is Archimedes (287 BC - 212 BC). He is considered the father of integral calculus. He is also considered a "polymath", somebody who mastered and innovated several topics: mathematics, physics, engineering and astronomy.

FIGURE 1. Archimedes working on some exam problem!

Seminar

27.2. In this review lecture we have the honor to have Archimedes as a special guest. Before the event, we talked to him using a technology called "quantum forward tunneling" which allows to interact with part of the past without running into a causality paradox. The actual Archimedes did not know about the interview. It is his "quantum spirit" which does it for us. How does it work? Quantum space-time produces sometimes tiny wormhole constellations in which a wave function can be trapped. By harvesting many of those trapped waves, we can rebuild and interact with an object or person from a previous time. The so established "time tunnel" is sustainable only for a short time as the trapped waves will fade within a half an hour. It is enough time however for a short interview. We take the opportunity and ask him about his theorems.

27.3. Math 22b: What a pleasure to have you here. Welcome! **Archimedes:** I'm glad to find myself in this lovely place. It must be a dream. I don't recognize the town but it feels like a 'Alexandria in the future'. **Math 22b:** yes, it is also a hot spot for science, but there are many now. We are eager to learn a bit about your proof expertise.

27.4. Math 22b: What result of yours do you consider the most important one? **Archimedes:** Definitely the formula for the volume of the sphere! **Math 22b:** Why? **Archimedes:** It was much harder to get this than the circumference of the circle or the surface area of the sphere. It was also harder to test the result experimentally. **Math 22b:** How did you measure? **Archimedes:** We build wood models of cylinders, cones and spheres of the same base radius and height and measured their volume ratios.

> **Problem A:** Explain how Archimedes can using wooden models measure their volumes. If you don't know, take a bath. Given a cylinder C, a cone O and a sphere S of base length 1. What ratios $|C|/|S|, |O|/|S|$ do the measurements show?

27.5. Math 22b: Was the comparison of the sphere with the complement of a cone in the cylinder historically the first proof? **Archimedes:** The relation had been conjectured before. It had been suspected that the ratio between the volume of a sphere and the volume of a cylinder is the fraction 2/3 but nobody had been able to prove this relation before I could see the slicing trick.

> **Problem B:** Explain why slicing the unit sphere at height z gives the same area as a ring of radius 1 in which a hole of size z has been has been drilled.

27.6. Math 22b: Do you remember the precise moment, when the discovery stuck? **Archimedes:** I don't recall directly but it must have been one of these "hot tub ideas".

27.7. Math 22b: This discovery must have occurred after you got the circle circumference computed. How difficult was the later? **Archimedes**: also this needed some time. It emerged pretty early that the circumference is somehow proportional to the radius. The measurement of the constant was then a bit trickier even so it remained

open what fraction it is. 22/7 was close. I got first the diameter/area ratio. I did that using the following picture.

Problem B: How does the picture below prove that the area A, radius R and diameter D of a circle satisfies $2A = RD$? How can you make this precise as in reality the circular sector does not have the same area as the triangle. (Hint: you can use modern tools like L'Hôpital's rule if you like).
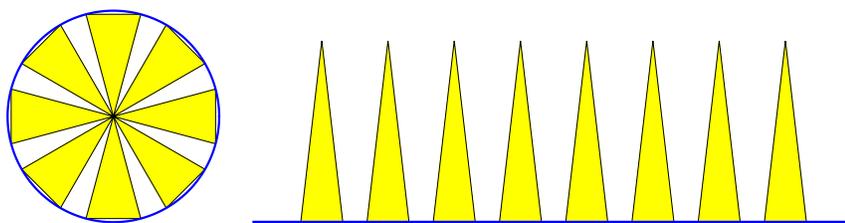

FIGURE 2. The circle proof.

27.8. Math 22b: We also wonder about your computation of the volume of the "hoof" which is the solid bound by the cylinder $x^2 + y^2 = 1$ and $z = x$ and $z = 0$. **Archimedes:** I don't recognize the symbols you just spelled out but I know what object you are talking about. It was exciting to see a solid bound partly by round parts to have a rational volume, which is 2 third of the height. One can see that the result is 2/3 in various ways.

Problem C: a) Take a hoof of height 1 and cut in triangular pieces which are obtained if y is constant. Show that the area of the triangle is $(1-y^2)/2$ and conclude from this the volume is 2/3. b) Cut the same hoof into rectangular pieces which are obtained if x is constant. Show that the area of the rectangle is $(1 - x^2)$ and conclude that the volume is 2/3.

27.9. Math 22b: Also very impressive is your computation of the surface area of the sphere by relating it with the surface area of a cylinder. What was the intuition there? **Archimedes:** Actually, a drawing which is accurate enough shows this pretty well. As both situations have circular symmetry, we only need to understand what happens with the lengths on a sphere when it is projected on the cylinder. There are similar triangles. Take a stick of some length and place it onto the sphere pointing to the north pole. As it gets closer to the pole and its height-length is one half of the actual length. then the radius of that position is also half etc. As the area of a small sphere strip is height times radius times about 22/7, this is also the area of a cylinder. In the sphere case, the factor one-half is applied to the radius. In the cylinder case it is applied to the height.

Problem C: Explain this in more modern terms. We have a unit sphere and a cylinder of radius 1. Look what the surface area of a strip $z, z + dz$ is in both cases. You can use the spherical angle ϕ which you know from spherical coordinates.

27.10. Math 22b. A last question: What is a function in mathematics? **Archimedes.** I don't know this expression: for me, mathematics deals with geometric objects and numbers which characterize those objects like length, area or volume. **Math 22b**: we interpret your formula for the volume of a sphere as $V(r) = 4\pi r^3/3$ which is a rule assigning to the radius r a number. We also have rules which tell how to compute rates of change. For the function $V(r)$ for example, its rate of change is $4\pi r^2$, the surface area of the sphere. The reason is that if we decrease the radius by a small unit, then this essentially means taking away a layer of area $4\pi r^2$. **Archimedes.** This is cool. Let me see: does this also work for the area and circumference of a disc? **Math 22b**. Certainly. Go ahead. **Archimedes.** Well, with this new language, we would say that a disc of radius r is $f(r) = \pi r^2$. I assume that for any integer n the rate of change of r^n is nr^{n-1}. **Math 22b**. Yes, that is correct. **Archimedes.** In that case the rate of change of πr^2 is $2\pi r$ and indeed this is my formula for the circumference of a circle. This is "Phaidros".

27.11. Math 22b Thank you very much for the interview. It will inspire us for the second midterm exam. Maybe you can visit and take the exam on Tuesday or review on Sunday? **Archimedes** 'It will be my pleasure."

Homework

27.1 Find a solid which has the property that if you project it on the xy-plane it is a half circle, if you project it on the yz plane it is a triangle and if you project it onto the xz-plane, it is a rectangle.

27.2 There are regions in the plane which have the property that their thickness is constant 1 but which are not circles. Find some. These curves are also known under the name curve of constant width.

27.3 There is a beautiful theorem of Pappus which gives the formula of a solid obtained by taking all points in distance d from a given curve $r(t)$ provided the thickened curve does not intersect: the formula is $L\pi d^2 + 4\pi d^3/3$, where L is the length of the curve. Archimedes designed a spiral pump in which a spiral $r(t) = [10\cos(t), 10\sin(t), t]$ plays an important role. Assume $0 \le t \le 100\pi$, what is the volume of the solid consisting of all points in distance 1 to the curve? You just need to compute that volume and not prove the formula.

27.4 Let A be the solid obtained by intersecting three perpendicular solid cylinders $x^2 + y^2 \le 1, x^2 + z^2 \le 1, y^2 + z^2 \le 1$. What is its volume?

27.5 Archimedes had another picture for the volume of a sphere. It is seen in the picture above to the right and was mentioned in lecture 24. Please explain in your own words.
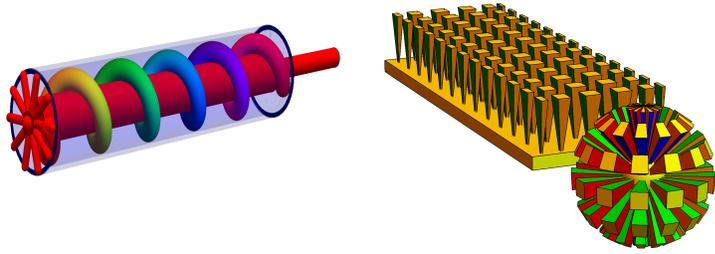
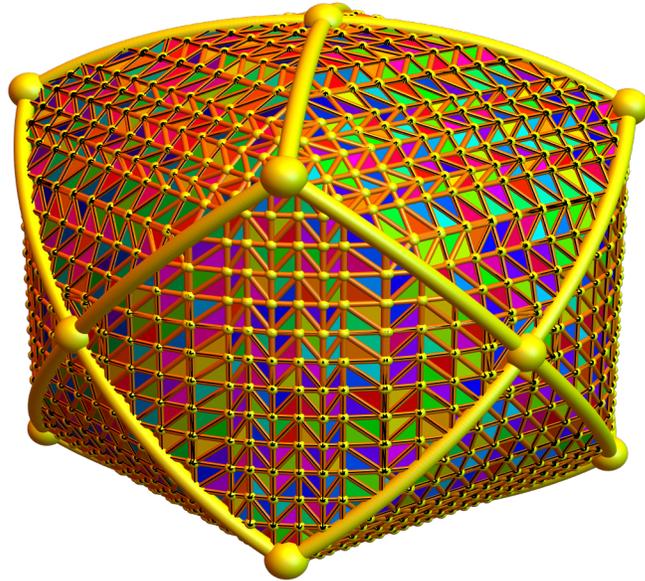FIGURE 3. The "Archimedes screw" and another sphere proof.

FIGURE 4. The intersection of three cylinders is a classical problem which Archimedes solved already.

Unit 28: Keywords for Second Hourly

This is a bit of a checklist. Make your own list. But here is a checklist which tries to be comprehensive. Check off the topics you know and check back with things you do not recall. You will need to have the following on your finger tips.

Partial Derivatives

- $f_x(x, y) = \frac{\partial}{\partial x} f(x, y)$ partial derivative
- $L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ linear approximation
- $Q(x, y) = L(x_0, y_0) + f_{xx}(x - x_0)^2/2 + f_{yy}(y - y_0)^2/2 + f_{xy}(x - x_0)(y - y_0)$ quadratic
- $L(x, y)$ estimates $f(x, y)$ near $f(x_0, y_0)$. The result is $f(x_0, y_0) + a(x - x_0) + b(y - y_0)$
- tangent line: $ax + by = d$ with $a = f_x(x_0, y_0), b = f_y(x_0, y_0)$, $d = ax_0 + by_0$
- tangent plane: $ax + by + cz = d$ with $a = f_x, b = f_y, c = f_z$, $d = ax_0 + by_0 + cz_0$
- estimate $f(x, y, z)$ by $L(x, y, z)$ near (x_0, y_0, z_0)
- $f_{xy} = f_{yx}$ Clairaut's theorem, if f_{xy} and f_{yx} are continuous.
- $r_u(u, v), r_v(u, v)$ tangent to surface parameterized by $r(u, v)$

Partial Differential Equations

- $f_t = f_{xx}$ heat equation
- $f_{tt} - f_{xx} = 0$ wave equation
- $f_x - f_t = 0$ transport equation
- $f_{xx} + f_{yy} = 0$ Laplace equation
- $f_t + f f_x = f_{xx}$ Burgers equation
- $f_x^2 + f_y^2 = 1$ Eiconal equation
- $f_t = f - x f_x - x^2 f_{xx}$ Black Scholes

Gradient

- $\nabla f(x, y) = df^T = [f_x, f_y]^T$, $\nabla f(x, y, z) = [f_x, f_y, f_z]^T$, gradient
- $D_v f = \nabla f \cdot v$ directional derivative
- $\frac{d}{dt} f(r(t)) = \nabla f(r(t)) \cdot r'(t)$ chain rule
- $\nabla f(x_0, y_0)$ is orthogonal to the level curve $f(x, y) = c$ containing (x_0, y_0)
- $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface $f(x, y, z) = c$ containing (x_0, y_0, z_0)
- $\frac{d}{dt} f(x + tv) = D_v f$ by chain rule
- $(x - x_0) f_x(x_0, y_0, z_0) + (y - y_0) f_y(x_0, y_0, z_0) + (z - z_0) f_z(x_0, y_0, z_0) = 0$ tangent plane
- $f(x, y)$ increases in the $\nabla f / |\nabla f|$ direction. Functions dance upwards.
- $f(x, y, z) = c$ defines $z = g(x, y)$, and $g_x(x, y) = -f_x(x, y, z)/f_z(x, y, z)$ implicit diff

Extrema

- [ ] $\nabla f(x,y) = [0,0]^T$, critical point or stationary point
- [ ] $D = f_{xx}f_{yy} - f_{xy}^2 = \det(df)$ discriminant, useful in second derivative test
- [ ] $f(x_0, y_0) \geq f(x,y)$ in a neighborhood of (x_0, y_0) local maximum
- [ ] $f(x_0, y_0) \leq f(x,y)$ in a neighborhood of (x_0, y_0) local minimum
- [ ] $\nabla f(x,y) = \lambda \nabla g(x,y), g(x,y) = c$, or $\nabla g = 0$ Lagrange equations
- [ ] second derivative test: $\nabla f = (0,0), D > 0, f_{xx} < 0$ **local max**, $\nabla f = (0,0), D > 0, f_{xx} > 0$ **local min**, $\nabla f = (0,0), D < 0$ **saddle point**
- [ ] $f(x_0, y_0) \geq f(x,y)$ everywhere, global maximum
- [ ] $f(x_0, y_0) \leq f(x,y)$ everywhere, global minimum
- [ ] f is Morse if the Hessian $H = d^2 f$ is invertible at every critical point

Double Integrals

- [ ] $\int\int_R f(x,y) \, dydx$ double integral
- [ ] $\int_a^b \int_{c(x)}^{d(x)} f(x,y) \, dydx$ bottom-to-top region
- [ ] $\int_c^d \int_{a(y)}^{b(y)} f(x,y) \, dxdy$ left-to-right region
- [ ] $\int\int_R f(r,\theta)\boxed{r} \, drd\theta$ polar coordinates
- [ ] $\int\int_R |r_u \times r_v| \, dudv$ surface area
- [ ] $\int_a^b \int_c^d f(x,y) \, dydx = \int_c^d \int_a^b f(x,y) \, dxdy$ Fubini
- [ ] $\int\int_R \boxed{1} \, dxdy$ area of region R
- [ ] $\int\int_R f(x,y) \, dxdy$ signed volume of solid bound by graph of f and xy-plane

Triple Integrals

- [ ] $\int\int\int_R f(x,y,z) \, dzdydx$ triple integral
- [ ] $\int_a^b \int_c^d \int_u^v f(x,y,z) \, dzdydx$ integral over rectangular box
- [ ] $\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y) \, dzdydx$ type I region
- [ ] $\int\int\int_R f(r,\theta,z) \, \boxed{r} \, dzdrd\theta$ integral in cylindrical coordinates
- [ ] $\int\int\int_R f(\rho,\theta,\phi) \, \boxed{\rho^2 \sin(\phi)} \, d\rho d\phi d\theta$ integral in spherical coordinates
- [ ] $\int_a^b \int_c^d \int_u^v f(x,y,z) \, dzdydx = \int_u^v \int_c^d \int_a^b f(x,y,z) \, dxdydz$ Fubini
- [ ] $V = \int\int\int_E \boxed{1} \, dzdydx$ volume of solid E
- [ ] $M = \int\int\int_E f(x,y,z) \, dzdydx$ mass of solid E with density f.

General advise

- [ ] Draw the region when integrating in in higher dimensions.
- [ ] Consider other coordinate systems if the integral does not work.
- [ ] Consider changing the order of integration if the integral does not work.
- [ ] For tangent planes, compute the gradient $[a,b,c]^T$ first then fix the constant.
- [ ] When looking at relief problems, mind the gradient.

Theorems

Clairaut, Taylor, Fubini, Island theorem, Sphere and Ball volumes, Morse theorem, chain rule, gradient theorem, change of variables

People

Clairaut, Fubini, Lagrange, Fermat, Riemann, Archimedes, Hamilton, Euler, Taylor, Morse, Hopf, Tao, Polya, Riemann

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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total :

Unit 28: Second Hourly (Practice A)

- You only need this booklet and something to write. Please stow away any other material and electronic devices. Remember the honor code.
- Please write neatly and give details. Except for problems 28.2 and 28.3, we want to see details, even if the answer should be obvious to you.
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Archimedes sends his good luck wishes. He unfortunately can not join us as he is "busy proving a new theorem". He just sent us his selfie. Oh well, these celebrities!

PROBLEMS

Problem 28A.1 (10 points):

a) (4 points) Prove that if x^3 is irrational, then x is irrational.

b) (3 points) Prove or disprove: the product of two odd integers is odd.

c) (3 points) Prove or disprove: the sum of two odd integers is odd.

Problem 28A.2 (10 points) Each question is one point:

a) What is the name of the partial differential equation $f_{tt} = f_{xx}$?

b) The series $f(x) = \sum_{k=0}^{\infty} x^k/k! = 1 + x + x^2/2! + x^3/3! + \cdots$ represents a function. Which one?

c) The implicit differentiation formula for $f(x, y(x)) = 1$ is $y'(x) = \ldots\ldots$

d) What is the name of the function $f(s) = \sum_{n=1}^{\infty} n^{-s}$?

e) On a circular island there are exactly 3 maxima and one minimum for the height f. Assuming f is a Morse function, how many saddle points are there?

f) Which mathematician first found the value for the volume of the ball $x^2 + y^2 + z^2 \leq 1$?

g) True or False: the directional derivative of f in the direction $\nabla f(x)/|\nabla f(x)|$ is negative at a point where ∇f is not zero.

h) The equation $f(x + t) = e^{Dt} f = f(x) + f'(x)t + f''(x)t^2/2 + \cdots$ solves a partial differential equation. Which one?

i) What is the formula for the surface area of a surface S parametrized by $r(u, v)$ over a domain R?

j) What is the integration factor (= distortion factor) when going to spherical coordinates (ρ, ϕ, θ)?

Problem 28A.3 (10 points) Each question is two points:

We see the level curves of a Morse function f. The circle through ABC will sometimes serve as a constraint $g(x,y) = x^2 + y^2 = 1$. In all questions, we only pick points from A,B,C,D,E,F,G,H,I,J,K,L,M.

a) Which points are local minima of f under the constraint $g(x,y) = 1$.

b) Which points are local maxima of f under the constraint $g(x,y) = 1$.

c) At which points do we have $f_x(x,y) \cdot f_y(x,y) \neq 0$?

d) At which points are $|\nabla f(x,y)|$ maximal?

e) At which points are $|\nabla f(x,y)|$ minimal?
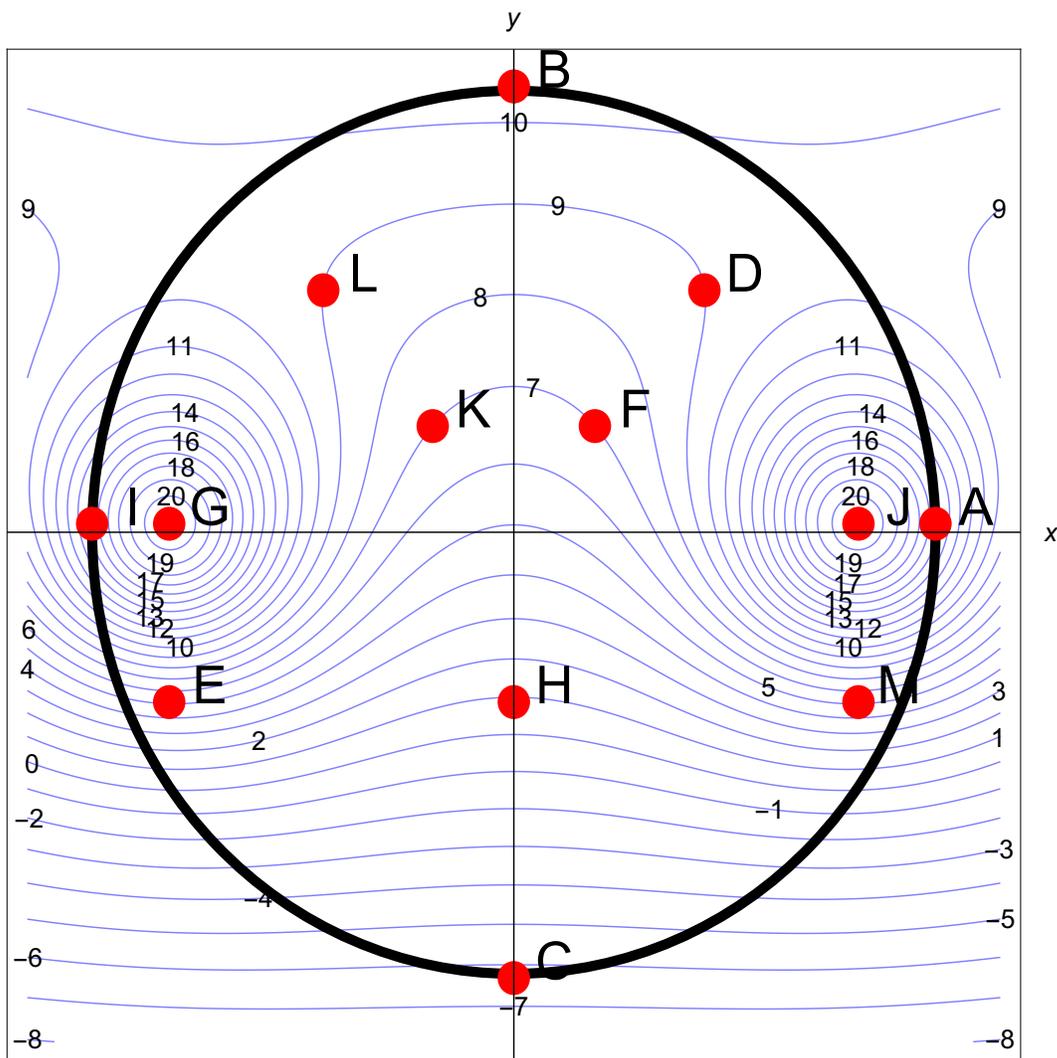
Problem 28A.4 (10 points):

a) (5 points) Find the tangent plane to the surface
$$f(x, y, z) = x^2 y - x^3 + y^2 + z^4 xy = -13$$
at the point $(2, -1, 1)$.

b) (5 points) Estimate $f(2.001, -0.99, 1.1)$ by linear approximation.

Problem 28A.5 (10 points):

a) (5 points) Find the quadratic approximation $Q(x, y)$ of
$$f(x, y) = 5 + x + y + x^2 + 3y^2 + \sin(xy) + e^x$$
at $(x, y) = (0, 0)$.

b) (5 points) Estimate the value of $f(0.001, 0.02)$ using quadratic approximation.

Problem 28A.6 (10 points):

a) (8 points) Classify the critical points of the function
$$f(x, y) = x^2 - y^3 + 2x + 3y$$
using the second derivative test.

b) (2 points) Does the function $f(x, y)$ have a global minimum or global maximum?

Problem 28A.7 (10 points):

Using the Lagrange optimization method, find the parameters (x, y) for which the area of an arch
$$f(x, y) = 2x^2 + 4xy + 3y^2$$
is minimal, while the perimeter
$$g(x, y) = 8x + 9y = 33$$
is fixed.

Problem 28A.8 (10 points):

a) (5 points) Find the moment of inertia
$$I = \iint_G (x^2 + y^2)\, dy\, dx$$
of the quarter disc $G = \{x^2 + y^2 \leq 1, x \geq 0, y \leq 0\}$.

b) (5 points) Evaluate the double integral
$$\int_1^e \int_{\log(x)}^1 \frac{y}{e^y - 1}\, dy\, dx ,$$
where log is the natural log as usual.

Problem 28A.9 (10 points):

Find the integral
$$\iiint_E f(x, y, z)\, dz\, dy\, dx$$
of the function
$$f(x, y, z) = x + (x^2 + y^2 + z^2)^4$$
over the solid
$$E = \{(x, y, z) \mid 1 \leq x^2 + y^2 + z^2 \leq 4,\ z \geq 0\}.$$

Problem 28A.10 (10 points):

Find the surface area of
$$r(x, y) = \begin{bmatrix} 2x \\ y \\ \frac{x^3}{3} + y \end{bmatrix}$$
with $0 \leq x \leq 2$ and $0 \leq y \leq x^3$.

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MATH 22B

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Unit 28: Second Hourly (Practice B)

PROBLEMS

Problem 28B.1 (10 points):
a) (4 points) You know the positive integer n^5 is odd. Prove that n is odd.
b) (3 points) Prove or disprove: if a and b are irrational, then ab is irrational.
c) (3 points) Prove or disprove: if a and b are irrational, then $a + b$ is irrational.

Problem 28B.2 (10 points, each sub problem is one point):
a) What is the name of the differential equation $f_t = f_{xx}$?
b) What assumptions need to hold so that $f_{xy} = f_{yx}$ is true?
c) The gradient $\nabla f(x_0)$ has a relation to $f(x) = c$ with $c = f(x_0)$. Which one?
d) The linear approximation of f at x_0 is $L(x) = f(x_0) + \ldots$ Complete the formula.
e) Assume f has a maximum on $g = c$, then either $\nabla f = \lambda \nabla g, g = c$ holds or ...
f) Which mathematician proved the switch the order of integration formula?
g) True or false: the gradient vector $\nabla f(x)$ is the same as $df(x)$.
h) The equation $u_t + uu_x = u_{xx}$ is an example of a differential equation. We have seen two major types (each a three capital letter acronym). Which type is it?
i) What is the formula for the arc length of a curve C?
j) What is the integration factor $|d\phi|$ when going into polar coordinates?

Problem 28B.3 (10 points, 2 points for each sub-problem):
We see the level curves of a Morse function f. Only pick points A-J.
a) Which point is critical with discriminant $D = \det(d^2 f) < 0$.
b) At which point is $f_x > 0, f_y = 0$?
c) At which point is $f_x > 0, f_y > 0$?
d) Which (x_0, y_0) are critical points of f when imposing the constraint $g(x, y) = y = y_0$?
e) Which (x_0, y_0) are critical points of f when imposing the constraint $g(x, y) = x = x_0$?
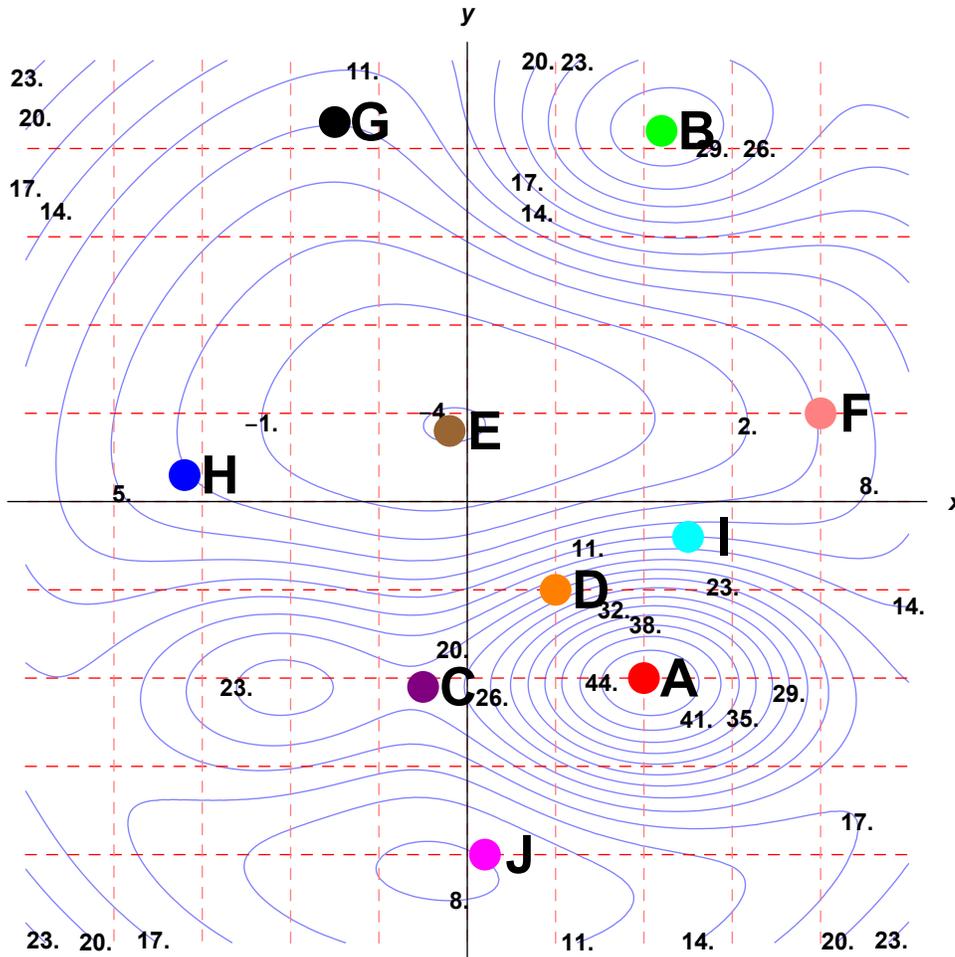
Problem 28B.4 (10 points):
a) (5 points) Find the tangent plane to the surface $xyz + x^5y + z = 11$ at $(1, 2, 3)$.
b) (5 points) Near $(x, y) = (1, 2)$, we can write $z = g(x, y)$. Find $g_x(1, 2), g_y(1, 2)$.

Problem 28B.5 (10 points):
a) Find the quadratic approximation of $f(x, y, z) = 1 + x + y^2 + z^3 + \sin(xyz)$ at $(0, 0, 0)$.
b) Estimate $f(0.01, 0.03, 0.05)$ using linear approximation.

Problem 28B.6 (10 points):
a) (8 points) Classify the critical points of the function $f(x, y) = x^{12} + 12x^2 + y^{12} + 12y^2$ using the second derivative test.
b) (2 points) Does f have a global minimum? Does f have a global maximum?

Problem 28B.7 (10 points):
On the top of a MIT building there is a radar dome in the form of a spherical cap. Insiders call it the **"Death star" radar dome**. We know that with the height h and base radius r, we have volume and surface area given by $V = \pi r h^2 - \pi h^3/3$, $A = 2\pi r h = \pi$. This leads to the problem to extremize

$$f(x, y) = xy^2 - \frac{y^3}{3}$$

under the constraint

$$g(x, y) = 2xy = 1 \ .$$

Find the minimum of f on this constraint using the Lagrange method!

Problem 28B.8 (10 points):
Find

$$\iint_R 5/(x^2 + y^2) \, dxdy \ ,$$

where R is the region $1 \le x^2 + y^2 \le 25$, $y^2 > x^2$.

Problem 28B.9 (10 points):
Integrate $f(x, y, z) = z$ over the solid E bound by

$$z = 0$$
$$x = 0$$
$$y = 0$$

and

$$x + y + z = 1 \ .$$

Problem 28B.10 (10 points):
What is the surface area of the surface

$$r(x, y) = \begin{bmatrix} 2y \\ x \\ \frac{y^3}{3} + x \end{bmatrix}$$

with $0 \le y \le 2$ and $0 \le x \le y^3$?

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LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

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Unit 28: Hourly 2

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Archimedes sends his good luck wishes again. He also sits here so that you can not read the first problem until we all start at noon.

PROBLEMS

Problem 28.1 (10 points):

a) (3 points) Prove or disprove that the product of a rational and an irrational number is irrational.

b) (3 points) Prove or disprove that the product of two irrational numbers is irrational.

c) (2 points) Prove or disprove that the product of two numbers of the form $4k - 1$ is a number of the form $4k - 1$.

d) (2 points) Prove or disprove that the product of two numbers of the form $4k + 1$ is a number of the form $4k + 1$.

Problem 28.2 (10 points) Each question is one point:

a) What is the name of the partial differential equation $f_t = f f_x$?

b) The series $f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$ represents a function. Which one?

c) The implicit differentiation formula for $f(x, y, z(x)) = 1$ is $z_x(x) = \ldots \ldots$

d) The problem to compute the value of $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $s = 2$ is called the .... problem.

e) Is it possible that we have a Morse function on the 2-sphere $x^2 + y^2 + z^2 = 1$ has 3 maxima, 1 minimum and 3 saddle points?

f) Who proved that one can change the order of integration on a rectangle? The result is called the .... theorem.

g) You measure progress with a Morse function f in a data space $\mathbb{R}^n$. You are located at a point which is not a critical point. In which direction do you have to change the parameters to make f larger?

h) The function $f(x, t) = \sin(x + t) + \sin(x - t)$ is a solution of of one of the basic partial differential equations. Which one?

i) What is the distortion factor of the coordinate change $r : \mathbb{R}^3 \to \mathbb{R}^3, (x, y, z) \to (3x, 4y, 7z)$?

j) You are on Elysium, a torus shaped artificial habitat on which the height function of the hills is a Morse function. There are 5 hills (maxima) and 2 sinks (minima). How many saddle points are there on Elysium?

Problem 28.3 (10 points) Each question is one point:

We see the level curves of a Morse function f. In every of the question, we pick **exactly one point** from A,B,C,D,E,F,G,H,I,J,K,L. Points might appear several times and some points might not appear.

a) Which point is a local maximum?
b) Which point is a local minimum?
c) Which point is a saddle point?
d) Which point is a local minima of f under the constraint $g(x, y) = y = 0$?
e) Which point is a local maxima of f under the constraint $g(x, y) = y = 0$?
f) At which point is $|\nabla f(x, y)|$ maximal among all the points?
g) At which point is $f_x(x, y)$ positive and $f_y(x, y) = 0$?
h) At which point is $f_y(x, y)$ positive and $f_x(x, y) = 0$?
i) At which point are both $f_x(x, y)$ and $f_y(x, y)$ positive?
j) At which point are both $f_x(x, y)$ and $f_y(x, y)$ negative?
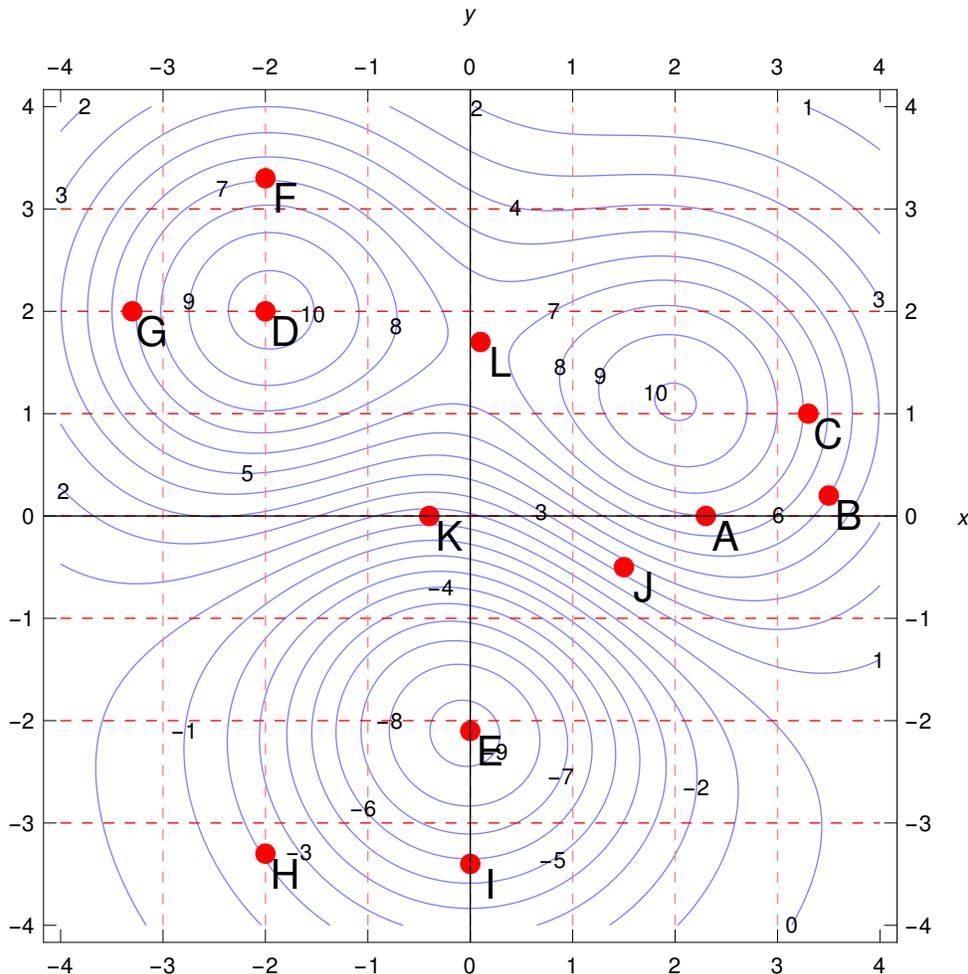
Problem 28.4 (10 points):

a) (5 points) Find the tangent hyper plane $ax + by + cz + dw = e$ to the hyper surface

$$f(x, y, z, w) = xy^2z^2 + w = 2$$

at the point $(x_0, y_0, z_0, w_0) = (2, -1, -1, 0)$.

b) (5 points) Estimate $f(2.001, -0.9, -1.01, 0.07)$ by linear approximation.

Problem 28.5 (10 points):

a) (6 points) Classify the critical points of the function

$$f(x, y) = 3 - 3x + x^2 - 3y + xy + y^2$$

using the second derivative test.

b) (2 points) Does the function $f(x, y)$ have a global minimum?

c) (2 points) Does the function $f(x, y)$ have a global maximum?

Problem 28.6 (10 points):

a) (4 points) Find the quadratic approximation $Q(x, y)$ of the function

$$f(x, y) = 3 - 3x + x^2 - 3y + xy + y^2$$

at $(x_0, y_0) = (1, 1)$. We have already seen this function in Problem 28.5).

b) (3 points) Is this function f a Morse function?

c) (3 points) Estimate the value of $f(1.03, 0.2)$ using quadratic approximation.

Problem 28.7 (10 points):

Using the Lagrange optimization method, find the parameters (x, y) for which

$$f(x, y) = 3 - 3x + x^2 - 3y + xy + y^2$$

is maximal or minimal under the constraint

$$g(x, y) = x^2 + y^2 = 2 .$$

Problem 28.8 (10 points):

a) (5 points) Evaluate the integral

$$I = \iint_G e^{x^2+y^2} \, dy dx$$

of the annular region $G = \{1 \le x^2 + y^2 \le 4\}$.

b) (5 points) Evaluate the double integral

$$\int_1^3 \int_0^{9-x^2} \frac{y^2}{\sqrt{9-y}-1} \, dy \, dx \, .$$

Problem 28.9 (10 points):

Integrate

$$\iiint_E (x^2 + y^2 + z^2)^2 \, dz dy dx$$

for

$$E = \{(x, y, z) \mid 4 \le x^2 + y^2 + z^2 \le 9, \ x^2 + y^2 < z^2 \} \, .$$

Problem 28.10 (10 points):

Compute the **surface area** of the surface

$$r(u, v) = \begin{bmatrix} 2v \cos(u) \\ 2v \sin(u) \\ u^2 \end{bmatrix}$$
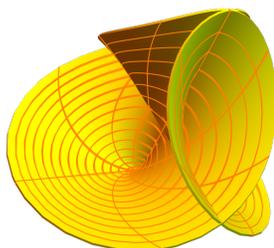
over the region $R = \{u^2 + v^2 \le 9\}$.

FIGURE 1. The surface in problem 10.

Unit 29: Line integrals

INTRODUCTION

29.1. Today, we learn already how to generalize the **fundamental theorem of calculus** $\int_a^b f'(t)\ dt = f(b) - f(a)$ to higher dimensions. The interval $[a, b]$ is now replaced by a curve and the derivative $f'(t)$ becomes $\frac{d}{dt} f(r(t))$ which by the chain rule is $\nabla f(r(t)) \cdot r'(t)$. If we integrate this from a to b we get the **fundamental theorem of line integrals**.

$$\int_a^b \nabla f(r(t)) \cdot r'(t)\ dt = \int_a^b \frac{d}{dt} f(r(t)) = f(r(b)) - f(r(a))\ .$$

The **gradient field** $\nabla f(x)$ can be generalized to a general vector field field $x \to F(x)$, a map which assigns to every point a vector.
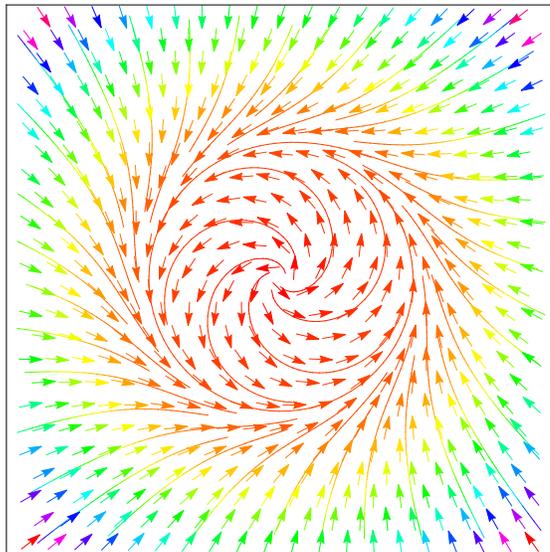
FIGURE 1.   The vector field $F(x, y) = [P(x, y), Q(x, y)]^T = [x - y - x(x^2 + y^2), x + y - y(x^2 + y^2)]^T$ is shown with some flow lines tracing the field. In this case there exists a single flow line which is a circle. Everything gets attracted to it. It is called a limit cycle. **Hilbert 16'th problem** asked to give an upper bound for the number of possible limit cycles if P, Q are polynomials in x, y of degree n. The problem is open.

29.2. One of the questions we want to answer is under which conditions a general vector field F is a gradient field $F = \nabla f$. The reason is that if this is the case, then the integral $\int_a^b F(r(t) \cdot r'(t) \, dt$ is easy to evaluate. If F is a gradient field, the result is $f(r(b)) - f(r(a))$. In general however, vector fields are not gradient fields. In the above figure we see an example. Not all hope is lost however. We will learn in the next two weeks that in some cases, like of the path is closed, we have other ways to compute the line integral.

29.3. A good way to think about line integral is to see it as **mechanical work**. The vector field F then is thought of as a force field and the product of the force with the velocity $F \cdot r'$ is **power**, which is a scalar. Integrating power over a time gives **work**. In the case when F was a gradient field $F = \nabla f$, then f is considered a **potential energy**. The fundamental theorem of line integrals now tells that the work done over some time is just the potential energy difference. It is not really necessary to adopt this picture. The set-up is purely mathematical but in order to remember it, it can be helpful to see it associated with concepts we know. If you bike for example, then both the force applied to the pedals as well as the velocity matters.

LECTURE

29.4. A **vector field** F assigns to every point $x \in \mathbb{R}^n$ a vector $F(x) = [F_1(x), \ldots, F_n(x)]^T$ such that every $F_k(x)$ is a continuous function. We think of F as a **force field**. Let $t \to r(t) \in \mathbb{R}^n$ be a curve parametrized on $[a, b]$. The integral

$$\int_C F \cdot \, dr = \int_a^b F(r(t)) \cdot r'(t) \, dt$$

is called the **line integral** of F along C. We think of $F(r(t)) \cdot r'(t)$ as **power** and $\int_C F \cdot \, dr$ as the **work**. Even so F and r are column vectors, we write in this lecture $[F_1(x), \ldots, F_n(x)]$ and $r' = [x'_1, \ldots, x'_n]$ to avoid clutter. Mathematically, $F : \mathbb{R}^n \to \mathbb{R}^n$ can also be seen as a coordinate change, we think about it differently however and draw a vector $F(x)$ at every point x.
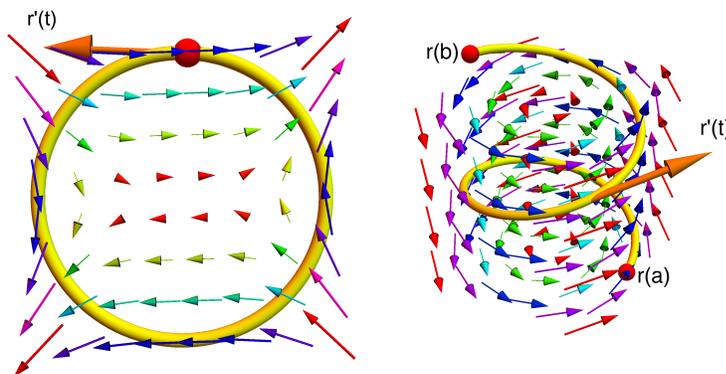

FIGURE 2.   A line integral in the plane and a line integral in space.

29.5. If $F(x,y) = [y, x^3]$, and $r(t) = [\cos(t), \sin(t)]$ a circle with $0 \le t \le 2\pi$, then $F(r(t)) = [\sin(t), \cos^3(t)]$ and $r'(t) = [-\sin(t), \cos(t)]$ so that $F(r(t)) \cdot r'(t) = -\sin^2(t) + \cos^4(t)$. The work is $\int_C F \cdot dr = \int_0^{2\pi} -\sin^2(t) + \cos^4(t) \, dt = -\pi/4$. Figure 1 shows the situation. We go more against the field than with the field.

29.6. A vector field F is called a **gradient field** if $F(x) = \nabla f(x)$ for some differentiable function f. We think of f as the **potential**. The first major theorem in vector calculus is the **fundamental theorem of line integrals** for gradient fields in $\mathbb{R}^n$:

> **Theorem:** $\int_a^b \nabla f(r(t)) \cdot r'(t) \, dt = f(r(b)) - f(r(a))$.

29.7. Proof: by the **chain rule**, $\nabla f(r(t)) \cdot r'(t) = \frac{d}{dt} f(r(t))$. The **fundamental theorem of calculus** now gives $\int_a^b \frac{d}{dt} f(r(t)) \, dt = f(r(b)) - f(r(a))$. QED.

29.8. As a corollary we immediately get path independence

> If C_1, C_2 are two curves from A to B then $\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$,

as well as the closed loop property:

> If C is a closed curve and $F = \nabla f$, then $\int_C F \cdot dr = 0$.

29.9. Is every vector field F a gradient field? Lets look at the case $n = 2$, where $F = [P, Q]$. Now, if this is equal to $[f_x, f_y] = [P, Q]$, then $P_y = f_{xy} = f_{yx} = Q_x$. We see that $Q_x - P_y = 0$. More generally, we have the following **Clairaut criterion**:

> **Theorem:** If $F = \nabla f$, then $\text{curl}(F)_{ij} = \partial_{x_j} F_i - \partial_{x_i} F_j = 0$.

Proof: this is a consequence of the Clairaut theorem.

29.10. The field $F = [0, x]$ for example satisfies $Q_x - P_y = 1$. It can not be a gradient field. Now, if $Q_x - P_y = 0$ everywhere in the plane, how do we find the potential f?

> Integrate $f_x = P$ with respect to x and add a constant $C(y)$.

> Differentiate f with respect to y and compare f_y with Q. Solve for $C(y)$.

29.11. Example: find the potential of $F(x,y) = [P, Q] = [2xy^2 + 3x^2, 2x^2y + 3y^2]$. We have $f(x,y) = \int_0^x 2xy^2 + 3x^2 \, dx + C(y) = x^3 + x^2y^2 + C(y)$. Now $f_y(x,y) = 2x^2y + C'(y) = 2x^2y + 3y^2$ so that $C'(y) = 3y^2$ or $C(y) = y^3$ and $f = x^3 + x^2y^2 + y^3$.

29.12. Here is a direct formula for the potential. Let C_{xy} be the straight line path which goes from $(0,0)$ to (x,y).

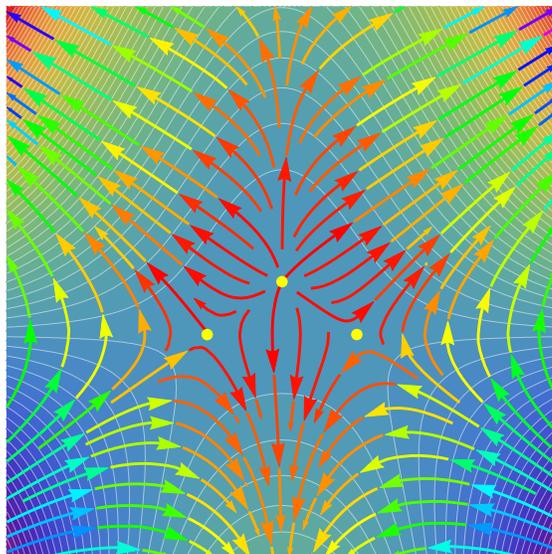> **Theorem:** If F is a gradient field then $f(x,y) = \int_{C_{xy}} F \cdot dr$.

FIGURE 3. The vector field $F = \nabla f$ for $f(x, y) = y^2 + 4yx^2 + 4x^2$. We see the **flow lines**, curves with $r'(t) = F(r(t))$. Going with the flow **increases** f because $F(r(t)) \cdot r'(t) = |\nabla f(t)|^2$ is equal to $d/dt f(r(t))$.

29.13. Proof: By the fundamental theorem of line integral, we can replace C_{xy} by a path $[t, 0]$ going from $(0, 0)$ to $(x, 0)$ and then with $[x, t]$ to (x, y). The line integral is $f(x, y) = \int_0^x [P, Q] \cdot [1, 0] dt + \int_0^y [P, Q] \cdot [0, 1] \, dt = \int_0^x P(t, 0) \, dt + \int_0^y Q(x, t) \, dt$. We see that $f_y = Q(x, y)$. If we use the path going $(0, 0)$ to $(0, y)$ and to (x, y) instead, the line integral is $f(x, y) = \int_0^y [P, Q] \cdot [0, 1] dt + \int_0^x [P, Q] \cdot [1, 0] \, dt = \int_0^y Q(0, t) \, dt + \int_0^x P(t, y) \, dt$. Now, $f_x = P(x, y)$. QED.

EXAMPLES

29.14. Find $\int_C [2xy^2 + 3x^2, 2x^2y + 3y^2] \cdot \, dr$ for a curve $r(t) = [t \cos(t), t \sin(t)]$ with $t \in [0, 2\pi]$. Answer: we found already $F = \nabla f$ with $f = x^3 + x^2y^2 + y^3$. The curve starts at $A = (1, 0)$ and ends at $B = (2\pi, 0)$. The solution is $f(B) - f(A) = 8\pi^3$.

29.15. If $F = E$ is an electric field, then the line integral $\int_a^b E(r(t)) \cdot r\,'(t) \, dt$ is an **electric potential**. In celestial mechanics, if F is the gravitational field, then $\int_a^b F(r(t)) \cdot r\,'(t) \, dt$ is a **gravitational potential** difference. If $f(x, y, z)$ is a temperature and $r(t)$ the path of a fly in the room, then $f(r(t))$ is the temperature, which the fly experiences at the point $r(t)$ at time t. The change of temperature for the fly is $\frac{d}{dt} f(r(t))$. The line-integral of the temperature gradient ∇f along the path of the fly coincides with the temperature difference.

29.16. A device which implements a non-gradient force field is called a **perpetual motion machine**. It realizes a force field for which the energy gain is positive along some closed loop. The **first law of thermodynamics** forbids the existence of such a machine. It is informative to contemplate the ideas which people have come up and to see why they don't work. We will look at examples in the seminar.

29.17. Let $F(x, y) = [P, Q] = [\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}]$. Its potential $f(x, y) = \arctan(y/x)$ has the property that $f_x = (-y/x^2)/(1 + y^2/x^2) = P, f_y = (1/x)/(1 + y^2/x^2) = Q$. In the seminar you ponder the riddle that the line integral along the unit circle is not zero:

$$\int_0^{2\pi} [\frac{-\sin(t)}{\cos^2(t) + \sin^2(t)}, \frac{\cos(t)}{\cos^2(t) + \sin^2(t)}] \cdot [-\sin(t), \cos(t)] \, dt = \int_0^{2\pi} 1 \, dt = 2\pi \; .$$
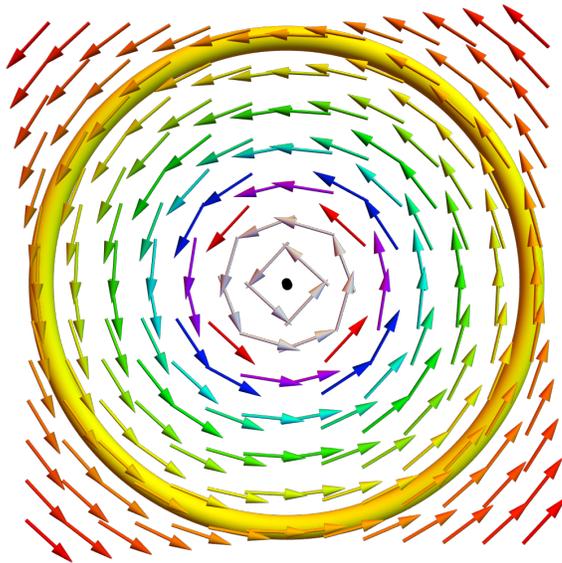
The vector field F is called the vortex.

FIGURE 4.   The vortex vector field has a singularity at $(0, 0)$. All the curl is concentrated at $(0, 0)$.

Homework

Problem 29.1: Let C be the space curve $r(t) = [\cos(t), \sin(t), \sin(t)]$ for $t \in [0, \pi/2]$ and let $F(x, y, z) = [y, x, 15]$. Calculate the line integral $\int_C F \cdot dr$.

Problem 29.2: What is the work done by moving in the force field $F(x, y) = [2x^3 + 1, 4\pi \sin(\pi y^4)y^3]$ along the quartic $y = x^4$ from $(-1, 1)$ to $(1, 1)$?

Problem 29.3: Let F be the vector field $F(x, y) = [-y, x]/2$. Compute the line integral of F along the curve $r(t) = [a\cos(t), b\sin(t)]$ with width $2a$ and height $2b$. The result should depend on a and b.

Problem 29.4: Archimedes swims around a curve $x^{22} + y^{22} = 1$ in a hot tub, in which the water has the velocity $F(x, y) = [3x^3 + 5y, 10y^4 + 5x]$. Calculate the line integral $\int_C F \cdot dr$ when moving from $(1, 0)$ to $(-1, 0)$ along the curve.

Problem 29.5: Find a closed curve $C : r(t)$ for which the vector field
$$F(x, y) = [P(x, y), Q(x, y)] = [xy, x^2]$$
satisfies $\int_C F(r(t)) \cdot r'(t) \, dt \neq 0$.

Unit 30: Perpetual motion machines

Introduction

30.1. Wouldn't it be nice to have a machine which produces energy from nothing? Humans have dreamed about this for centuries. There is no mathematical proof that such a machine can not exist. It is an experimental fact that all isolated physical process we know preserve energy. [1] In experiments, we see that all basic forces of nature are gradient fields. So, how come we can harvest energy from the wind force for example? Wind energy is driven by external sources, in particular the solar energy which heats up different parts of the earth surface. The sun energy comes from nuclear processes, mainly the fusion process.

30.2. It is a nice sport to come up with machines which seem to work or then to analyze a given machine which has been constructed and to find why it fails.

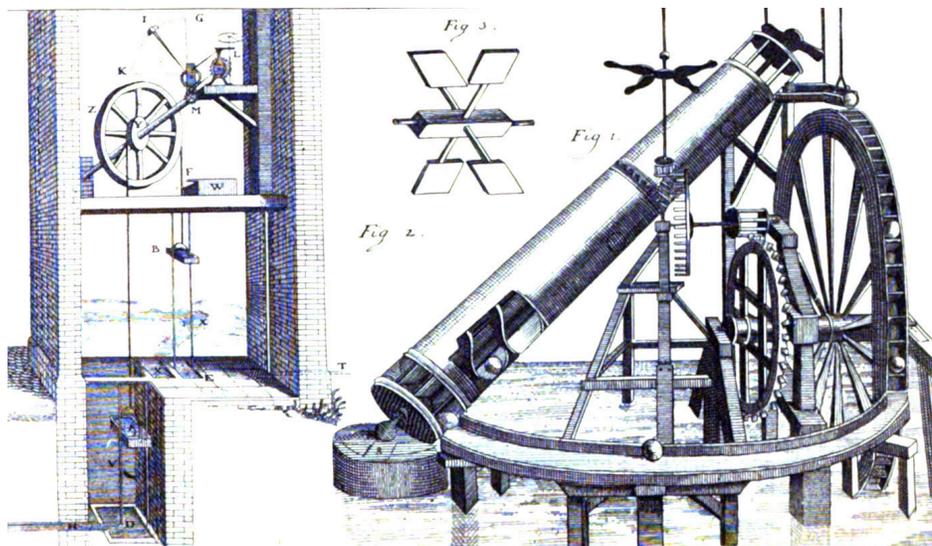FIGURE 1. A perpetual motion machine by Ulrich von Kranach in 1664 is a variant of Robert Fludd's Water wheel from 1618. Picture from Gentleman's Magazine in 1747, where it is stated that the inventor spent 30 years before bringing the machine to perfection. University of Michigan Archive. Public domain.

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[1] Except for very short time, where virtual particles can appear and disappear in a short time frame.

SEMINAR

30.3. Our first machine is a circular pipe which is half filled with water. On the side without water, the **gravitational force** pulls a wooden ball down. On the water side, the **buoyancy force** pulls the ball up. Valves are in place so that the water stays in place.

> **Problem A:** Analyse the pipe machine. You can assume that operating the valves uses arbitrary little energy and when opening one of the valves, the water stays in place.
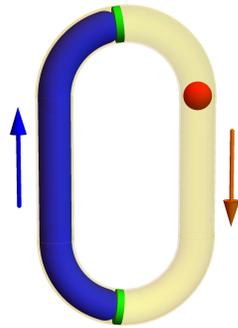
FIGURE 2. A half filled pipe produces a non-conservative force field.

30.4. An other class of machines uses magnets. Magnets are arranged in a circular way to produce a circular non-conservative force field in which a magnet is pushed forward.
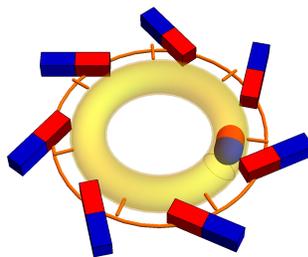

FIGURE 3. Magnets arranged so that a magnet always gets pushed forward (positive parts of magnets repel, equal parts attract).

> **Problem B:** Analyze the magnet machine. Experiment with real magnets.
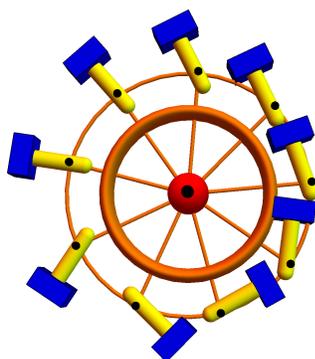
FIGURE 4. Mechanical perpetual motion machine: the torque on the left hand side is larger as the hammers are further out. The wheel moves counter clockwise.

30.5. And then there are mechanical machines. Here is an example with weights.

Problem C: Analyze the hammer machine using line integrals using the gravitational potential $f(x, y, z) = z$.

30.6. You all know that a sponge, a paper or a plant put into water lifts up the water using the **capillary effect**. In narrow spaces, this force can beat gravity.
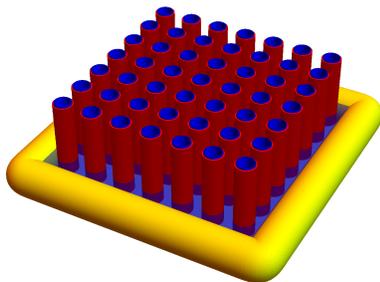
FIGURE 5. The capillary effect lifts the water level.

Problem D: Why does the "capillary lifting machine" not work?

30.7. Why are there no "perpetual motion machines"? There is no fundamental principle which forbids it. We could certainly produce a computer simulation of a world, where energy conservation fails. But it is like with "time machines". If such a machine would exist in our physical world, there would be serious dangers luring for a physicist who studies it. Benjamin Peirce refers in his book "A system of analytic mechanics" of 1855 to the **"Antropic Principle"**: *"Such a series of motions would receive the technical name of a 'perpetual motion' by which is to be understood, that of a system*

which would constantly return to the same position, with an increase of power, unless a portion of the power were drawn off in some way and appropriated, if it were desired, to some species of work. A constitution of the fixed forces, such as that here supposed and in which a perpetual motion would possible, may not, perhaps, be incompatible with the unbounded power of the Creator; but, if it had been introduced into nature, it would have proved destructive to human belief, in the spiritual origin of force, and the necessity of a First Cause superior to matter, and would have subjected the grand plans of Divine benevolence to the will and caprice of man".

Problem E: Can you reformulate this "anthropic principle" in more modern terms? What could be the fate of a universe in which energy conservation does not hold in macroscopic physics?

30.8. Non-conservative fields can also be generated by **optical illusion** as **M.C. Escher** did. The illusion suggests the existence of a force field which is not conservative. Can you figure out how Escher's pictures "work"? This is part of the homework. Here is a last possible task for the seminar:

Problem F: Find some perpetual motion machine on the web (i.e. youtube). If you find something interesting, share with others. Why does it not work?

Problem G: Finally, if you have adventure spirit, Come up with a machine which actually works, get some seed money from investors or by crowd sourcing and then start production.

FIGURE 6. Escher Stairs.

30.1 The force field $F(x,y) = [-y/(x^2+y^2), x/(x^2+y^2)]$ produces a non-conservative force. A body put near the vortex will spin around it. Check that that the field is a gradient field $F = \nabla f$ with $f = \arctan(y/x)$. Draw the level curves of f, take a path $r(t) = [\cos(t), \sin(t)]$ and verify that $d/dt f(r(t)) = \nabla f(r(t)) \cdot r'(t) = 1$ for all t so that $\int_0^{2\pi} \nabla f(r(t)) \cdot r'(t)\ dt = 2\pi$. Why does this not contradict the fundamental theorem?

30.2 If $H(x,y)$ is a function of two variables, then a curve $r(t) = [x(t), y(t)]$ satisfying $x'(t) = H_y(x,y)$ and $y'(t) = -H_x(x,y)$ is called a solution of the Hamiltonian system. The function $H(x,y)$ is called the energy of the system. Verify that $d/dt H(x(t), y(t)) = 0$ meaning that energy is conserved.

30.3 Design a vector field $F(x,y) = [P(x,y), Q(x,y)]$ which has the property such that for any closed curve $C : r(t)$ in $\{x^2 + y^2 > 1\}$ winding once around the hole $\{x^2 + y^2 \leq 1\}$, the line integral $\int_C F(r(t)) \cdot r'(t)\ dt$ is a multiple of 6π. An example of a curve winding once around is $r(t) = [2\cos(t), 2\sin(t)]$ with $0 \leq t \leq 2\pi$.

30.4 A heat engine is a system that convert heat energy into mechanical energy. We have seen such a machine in class. How does it work?

30.5 Explain the Escher waterfall illusion.

Unit 31: Green's theorem

Introduction

31.1. You might have seen the movie "Good Will Hunting" or the movie "The man who knew infinity". The former movie was inspired by Ramanujan, the main character in the second. Unlike "Good Will Hunting", which is pure fiction, the story of Ramanujan is real. He was a self-taught mathematician who made amazing discoveries. There is an older story, which also is true. **George Green 1893-1841)** was a British mathematician who first described a mathematical framework for electricity and magnetism paving the way for Clerk Maxwell and Lord Kelvin.

31.2. The theorem we are going to look at is a theorem about vector fields $F = [P, Q]$ in the plane. Its derivative dF is called the curl of F which is the scalar field $Q_x - P_y$. The theorem tells that

$$\iint_G dF \, dA = \int_{\delta G} F \cdot dr \ .$$

This is completely analog to $\int_I f'(x)dx = f(b) - f(a)$ because the later is the integral of f along the boundary δG of $I = [a, b]$.

FIGURE 1. Good Will Hunting and The Man who knew Infinity.

31.3. Green's theorem has been first described by Cauchy. Since Green discovered Gauss Theorem first, the nomenclature of the theorem is a bit strange. Still, since Green saw the general structure of the integral theorems first: integrating the derivative of a field over a manifold is the same than integrating the field over the boundary. In short $\int_G dF = \int_{\delta G} F$. When looking at two dimensional objects the derivative of a field $F = [P, Q]^T$ is $Q_x - P_y$. The boundary of a planar region is the rim, the curve bounding the region. It is important that the orientation of the curve matches the orientation of the region. We transverse the curve in such a way that the region is to our left.

31.4. A remark about notation: we will often also write just F as a row vector field $F = [P, Q]$. This is also called a **differential 1-form**. It is technically correct to write the matrix product $F dr$ instead of the dot product $F^T \cdot dr$. Also, remember that df is a row vector field and $\nabla f = df^T$ a column vector field. Also about notation we should note that it is custom to call continuous functions or fields or curves C^0 and functions, fields or curves which are continuously differentiable C^1. We most of the time assume in calculus that all objects are at least piecewise C^1. Regions can be squares for example but not a region bound by a Koch snowflake.

LECTURE

31.5. For a C^1 vector field $F = [P, Q]$ in a region $G \subset \mathbb{R}^2$, the **curl** is defined as $\mathrm{curl}(F) = Q_x - P_y$. Assume the boundary C of G oriented so that the region G is **to the left** (meaning that if $r(t) = [x(t), y(t)]$ is a parametrization, then the turned velocity $[-y'(t), x'(t)]$ cuts through G close to $r(t)$). **Green's theorem** assures that if C is made of a finite collection of smooth curves, then

Theorem: $\iint_G \mathrm{curl}(F)\, dxdy = \int_C F(r(t)) \cdot dr(t)$.

31.6. Proof. It is enough to prove the theorem for $F = [0, Q]$ or $F = [P, 0]$ separately and for regions G which are both "bottom to top" $G = B = \{a \le x \le b, c(x) \le y \le d(x)\}$ and "left to right" $G = L = \{c \le y \le d, a(y) \le x \le b(y)\}$. For $F = [P, 0]$, use a bottom to top integral, where the two vertical integrals along $r(t) = [b, t]$ and $r(t) = [a, t]$ are zero. The integrals along $r(t) = [t, c(t)]$ and $r(t) = [t, d(t)]$ give

$$\int_b^b P(s, c(s))\, ds - \int_a^b P(s, d(s))\, ds = \int_a^b \int_{c(t)}^{d(t)} -P_y(t, s)\, dsdt = \iint_G -P_y\, dsdt \ .$$

For $F = [Q, 0]$, use a left to right integral, where the bottom and top integrals are zero and where

$$\int_c^d Q(b(t), t)\, dt - \int_c^d Q(a(t), t)\, dt = \int_c^d \int_{a(s)}^{b(s)} Q_x(t, s)\, dtds = \iint_G Q_x\, dsdt \ .$$

Together, write $F = [0, Q] + [P, 0]$, use the first computation for $[P, 0]$ and the second computation for $[0, Q]$. In general, cut G along a small grid so that each part is of both types. When adding the line integrals, only the boundary survives. QED.
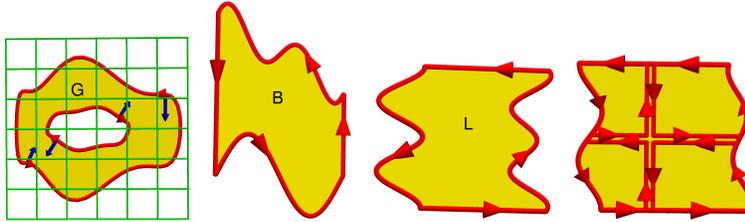
FIGURE 2. To prove Green cut the region into regions which are "bottom to top" and "left to right". Interior cuts cancel.

31.7. To see that we can cut G into regions of both types, turn the coordinate system first a tiny bit so that no horizontal nor vertical line segments appear at the boundary. This is possible because we assume the boundary to consist of finitely many smooth pieces. Now also use a slightly turned grid to chop up the region into smaller parts. Now we have a situation where each piece has the form $G = \{(x,y) \mid c(x) \leq y \leq d(x)\} = \{(x,y) \mid a(y) \leq x \leq b(y)\}$, where a, b, c, d are piecewise smooth functions.
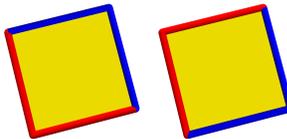

FIGURE 3. A case where we integration bottom to top and a case where we integrate left to right.

31.8. Green assures:

Theorem: If F is irrotational in $\mathbb{R}^2$, then F is a gradient field.

31.9. There are four properties which are equivalent if F is differentiable in $\mathbb{R}^2$: A) F is a gradient field, B) F has the closed loop property, C) F has the path independence property, and D) F is irrotational. We have seen in the proof seminar that the vortex vector field $F = [-y, x]/(x^2 + y^2)$ is a counter example to a more general theorem if the field is not differentiable at some point.

APPLICATIONS

31.10. Green's theorem allows to compute **areas**. If $\text{curl}(F) = 1$ and C is a curve enclosing a region G, then $\text{Area}(G) = \int_C F(r(t)) \cdot r'(t) \; dt$. For example, with $F = [-y, x]/2$, and $r(t) = [a\cos(t), b\sin(t)]$, then

$$\int_C F \cdot dr = \int_0^{2\pi} \begin{bmatrix} -b\sin(t) \\ a\cos(t) \end{bmatrix} \cdot \begin{bmatrix} -a\sin(t) \\ b\cos(t)]/2 \end{bmatrix} = \int_0^{2\pi} ab/2 \; dt = \pi ab$$

is the **area of the ellipse** $x^2/a^2 + y^2/b^2 = 1$.

31.11. What is the area of the region enclosed by $r(t) = [\cos(t), \sin(t) + \cos(22t)/22]$? Take $F(x,y) = [0, x]$. The line integral is $\int_0^{2\pi} [0, \cos(t)] \cdot [-\sin(t), \cos(t) - \sin(22t)] \; dt = \pi$.

31.12. The **planimeter** is an analogue computer which computes the area of regions. It works because of Green's theorem. The vector $F(x, y)$ is a unit vector perpendicular to the second leg $(a, b) \to (x, y)$ if $(0, 0) \to (a, b)$ is the second leg. Given (x, y) we find (a, b) by intersecting two circles. The magic is that the curl of F is constant 1. The following computer assisted computation proves this:

```
s=Solve[{(x−a)^2+(y−b)^2==1,a^2+b^2==1},{a,b}];
{A,B}=First[{a,b}/.s];F={−(y−B),x−A};Simplify[Curl[F,{x,y}]]
```
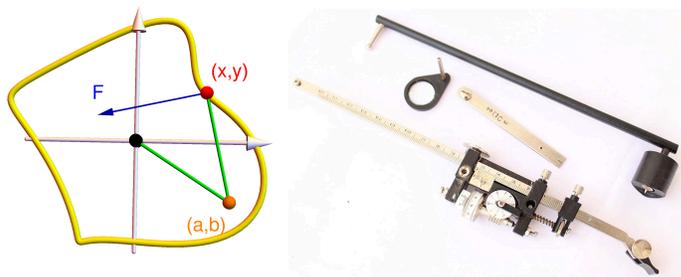

FIGURE 4. The planimeter is an analog computer which allows to compute the area of a region enclosed by a curve. The mechanical planimeter we will see in class.

EXAMPLES

31.13. Problem: Compute $F(x, y) = [x^2 - 4y^3/3, 8xy^2 + y^5]$ along the boundary of the rectangle $[0, 1] \times [0, 2]$ oriented counter clockwise. **Solution:** Since $\mathrm{curl}(F) = Q_x - P_y = 8y^2 + 4y^2 = 12y^2$ we have $\int_C F \cdot dr = \int_0^1 \int_0^2 12y^2 \, dy dx = 32$.

31.14. Problem: Find the line integral of the vector field

$$F(x, y) = \begin{bmatrix} x + y \\ 3x + 3y^2 \end{bmatrix}$$

along the boundary C of the quadratic **Koch island**. The counter clockwise oriented C encloses the island G which has 289 unit squares. **Solution:** $\mathrm{curl}(F) = 2$, so that $\iint_G 2dA = 2\mathrm{Area}(G) = 578$.

FIGURE 5. Koch islands constructed by a **Lindenmayer system**, a recursive grammar. It starts with $F + F + F + F$ and recursion $F \to F - F + F + FFF - F - F + F$. [F="moving forward by 1", $+$ = "turn by 90 degrees", $-$ = "turn by (-90) degrees".]

HOMEWORK

Problem 31.1: Calculate the line integral $\int_C F \cdot dr$ with $F = [-11y + 3x^2 \sin(y) + e^{7777 \sin(x^6)}, 11x + x^3 \cos(y) + 2ye^{22\sin(y)}]^T$ along a triangle C which traverses the vertices $(0,0)$, $(7,0)$ and $(7,11)$ back to $(0,0)$ in this order.

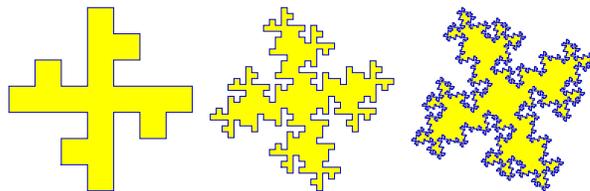**Problem 31.2:** A classical problem asks to compute the area of the region bounded by the **hypocycloid**

$$r(t) = [4\cos^3(t), 4\sin^3(t)], 0 \le t \le 2\pi .$$

We can not do that directly so easily. Guess which theorem to use, then use it!

Problem 32.3: Find $\int_C [\sin(\sqrt{1+x^3}), 7x] \cdot dr$, where C is the boundary of the region $K(n)$. You see in the picture $K(0), K(1), K(2), K(3), K(4)$. The first $K(0)$ is an equilateral triangle of length 1. The second $K(1)$ is $K(0)$ with 3 equilateral triangles of length $1/3$ added. $K(2)$ is $K(1)$ with $3*4^1$ equilateral triangles of length $1/9$ added. $K(3)$ is $K(2)$ with $3*4^2$ of length $1/27$ added and $K(4)$ is $K(3)$ with $3*4^3$ triangles of length $1/81$ added. What is the line integral in the Koch Snowflake limit $K = K(\infty)$? The curve K is a **fractal** of dimension $\log(4)/\log(3) = 1.26\ldots$.
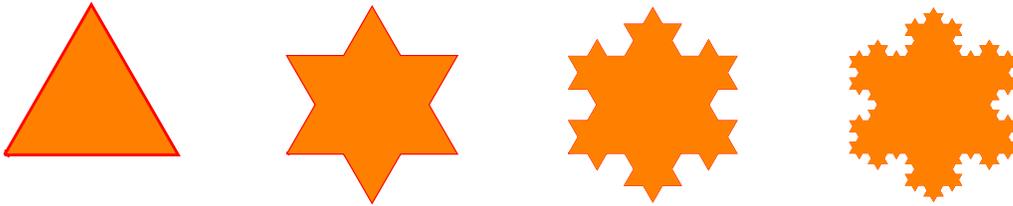
FIGURE 6. The first 4 approximations of the Koch curve.

Problem 32.4: Given the scalar function $f(x,y) = x^5 + xy^4$, compute the line integral of

$$F(x,y) = [5y - 3y^2, -6xy + y^4] + \nabla(f)$$

along the boundary of the **Monster region** given in the picture. There are four boundary curves, oriented as shown in the picture: a large ellipse of area 16, two circles of area 1 and 2 as well as a small ellipse (the mouth) of area 3. "Mike" from **Monsters, Inc.** warns you about orientations!

Problem 32.5: Let C be the boundary curve of the white Yang part of the Yin-Yang symbol in the disc of radius 6. You can see in the image that the curve C has three parts, and that the orientation of each part is given. Find the line integral of the vector field

$$F(x, y) = [-y + \sin(e^x), x]^T$$

along C. There are three separate line integrals.
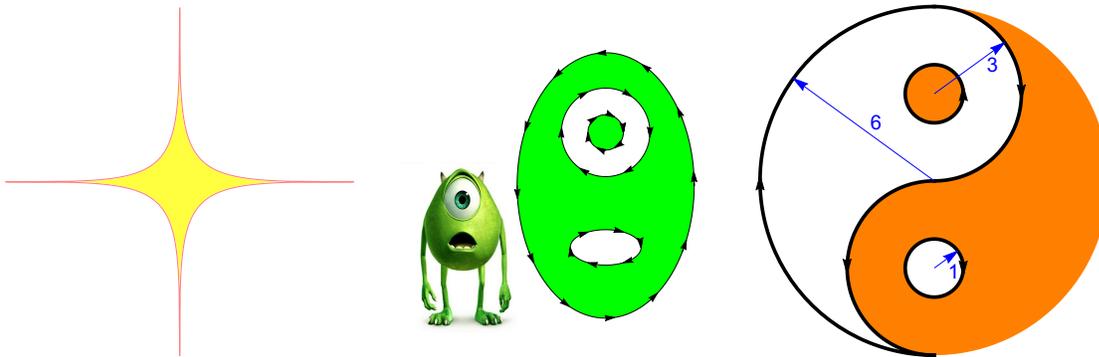
FIGURE 7. Hypocycloid, Monster and Yin-Yang

Unit 32: Stokes theorem

INTRODUCTION

32.1. Stokes theorem is a mountain peak in mathematics. You have not really lived before having climbed that mountain. The theorem was developed first in a physics context but it is important for other reasons. First, it is a place where many multi-variable concepts come together: it involves curves, surfaces, the dot and cross products, various derivatives like Jacobean or gradient, integrals or coordinate changes. If you master this theorem you own the bulk of this course. The theorem is also a prototype for a method in science: a theorem helps to solve problems which otherwise would be inaccessible. We will see quite many integrals which are not reachable without the theorem. Also, like mountain climbing, it produces some satisfaction top-out on something that important. The theorem is also **beautiful** $\boxed{\int_G dF = \int_{dG} F}$ and so art.

FIGURE 1. The **Matterhorn** in the southern part of Switzerland. Starting in the Hörnli hut (3262 meters, vectors, lines planes, curves, surfaces) one reaches the Solvay Bivouac at (4003 meters, extrema, Lagrange, integration) and arrives at the peak (4478 meters, Green, Stokes and Gauss). Image source: Wikimedia, CC BY-SA).

32.2. Proving the theorem was an exam problem given by George Stokes. James Clerk Maxwell who was a student there would later use it to formulate the **Maxwell equations** $\boxed{dF = 0, d^*F = j}$ for the **electromagnetic field** F and **charge-current** j. When **space-time** $\mathbb{R}^4$ is split into space and time, there are 4 equations. One of them is $\mathrm{curl}(E) = -\frac{\partial}{\partial t}B/c$. It explains how an **electric potential** $\int_C E\,dr$ emerges from **flux changes** of a **magnetic field** B when turning a wire C, allowing us to generate **electricity** from **motion**. When reversed, it turns electricity back into **mechanical energy**. Think about Stokes theorem next time you are using an **electric motor**!

LECTURE

32.3. Given a C^1 surface $S = r(G)$ in $\mathbb{R}^3$ using a parametrization $r = [x, y, z]$ and a C^1 vector field $F = [P, Q, R]$, we can form the **flux integral**

$$\iint_S F \cdot dS = \iint_G F(r(u, v)) \cdot r_u \times r_v \, dudv \ .$$

For $F = [P, Q, R]$, the **curl** is defined as $\nabla \times F = [R_y - Q_z, P_z - R_x, Q_x - P_y]$. The **Stokes theorem** tells that if $C = r(I)$ is the boundary of $S = r(G)$ and I is oriented so that G is to the left of C, then

> **Theorem:** $\iint_S \mathrm{curl}(F) \cdot dS = \int_C F \cdot dr.$

32.4. Proof. The key is the following **"important formula"**

> $$\mathrm{curl}(F)(r(u, v)) \cdot (r_u \times r_v) = F_u \cdot r_v - F_v \cdot r_u.$$

This is straightforward and done in class. Now define the field $\tilde{F}(u, v) = [\tilde{P}, \tilde{Q}] = [F(r(u, v)) \cdot r_u(u, v), F(r(u, v)) \cdot r_v(u, v)]$ in the uv-plane. The 2-dimensional curl of $\tilde{F}$ is $\tilde{Q}_u - \tilde{P}_v = F_u \cdot r_v - F_v \cdot r_u$ as we can see by using Clairaut $r_{uv} = r_{vu}$. The Stokes theorem is now a direct consequence of **Green's theorem** proven last time. QED. [1]
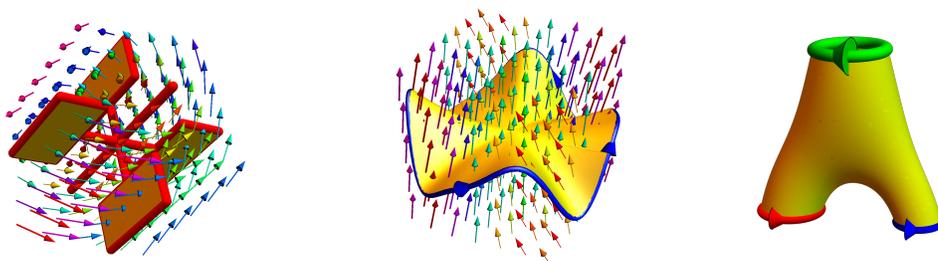
FIGURE 2. The paddle wheel measures curl. The boundary C has S "to the left". The pant surface illustrates a "cobordism". You definitely need to contemplate Stokes the next time you dress up your underpants!

EXAMPLES

32.5. Problem: Compute the flux of $F(x, y, z) = [0, 0, 8z^2]^T$ through the upper half unit sphere S oriented outwards. **Solution:** we parametrize the surface as $r(u, v) = [\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)]^T$. Because $r_u \times r_v = -\sin(v)r$, this parametrization has the wrong orientation! We continue nevertheless and just change the sign at the end. We have $F(r(u, v)) = [0, 0, 8\cos^2(v)]^T$ so that

$$\int_0^{2\pi} \int_0^{\pi/2} -[0, 0, 8\cos^2(v)]^T \cdot [\cos(u)\sin^2(v), \sin(u)\sin^2(v), \cos(v)\sin(v)]^T \, dvdu \ .$$

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[1] Mathematicians say: "we pulled back the field from $\mathbb{R}^3$ to $\mathbb{R}^2$ along the parametrization".

The flux integral is $\int_0^{2\pi}\int_0^{\pi/2}$ - $8\cos^3(v)\sin(v)\,dvdu$ which is $2\pi \cdot 8\cos^4(v)/4|_0^{\pi/2} = -4\pi$. The flux with the outward orientation is $+4\pi$. We could **not** use the Stokes theorem here because we don't deal with the flux of the curl but the flux of F itself.

32.6. Problem: What is the value of $\int_C F \cdot dr$ if $F = [\sin(\sin(x)) + z^2, e^y + x^3 + y^2, \sin(y^2) + z^2]$ and C is the unit polygon $(0,0,0) \to (1,0,0) \to (1,1,0) \to (0,1,0) \to (0,0,0)$? **Solution:** use Stokes theorem. The curl of F is $[2y\cos(y^2), 2z, 3x^2]$. The surface $S : r(u,v) = [u,v,0]$ with $0 \le u \le 1$ and $0 \le v \le 1$ has C as boundary. Stokes allows to compute $\iint_S \text{curl}(F) \cdot dS$ instead. Since $r_u \times r_v = [0,0,1]$, the flux integral is $\int_0^1\int_0^1 3u^2 dvdu = 1$. The computation of the line integral would have been more painful.

32.7. Problem: Compute the flux of the curl of $F(x,y,z) = [0,1,8z^2]^T$ through the upper half sphere S oriented outwards. **Solution:** Great, it is here, where we can use Stokes theorem $\iint_S \text{curl}(F) \cdot dS = \int_C F \cdot dr$, where C is the boundary curve which can be parametrized by $r(t) = [\cos(t), \sin(t), 0]^T$ with $0 \le t \le 2\pi$. Before diving into the computation of the line integral, it is good to check, whether the vector field is a gradient field. Indeed, we see that $\text{curl}(F) = [0,0,0]$. This means that $F = \nabla f$ for some potential f implying by the **fundamental theorem of line integrals** that $\int_C F \cdot dr = 0$. But wait a minute, if the curl of F is zero, couldn't we just have seen directly that the flux of the curl through the surface is zero? Yes, we could have seen that before: for a gradient field, the flux of the curl of F through a surface is always zero, for the simple reason that the curl of such a field is zero.

32.8. Problem. What is the flux of the curl of $F(x,y,z) = [\sin(xyz), ze^{\cos(x+y)}, zx^5 + z^{22}]$ through the lower ellipsoid S given by $x^2/4 + y^2/9 + z^2/16 = 1, z < 0$? **Solution:** by Stokes theorem, it is the line integral $\int_C F \cdot dr$. Through the boundary $r(t) = [2\cos(t), 3\sin(t), 0]$. But in the xy-plane $z = 0$, the field F is zero. The result is zero.

32.9. Problem: What is the flux of the curl of F through an ellipsoid $x^2/4 + y^2/9 + z^2/16 = 1$? **Solution:** We can cut the ellipsoid into two parts to get two surfaces with boundary. The upper part $S_+ = \{(x,y,z) \in S, z > 0\}$ has the boundary $C_+ : r(t) = [2\cos(t), 3\sin(t), 0]$ which matches the orientation of the surface. Stokes theorem tells that $\iint_{S_+} \text{curl}(F) \cdot dS = \int_{C_+} F \cdot dr$. The lower part $S_- = \{(x,y,z) \in S, z < 0\}$ has the boundary $C_- : r(t) = [2\cos(t), -3\sin(t), 0]$ which matches the orientation of the lower part. Stokes theorem tells that $\iint_{S_-} \text{curl}(F) \cdot dS = \int_{C_-} F \cdot dr$. Together we have $\int_{C_-} F \cdot dr + \int_{C_+} F \cdot dr = 0$ as the line integrals have just different signs. The result is zero.

Remarks

32.10. The left hand side of the **important formula** (it "imports" the curl) [2] is defined only in three dimensions. But the right hand side also makes sense **in $\mathbb{R}^n$**. It is $\operatorname{tr}((dF)^* dr)$, where * rotates the 2-frame by 90 degrees. The Stokes theorem for 2-surfaces works for $\mathbb{R}^n$ if $n \geq 2$. For $n = 2$, we have with $x(u,v) = u, y(u,v) = v$ the identity $\operatorname{tr}((dF)^* dr) = Q_x - P_y$ which is Green's theorem. Stokes has the general structure $\boxed{\int_G \delta F = \int_{\delta G} F}$, where δF is a derivative of F and δG is the boundary of G.

> **Theorem:** Stokes holds for fields F and 2-dimensional S in $\mathbb{R}^n$ for $n \geq 2$.

32.11. Why are we interested in $\mathbb{R}^n$ and not only in $\mathbb{R}^3$? One example is that 2-dimensional surfaces appear as "paths" which a **moving string** in 11 dimension traces. More important maybe is that statisticians work by definition in high dimensional spaces. When dealing with n data points, one works in $\mathbb{R}^n$. Why would you care about theorems like Stokes in statistics? As a matter of fact, integral theorems in general allow to **simplify computations**. As we have seen in Green's theorem, when computing the sum over all the curls, there are **cancellations** happening in the inside. Integral theorems "see these cancellations" and allow to **bypass and ignore stuff which does not matter**.

32.12. The fundamental theorem of line integrals $\int_a^b \operatorname{tr}(df(r(t)) dr(t)) dt = f(r(b)) - f(r(a))$ holds also in $\mathbb{R}^n$. The flux integral

$$\iint_G \operatorname{tr}(F^*(r(u,v)) dr(u,v)) \, du dv$$

is the analogue of a line integral in two dimensions. Written like this, we don't need the cross product. And not yet the language of **differential forms**.

32.13. Stokes deals with "fields" and "space". What happens if the field is space itself, that is if $F^* = dr$? It is of interest. For $m = 1$, and $F = dr^T$, then $\int_a^b |dr|^2 \, dt$ is the **action integral** in physics. A general **Maupertius principle** assures that it is equivalent to the **arc length** $\int_a^b |dr| \, dt$ in the sense that minimizing arc length between two points is equivalent to minimize the action integral (which is more like the energy one uses to get from the first point to the second). Now, in two dimensions we have $\iint_G \operatorname{tr}(dr^T dr) \, du dv$. We can compare this with $\iint_G \det(dr^T dr) \, du dv$ which is called the **Nambu-Goto action**, which resembles the **surface area** $\iint_G \sqrt{\det(dr^T dr)} \, du dv$ also called the **Polyakov action**. Nature likes to minimize. Free particles move on shortest paths, minimize the arc length. Maupertius tells that minimizing the length $\int_A^B |r'(t)| \, dt$ of a path equivalent to minimizing $\int_A^B r'(t) \cdot r'(t) \, dt$ which essentially is the integrated kinetic energy or gasoline use to go from A to B. For the purpose of minimizing stuff this also works for two dimensional actions. Minimizing the surface area $\iint_G |r_u \times r_v| \, du dv$ among all surfaces connecting two one dimensional curves is equivalent to minimize $\iint_G |r_u \times r_v|^2 \, du dv$. Also in higher dimensions, Nambu-Goto and Polyakov are equivalent.

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[2] I learned the "important formula" from Andrew Cotton-Clay in 2009: http://www.math.harvard.edu/archive/21a_fall_09/exhibits/stokesgreen

Problem 32.1: Use Stokes to find $\int_C F \cdot dr$, where $F(x, y, z) = [12x^2y, 4x^3, 12xy+e^{(e^z)}]$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.

Problem 32.2: Evaluate the flux integral $\int\int_S \text{curl}(F) \cdot dS$, where

$$F(x, y, z) = [xe^{y^2}z^3 + 2xyze^{x^2+z}, x + z^2e^{x^2+z}, ye^{x^2+z} + ze^x]^T$$

and where S is the part of the ellipsoid $x^2 + y^2/4 + (z + 1)^2 = 2$, $z > 0$ oriented so that the normal vector points upwards.

Problem 32.3: Find the line integral $\int_C F \, dr$, where C is the circle of radius 3 in the xz-plane oriented counter clockwise when looking from the point $(0, 1, 0)$ onto the plane and where F is the vector field

$$F(x, y, z) = [4x^2z + x^5, \cos(e^y), -4xz^2 + \sin(\sin(z))]^T .$$

Use a convenient surface S which has C as a boundary.

Problem 32.4: Find the flux integral $\int\int_S \text{curl}(F) \cdot dS$, where $F(x, y, z) =$

$$[y + 2\cos(\pi y)e^{2x} + z^2, x^2\cos(z\pi/2) - \pi\sin(\pi y)e^{2x}, 2xz + (z - 1)^{22}]^T$$

and S is the surface parametrized by

$$r(s, t) = [(1 - s^{1/3})\cos(t) - 4s^2, (1 - s^{1/3})\sin(t), 5s]^T$$

with $0 \leq t \leq 2\pi, 0 \leq s \leq 1$ and oriented so that the normal vectors point to the outside of the thorn.

FIGURE 3. Problem 32.4 is a thorny problem! You might definitely have to discuss this with somebody else.

Problem 32.5: Assume S is the surface $x^{22} + y^8 + z^6 = 100$ and $F = [e^{e^{22z}}, 22x^2yz, x - y - \sin(zx)]$. Explain why $\int\int_S \text{curl}(F) \cdot dS = 0$.

APPENDIX: APPLICATIONS

32.14. A region E in $\mathbb{R}^n$ is called **simply connected** if it is connected and for every closed loop C in E there is a continuous deformation C_s of C **within G** such that $C_0 = C$ and $C_1(t) = P$ is a point. For example, $C(t) = [\cos(t), \sin(t), 0]$ can be deformed in $E = \mathbb{R}^3$ to a point with $C_s(t) = [(1-s)\cos(t), (1-s)\sin(t), 0]$ as $C_1(t) = P = [0, 0, 0]$ for all t. Each Euclidean space $\mathbb{R}^n$ is simply connected. The region $G = \{x^2 + y^2 > 0\} \subset \mathbb{R}^3$ is not simply connected as the circle $C : r(t) = [\cos(t), \sin(t), 0]$ winding around the z-axis can not be pulled together to a point **within** G. The region $G = \{x^2 + y^2 + z^2 > 0\} \subset \mathbb{R}^3$ is simply connected, but $G = \{x^2 + y^2 > 0\}$ in $\mathbb{R}^2$ is not. Remember that F was called **irrotational** if $\mathrm{curl}(F) = 0$ everywhere.

Theorem: If F is irrotational on a simply connected E then $F = \nabla f$ in E.

32.15. Proof: since E is simply connected and $\mathrm{curl}(F) = 0$, every closed loop C can be filled in by a surface $S = \bigcup_{0 \le s \le 1} C_s$ which has the boundary C. Stokes theorem gives $\int_S F \cdot dr = \iint_S \mathrm{curl}(F) \cdot dS = 0$. The closed loop property implies path independence. A potential f can be obtained by fixing a base point p in E, then define for any other point x a path C_{px} going from p to x. The potential function f is then defined as $f(x) = \int_{C_{px}} F \cdot dr$. QED

32.16. The field $F(x, y, z) = [-y/(x^2 + y^2), x/(x^2 + y^2), 0]$ is defined everywhere except on the z-axis. The domain E, where F is defined is not simply connected. There is no global function f which is a potential for F.

32.17. The notion of "simply connectedness" is important in topology. The first solved **Millenium problem**, the **Poincaré conjecture**, is now a theorem. It tells that a 3-dimensional manifold which is simply connected is topologically equivalent to the 3-sphere $\{x^2 + y^2 + z^2 + w^2 = 1\} \subset \mathbb{R}^4$. In two dimensions, the result was known for a long time already, because the structure of 2-dimensional connected manifolds is known.

ELECTROMAGNETISM

32.18. The **Maxwell-Faraday equation** in electromagnetism relates the **electric field** E and the **magnetic field** B with the partial differential equation $\mathrm{curl}(E) = -\frac{d}{dt} B$. Given a surface S, the flux integral $\iint_S B \cdot dS$ is called the **magnetic flux** of B through the surface. If we integrate the Maxwell-Faraday equation, we see that $\iint_S \mathrm{curl}(E) \cdot dS$ is equal to minus the rate of change of the magnetic flux $-\frac{d}{dt} \iint_S B \cdot dS$. Stokes theorem now assures that $\iint_S \mathrm{curl}(E) \cdot dS = \int_C E \cdot dr$ is the line integral of the electric field along the boundary. But this is **electric potential** or voltage. We see:

We can generate an electric potential by changing the magnetic flux.

32.19. Changing the magnetic flux can happen in various ways. We can generate a changing magnetic field by using **alternating current**. This is how **transformers work**. An other way to change the flux is to **rotate a wire** in a fixed magnetic field. This is the **principle of the dynamo**:
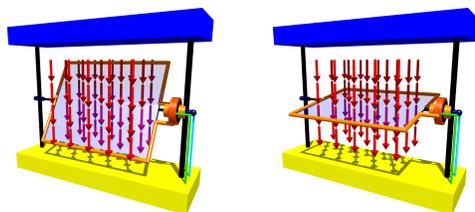
FIGURE 4.   The dynamo, implemented using the ray tracer Povray. Electric current is generated by moving a wire in a fixed magnetic field.

32.20. The vector field $A(x, y, z) = \frac{[-y, x, 0]}{(x^2 + y^2 + z^2)^{3/2}}$ is called the **vector potential** of a magnetic field $B = \text{curl}(A)$. The picture shows some flow lines of this **magnetic dipole field** B. **Problem:** Find the flux of B through the lower half sphere $x^2 + y^2 + z^2 = 1, z \leq 0$ oriented downwards. **Solution:** Since we have an integral of the curl of the vector field A, we use **Stokes theorem** and integrate $A(r(t))$ along the boundary curve $r(t) = [\cos(t), -\sin(t), 0]$. First of all, we have $A(r(t)) = [\sin(t), \cos(t), 0]$. The velocity is $r'(t) = [-\sin(t), \cos(t), 0]$. The integral is $\int_0^{2\pi} -1 \, dt = -2\pi$.
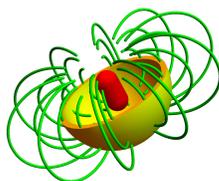
FIGURE 5.   The flux of the magnetic field B through a surface can be computed with Stokes by computing a line integral of the vector potential A.

32.21. Here are all the four magical **Maxwell equations** for the **electric field** E and **magnetic field** B related to the **charge density** σ and the **electric current** j. The constant c is the speed of light. (By using suitable coordinates, one can assume $c = 1$.)

$$\boxed{\text{div}(E) = 4\pi\sigma, \text{div}(B) = 0, c \cdot \text{curl}(E) = -B_t, c \cdot \text{curl}(B) = E_t + 4\pi j}.$$

FLUID DYNAMICS

32.22. If F is the fluid velocity field and C is a closed curve, then $\int_C F \cdot dr$ is called the **circulation** of F along C. The curl of F is called the **vorticity** of F. A **vortex line** is a flow line of $\text{curl}(F)$. Given a curve C, we can let any point in C flow along the vorticity field. This produces a **vortex tube** S. The flux of the vorticity though a surface S is the **vortex strength** of F through S. Stokes theorem implies the **Helmholtz theorem**.

Theorem: If C_s flows along F, then $\int_{C_s} F \cdot dr$ stays constant.

32.23. Proof: Let C be a closed curve and $C_s(t)$ be the curve after letting it flow using a deformation parameter s. The deformation produces a **tube surface** $S = \bigcup_{s=0}^{t} C_s$ which has the boundary C and C_t. Since the curl of F is always tangent to the surface S, the flux of the curl of F through S is zero. Stokes theorem implies that $\int_C F \cdot dr - \int_{C_s} F \cdot dr = 0$. The negative sign is because the orientation of C_s is different from the orientation of C if the surface has to be to the left.
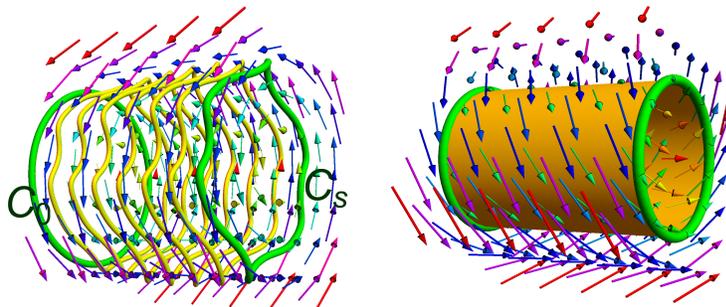
FIGURE 6. The Helmholtz theorem assures that the circulation along a flux tube is constant. This is a direct application of Stokes theorem: because the curl of F is tangent to the tube, there is no flux through the tube.

COMPLEX ANALYSIS

32.24. An application of Green's theorem is obtained, when integrating in the complex plane $\mathbb{C}$. Given a function $f(z) = u(z) + iv(z)$ from $\mathbb{C} \to \mathbb{C}$ and a closed path C parametrized by $r(t) = x(t) + iy(t)$ in $\mathbb{C}$, define the **complex integral** $\int_a^b (u(x(t) + iy(t)) + iv(x(t) + iy(t)))(x'(t) + iy'(t))\ dt$. This is $\int_a^b u(r(t))x'(t) - v(r(t))y'(t)\ dt + i\int_a^b v(r(t))x'(t) + u(r(t))'(t)\ dt$. These are two line integrals. The real part is $F = [u, -v]$, the imaginary part is $F = [v, u]$. Assume C bounds a region G, then Green's theorem tells that the first integral is $\iint_G -v_x - u_y dxdy$ and the second integral is $\iint_G u_x - v_y\ dxdy$. It turns out now that for nice functions f like polynomials, the **Cauchy-Riemann** differential equations $\boxed{u_x = v_y, v_x = -u_x}$ hold so that these line integrals are zero. We have therefore

Theorem: If f is a polynomial and C a closed loop, $\int_C f(z)\ dz = 0$

Unit 33: Discrete Vector Calculus

Introduction

33.1. In this seminar as well as the one next week, we redo calculus on finite networks. This is how multi-variable calculus of the future might look like. There is no infinity, there are no limits. The mathematics is the same. We will formulate first the fundamental theorem of line integral, Green and Stokes theorem $\int_G dF = \int_{dG} F$ on such a world.
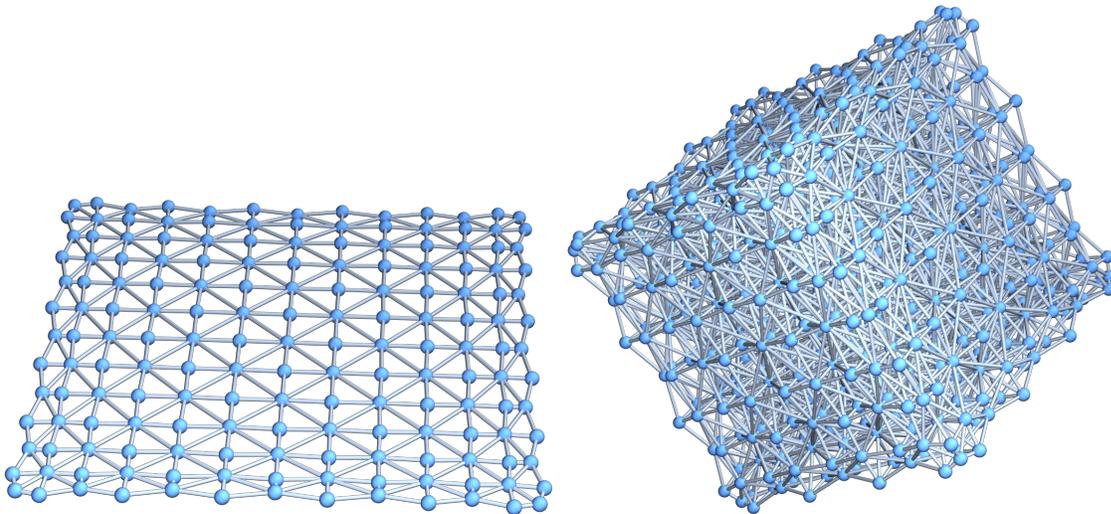
FIGURE 1. Two networks. The first one is a two dimensional surface, the second a three dimensional solid.

33.2. A finite network is a graph (V, E), where V is a finite set of nodes called vertices and where E are connections between nodes. There are no loops in that connections connect different nodes and also, there is only one connection possible between two points. One calls (V, E) a **finite simple graph**. Now similarly as we introduce coordinate systems in our space telling what is north, south, up or down etc, we assign when doing computations an orientation on the edges. This is pretty arbitrary but usually done any way when we implement a graph on a computer. For example $(G, V) = (\{1, 2, 3, 4\}, \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\})$ is a graph with 4 nodes, where all nodes are connected to each other. When doing calculus, we look at functions

on nodes, functions on edges, functions on triangular sugraphs as well as functions on tetrahedral subgraphs.

SEMINAR

33.3. In this seminar, we replace the space $\mathbb{R}^n$ with a finite graph $G = (V, E)$, where V is a set of vertices called **nodes** and E is a set of **edges** called connections. A **scalar field** is a function f which assigns to every vertex x a function value $f(x)$. We assume the vertices to be ordered leading to an order of the edges: draw an arrow $a \to b$ if $a < b$. This a priori order has no effect on any of the theorems. A **vector field** assigns to every edge a number $F(x)$. A **curve** is a list of nodes $x_1, x_2, \ldots, x_n$ such that x_1 is connected to x_2, x_2 is connected to x_3 etc. The **gradient** ∇f of a scalar function f is the vector field $F(a, b) = f(b) - f(a)$. The **line integral** $\int_C F \cdot dr$ is defined as $\sum_{e \in C} F(e) de$. We just add up the function values of F along the curve C, positive $de = 1$ if we go with the arrow, negative $de = -1$ if we go against the arrow.

> **Problem A:** Check the **closed loop property** of the gradient field ∇f shown in the graph of Figure 2.
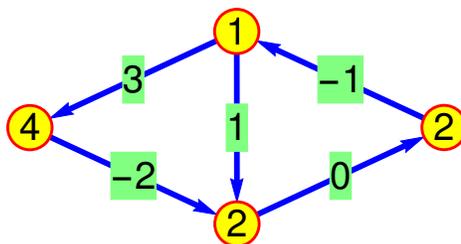
FIGURE 2. We see a graph with 4 vertices and 5 edges. The scalar function f is given by the values on the round vertices. It defines a gradient vector field $F = \nabla f$ which is a function on edges.

33.4. The **discrete fundamental theorem of line integrals** is:

> **Theorem:** If $F = \nabla f$ is a gradient field and C is a curve from a to b, then $\int_C \nabla f \cdot dr = f(b) - f(a)$.

> **Problem B:** Prove the discrete fundamental theorem of line integrals by induction on the length of the curve C.

33.5. Let's look at some terminology. Given a vertex x in a graph G, the **unit sphere** $S(x)$ of x is the sub-graph generated by the set of vertices directly attached to x. The unit sphere of the vertex labeled 11 in Figure 3 for example is the **circular graph** generated by the vertices $\{2, 4, 9, 8, 7, 9\}$. It is a "circle". The unit sphere of the vertex with label 4 in that figure is the graph generated by the vertices $\{11, 7, 1\}$. It is an linear graph, a half circle.

33.6. A graph is called a **discrete two-dimensional region**, if every unit sphere $S(x)$ is a circular graph with 4 or more vertices or a linear graph with 2 or more vertices. The set of vertices for which the unit sphere is circular form the interior of the region. The other vertices form the **boundary** of the region. A two dimensional region without boundary is called **closed**. In Figure 3 for example, there are 4 interior points and 9 boundary points. In Figure 5, we see a closed region.

33.7. The **curl** of a vector field F is a function on the triangles T of G. To get the value of the triangle (a, b, c) we form the line integral of F along the curve $C : a \to b \to c \to a$. Each triangle is assumed to be oriented (if drawn in the plane, then counter clockwise).

33.8. Given a function F on the triangles of a region G which is oriented, the **flux integral** $\iint_G F(x) \, dA$ is defined as $\sum_{t \in T} f(t)$, where T is the set of triangles in G.
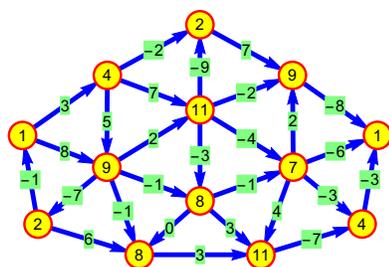
FIGURE 3. A gradient field on a two-dimensional region with boundary. Check that the curl is zero everywhere.

33.9. Here is the discrete **Green theorem**:

Theorem: If F is a vector field on a 2-dimensional discrete region G, and the boundary C is oriented in a compatible way with the region, then $\iint_G \text{curl}(F) \, dA = \int_C F \cdot dr$.

33.10. Figure 4 shows a region equipped with a vector field F.

Problem C: Write in the curl of the vector field in Figure 4.
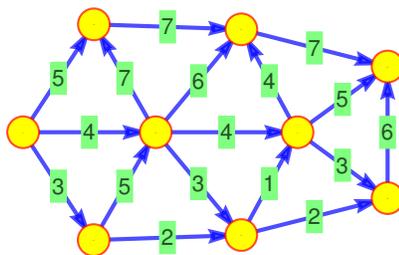
FIGURE 4. A vector field on a two-dimensional discrete region.

Problem D: Prove the discrete Green theorem by induction on the number of triangles.

HOMEWORK

33.1 Check that the curl of a gradient field is zero: $\text{curl}(\text{grad}(f)) = 0$ for a general triangle: draw an oriented triangle. Define $F = \nabla f$ for a scalar function f,l then compute $\text{curl}(F)$.

33.2 Figure 5 shows a vector field on the octahedron a two dimensional discrete sphere. Determine all the curls and check that the sum of all curls is zero. You have checked $\int_S \text{curl}(F)\, dS = 0$ for a closed surface.
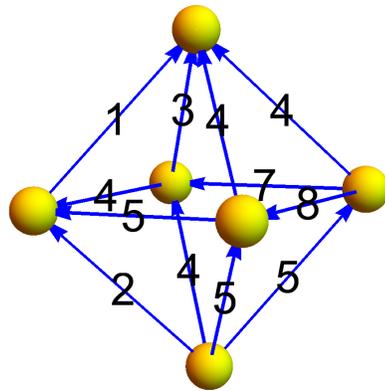
FIGURE 5. On a closed discrete 2-dimensional region like an octahedron, the sum of the curls of a vector field are zero.

33.3 Figure 6 shows a tree, a graph without closed loops. Find a potential function f. You can assume that the value at the top node is 0. You see then that the function value right below is 1. Get all the function values of the potential.

33.4 Find a vector field on a circular graph with 5 vertices which is not a gradient field.
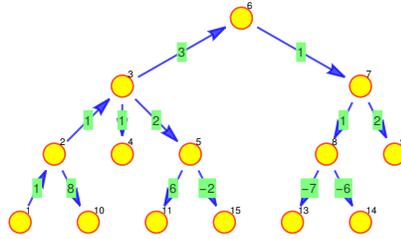
FIGURE 6. On a tree, every vector field is a gradient field.
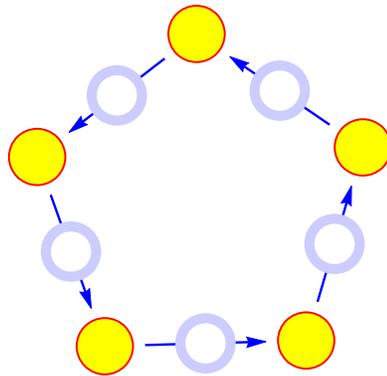
FIGURE 7. Fill in a vector field which is not a gradient field

33.5 Check Green's theorem in the following annular region. Compute both the line integral $\int_C F \, dr$ along the boundary (which has two components) as well as $\iint_G \text{curl}(F) \, dA$, the sum of the curvatures over all triangles.
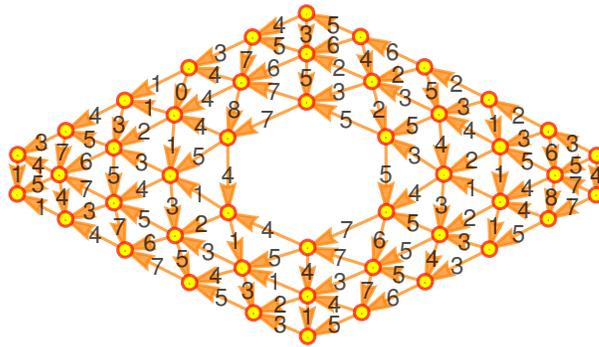
FIGURE 8. The region for Problem 33.5

Unit 34: Gauss theorem

34.1. The fastest way to compute the volume of a complicated solid is to use Gauss theorem. It turns out that the flux of the vector field $F = [x, 0, 0]$ through the boundary surface S of a solid E is the volume. Gauss theorem equates this flux with the creation of the field inside. If you draw out the vector field $F = [x, 0, 0]$ you see that it expands things out. Look at the unit cube. The field does not flow through the y=0,y=1 or z=0 and z=1 faces because the field is parallel there. On $x = 0$, the field is zero. The only face of the cuboid where some field passes is $x = 1$ and the field is leaving there. So, something must be created inside. This creation of field is called divergence. If $F = [P, Q, R]$, then $\text{div}(F) = P_x + Q_y + R_z$. In the case $F = [x, 0, 0]$ we have constant divergence 1. We have seen that $\iint_E \text{div}(F) \, dV = \iint_S F \cdot dS$. You can also see that for fields like $F = [0, x, 0]$ the total flux is zero as what comes in on one side goes out on the other side. Any linear linear field F as a linear combination of fields in the class $[x, 0, 0], [y, 0, 0], [z, 0, 0], [0, x, 0], [0, y, 0], [0, z, 0], [0, 0, x], [0, 0, y], [0, 0, z]$ for which the divergence theorem is satisfied.
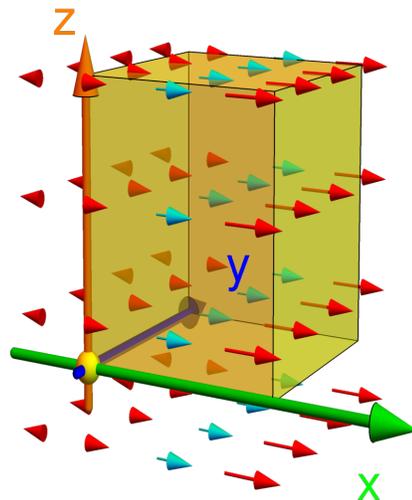
FIGURE 1.   The vector field $F = [x, 0, 0]$ has constant divergence 1. The flux through $\iint_S F \, dS$ through the boundary is $\iiint_E \text{div}(F) \, dV$.

34.2. Gauss law $\text{div}(F) = f = 4\pi G\rho$ describes the **gravitational field** induced from a **mass density** ρ and gravitational constant G. The picture is that mass is a source for the field. We will see that with the help of the divergence theorem, this equation implies **Newton's law** of gravity $F = MG/r^2$ induced by a mass M. Since the gravitational field does not to allow perpetual motion, it is a gradient field and $F = \nabla V$. The combination $\Delta = \text{div}(\text{grad}$ is called the **Laplacian** Gauss law now produces the Poisson equation $\Delta V = f$ which determines potential $V = \Delta^{-1} f$ from the mass density. The inverse of Δ also called Green function. Once we have such a description, we have now gravity on any space with a Laplacian. We can study gravity on a surface like the sphere or in $\mathbb{R}^n$, where the force is proportional to $1/r^{n-1}$. We will see in the proof part how we can so define gravity on any finite network.

LECTURE

34.3. The **divergence** of a vector field $F = [P, Q, R]$ in $\mathbb{R}^3$ is defined as $\text{div}(F) = \nabla \cdot F = P_x + Q_y + R_z$. Let G be a solid in $\mathbb{R}^3$ bound by a surface S made of finitely many smooth surfaces, oriented so the normal vector to S points outwards. The **divergence theorem** or **Gauss theorem** is

Theorem: $\iiint_G \text{div}(F) \, dV = \iint_S F \cdot dS$.
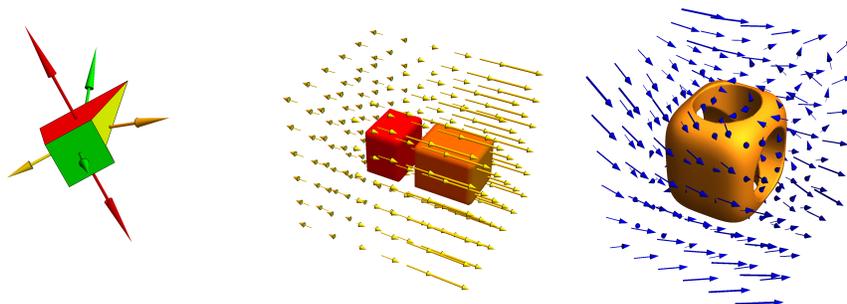
FIGURE 2. The boundary of a solid is oriented outwards. The divergence measures the expansion of a box flowing in the field. The flux of $\text{curl}(F)$ through a closed surface is 0. No field is created inside.

34.4. Proof. If G is a solid of the form $G = \{(x, y, z) | (x, y) \in U, g(x, y) \le z \le h(x, y)\}$ and $F = [0, 0, R]$, then $\iiint_G \text{div}(F) \, dV = \iint_U \int_{g(x,y)}^{h(x,y)} R_z \, dzdydx$ which is $\iint_G R(x, y, h(x, y)) - R(x, y, g(x, y)) \, dydx$. The flux of $F = [0, 0, R]$ through a surface $r(u, v) = [u, v, h(u, v)]$ is

$$\iint_G [0, 0, R(u, v, h(u, v))] \cdot [-g_u, g_v, 1] dvdu = \iint_G R(x, y, h(x, y)) \, dxdy \ .$$

Similarly, the flux through the bottom surface is $-\iint_G R(x, y, g(x, y)) \, dxdy$. In general, write $F = [P, Q, R] = [P, 0, 0] + [0, Q, 0] + [0, 0, R]$ to get the claim for solid which are simultaneously bound by graphs of functions in x and y, or y and z or x and z. A general solid can be cut into such solids.

34.5. The theorem gives meaning to the term divergence. The total divergence over a small region is equal to the flux of the field through the boundary. If this is positive, then more field leaves than enters and field is "generated" inside. The divergence measures the expansion of the field. The field $F(x, y, z) = [x, 0, 0]$ for example expands, while $f(x, y, z) = [-x, 0, 0]$ compresses. $F(x, y, z) = [y, z, x]$ is "incompressible".

34.6. The divergence theorem holds in any dimension m. If $F = [F_1, \cdots, F_m]$ is the vector field, then $\partial_{x_1} F_1 + \cdot + \partial_{x_m} F_m$ is defined as the **divergence** of F. If G is an m-dimensional region with boundary $S = s(G)$, then the flux of F through S is defined as $\int_G F(s(u)) \cdot n(s(u)) |ds(u)|$, where $n(s(u))$ is a unit normal vector. This can be explained a bit better using the language of differential forms which is introduced next time.

34.7. The divergence of $F = [P, Q]$ is defined as $P_x + Q_y$. If $F^\perp = [Q, -P]$ is the turned vector field, then $\text{div}(F^\perp) = Q_x - P_y$ is the curl of F. Green's theorem tells that $\iint_G \text{curl}(F)\, dxdy$ which is $\iint_G \text{div}(F^\perp)\, dxdy$ is the line integral $\int_C F \cdot dr$. The line integral for F is the flux integral for $F^\perp$. The two dimensional divergence theorem is Green's theorem "turned".

<center>EXAMPLES</center>

34.8. Problem: Compute the flux of $F = [x, y, z]$ through the sphere of radius ρ bounding a ball G, oriented outwards. **Solution:** As $\text{div}(F) = 3$ we have $\iiint_G \text{div}(F)dV = 3\text{Vol}(G) = 3 \cdot 4\pi\rho^3/3$. The flux through the boundary is $\iint_S F \cdot dS$. As in spherical coordinates, $F(r(\phi, \theta)) \cdot r_\phi \times r_\theta = \rho^3 \sin(\phi)$, the flux is $\int_0^{2\pi} \int_0^\pi \rho^3 \sin(\phi)\, d\phi d\theta = 4\pi\rho^3$ also.

34.9. Problem: What is the flux of the vector field $F(x, y, z) = [6x + y^3, 3z^2 + 8y, 22z + \sin(x)]$ through the solid $G = [0, 3] \times [0, 3] \times [0, 3] \setminus ([0, 3] \times [1, 2] \times [1, 2] \cup [1, 2] \times [0, 3] \times [1, 2] \cup [0, 3] \times [0, 3] \times [1, 2])$ which is a cube with three perpendicular cubic holes which is the first stage of the **Menger sponge construction**? **Solution:** As $\text{div}(F) = 22 + 8 + 6 = 36$, the result is 36 times the volume of the solid which is $36(27 - 7) = 720$.
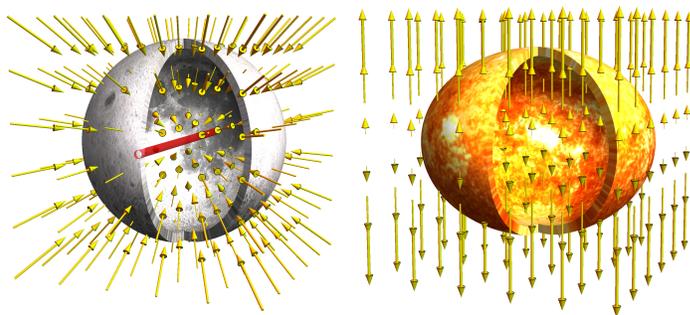

FIGURE 3.   The gravity inside the moon is such that an elevator crossing the moon oscillates like a harmonic oscillator. The flux of $F = [0, 0, z]$ through a surface is the volume inside.

34.10. Problem. How does the gravitational field look like inside the moon in distance ρ to the origin? **Solution.** A direct computation of summing up all the field values $F(x) = \iint_G (x - y)/|x - y|^3 dy$ is difficult as we can not compute in spherical coordinates. Fortunately we have the divergence theorem. The field $F(x)$ has constant length $F(\rho) = |F(x)|$ for x on a sphere $S(\rho)$ of radius ρ and points inwards. So $\iint_{S(\rho)} F \cdot dS = -4\pi \rho^2 F(\rho)$. Gauss was able to write down the gravitational field as a partial differential equation $\boxed{\operatorname{div}(F(x)) = 4\pi\sigma(x)}$, where $\sigma(x)$ is the mass density of the solid. We see then with the divergence theorem that $\iiint_{B(\rho)} 4\pi\sigma(x) \, dx$ is equal to $-4\pi\rho^2 F*(\rho)$. Assuming σ to be constant, we have $4\pi(4\pi\rho^3/3)\sigma = -4\pi\rho^2 F(\rho)$ which gives $F(\rho) = (4\sigma/3)\rho$. The field grows linearly inside the body. If ρ is bigger than the radius of the moon, then $\iiint_{B(\rho)} 4\pi\sigma(x) \, dx$ is $4\pi M$, where $M = \iiint_G \sigma(x) \, dx$ is the mass of the moon. We see that in that case $F(\rho) = M/\rho^2$, which is the Newton law.

34.11. Problem: Compute using the divergence theorem the flux of the vector field $F(x, y, z) = [2342434y, 2xy, 4yz]^T$ through the unit cube $[0, 1] \times [0, 1] \times [0, 1]$ which is opened on the top. **Solution:** the divergence of F is $2x + 4y$. Integrating this over the unit cube gives $1 + 2 = 3$. The flux through all 6 faces is 3. The flux through the face $z = 1$ is $\int_0^1 \int_0^1 4y \, dxdy = 2$. We have to subtract this and get $3 - 2 = 1$.

34.12. Similarly as Green's theorem allowed area computation using line integrals the volume of a region can be computed as a flux integral: take a vector field F with constant divergence 1 like $F(x, y, z) = [0, 0, z]$. We have $\int \int_S [0, 0, z] \cdot dS = \operatorname{Vol}(G)$.

34.13. Example: For an ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2$, where the parametrization is $r(\phi, \theta) = [a \sin(\phi)\cos(\theta), b \sin(\phi)\sin(\theta), c \cos(\phi)]$, we have $[0, 0, c \cos(\phi)][ab \sin(\phi)\cos(\phi)] = abc \sin(\phi)\cos^2(\phi)$ leading to $2\pi abc2/3 = 4\pi abc/3$.

34.14. A computer can determine the volume of a solid enclosed by a triangulated surface by computing the flux of the vector field $F = [0, 0, z]$ through the surface. The vector field has divergence 1 so that by the divergence theorem, the flux gives the volume. A computer stores a geometric object using triangles. Assume ABC is that triangle. If $n = AB \times AC$ points outside the region, then the flux is $F \cdot n/2$. A computer can now add up all these values and get the volume.
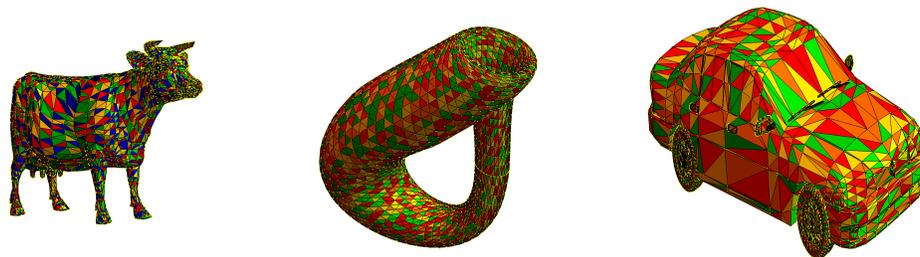
FIGURE 4. A cow, a Klein bottle and a car from the Mathematica example files and produce closed surfaces. The Klein bottle does not have an interior however.

Problem 34.1: Use the divergence theorem to calculate the flux of $F(x, y, z) = [x^3, y^3, z^3]^T$ through the sphere $S : x^2 + y^2 + z^2 = 1$, where the sphere is oriented so that the normal vector points outwards.

Problem 34.2: Assume the vector field

$$F(x, y, z) = [5x^3 + 12xy^2, y^3 + e^y \sin(z), 5z^3 + e^y \cos(z)]^T$$

is the magnetic field of the **sun** whose surface is a sphere of radius 3 oriented with the outward orientation. Compute the magnetic flux $\iint_S F \cdot dS$.

Problem 34.3: Find the flux of the vector field $F(x, y, z) = [xy, yz, zx]^T$ through the solid cylinder $x^2 + y^2 \leq 1$, $0 \leq z \leq 2$.

Problem 34.4: Find the flux of $F(x, y, z) = [x + y + z, x + z, z + y]^T$ through the Menger sponge M_n defined in the unit cube and take the limit $n \to \infty$.
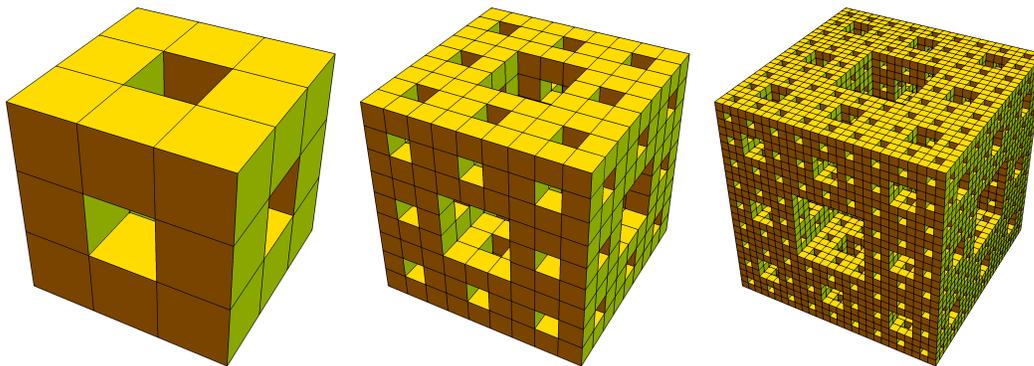
FIGURE 5. Approximations to the Menger sponge.

Problem 34.5: Compute the flux of the vector field $F(x, y, z, w) = [x + 2y^2, 3x + 4z^5, 6z + 8w^9, 7w + 9x^{10}]^T$ through the three 3-sphere $x^2 + y^2 + z^2 + w^2 = 1$ in $\mathbb{R}^4$, oriented outwards.

Unit 35: General Stokes

INTRODUCTION

35.1. Having seen a fundamental theorem (FTC) in dimension 1, two theorems (FTLI, GREEN) in dimension two and three theorems (FTLI,STOKES,GAUSS) in dimension 3, we expect 4 theorems in dimensions 4. This is indeed the case, but how do we formulate such a theory? How would you formulate this in 4 dimensions where points have coordinates (x, y, z, w)?
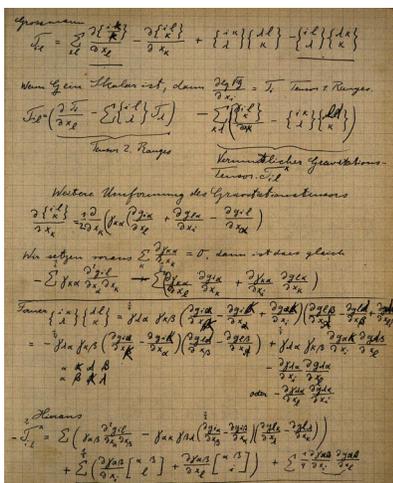
FIGURE 1. A page from Einstein's Zürich notebook features tensors.

35.2. Élie Cartan introduced **forms**. In three dimensions, a **0-form** is just a scalar function $f(x, y, z)$. A **1-form** is $F = Pdx + Qdy + Rdz$, where P, Q, R are scalar functions and dx, dy, dz are formal expressions. A **2-form** is an expression of the form $F = Pdydz + Qdzdx + Rdxdy$, where dx, dy, dz are again symbols but satisfy rules like $dxdy = -dydx, dxdz = -dzdx$ and $dydz = -dzdy$. A **3-form** finally is written as $fdxdydz$, where $dxdydz$ as a volume form. Most calculus books treat 0-forms and 3-form f as a scalar functions and 1-forms and 2-forms as vector fields. But what is dx? It is a linear map from $\mathbb{R}^3 \to \mathbb{R}$ which maps a vector $[v_1, v_2, v_3]$ to v_1. The expression $dxdy$ as a multi-linear anti-symmetric map from $\mathbb{R}^3 \times \mathbb{R}^3$ to $\mathbb{R}$: the object $dxdy$ assigns to two vectors v, w the determinant of the matrix $v, w, [0, 0, 1]^T$ as column vectors. which is equal to $v \times w \cdot k = v_1w_2 - v_2w_1$. Switching v and w changes the sign

so that $dxdy = -dydx$ and especially $dxdx = 0$. The object $dxdydz$ is a multi-linear map from $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ to $\mathbb{R}$ assigning to 3 vectors u, v, w the determinant of the matrix in which u, v, w are the columns. Again, switching two elements changes the sign. For example $dxdydz = -dxdzdy$, or $dxdxdz = 0$.

35.3. Breaking away from notions like cross product, we get now objects which can be defined in arbitrary dimensions $\mathbb{R}^n$. A k-**form** is a rule which at every point defines a multi-linear and anti-symmetric map to the reals. Let us look how this is defined in 4 dimensions: a **0**-form is a scalar function f. It assigns to every point (x, y, z, w) a number $f(x, y, z, w)$. A **1-form** is an expression $F = Pdx + Qdy + Rdz + Sdw$ which can be thought of as a vector field $F = [P, Q, R, S]$. A **2-form** is an expression $F = Adxdy + Bdxdz + Cdxdw + Pdydz + Qdydw + Rdzdw$. It is a field with 6 components. A **3-form** is an expression $F = Adydzdw + Bdxdzdw + Cdxdydw + Ddxdydz$. As it is a field with 4 components we can again see it as a "vector field". A **4-form** is an expression $F = fdxdydzdw$. As it has only one component, we can again think of it as a "scalar function" even so this a **lie**. A 4-form is a different object than a 0-form.

35.4. The **exterior derivative** produces from a k form a $(k + 1)$-form. First define the 1-form $df = f_x dx + f_y dy + f_z dz + f_w dw$ for a 0-form f, then use this for general k forms. Given a 1-form $F = Pdx + Qdy + Rdz + Sdw$ define $dF = (P_x dx + P_y dy + P_z dz + P_w dw)dx + (Q_x dx + Q_y dy + Q_z dz + Q_w dw)dy + (R_x dx + R_y dy + R_z dz + R_w dw)dz + (S_x dx + S_y dy + S_z dz + S_w dw)dw$ which simplifies to $dF = (Q_x - P_y)dxdy + (R_x - P_z)dxdz + (S_x - P_w)dxdw + (R_y - Q_z)dydz + (S_y - Q_w)dydw + (R_w - S_z)dwdz$. If $F = Adxdy + Bdxdz + Cdxdw + Pdydz + Qdydw + Rdzdw$ is 2-form, then $dF = (A_x dx + A_y dy + A_z dz + A_w dw)dxdy + (B_x dx + B_y dy + B_z dz + B_w dw)dxdz + (C_x dx + C_y dy + C_z dz + C_w dw)dxdw + (P_x dx + P_y dy + P_z dz + P_w dw)dydz + (Q_x dx + Q_y dy + Q_z dz + Q_w dw)dydw + (R_x dx + R_y dy + R_z dz + R_w dw)dzdw$ simplifies to $(P_x - B_y + A_z)dxdydz + (Q_x - C_y + A_w)dxdydw + (R_x - C_z + B_w)dxdzdw + (R_y - Q_z + P_w)dydzdw$. Finally for $F = Adydzdw + Bdxdzdw + Cdxdydw + Ddxdydz$ we have $dF = (A_x dx + A_y dy + A_z dz + A_w dw)dydzdw + (B_x dx + B_y dy + B_z dz + B_w dw)dxdzdw + (C_x dx + C_y dy + C_z dz + C_w dw)dxdydw + (D_x dx + D_y dy + D_z dz + D_w dw)dxdydz = (A_x + B_y + C_z + D_w)dxdydzdw$.

35.5. We can integrate a $(k + 1)$-form dF over a $(k + 1)$-manifold G and a k-form F over the k-manifold dG, the boundary dG of G. We write $\int_G dF$. To see the general Stokes theorem $\boxed{\int_G dF = \int_{dG} F}$, we need to know that a tensor is. **Machine learning** can justify to introduce the concept. [1] Let E be a space of column vectors and E^* a space of row vectors.

$\boxed{\textbf{A } (p, q)\textbf{-tensor on } E \textbf{ as a multi-linear map from } (E^*)^p \times (E^q) \textbf{ to } \mathbb{R}.}$

Column vectors are tensors of the type $(1, 0)$, row vectors are tensors of the type $(0, 1)$, matrices are tensors of the type $(1, 1)$. The k-th Jacobean derivative of a function f is a tensor of type $(0, k)$. A tensor of type $(0, 3)$ for example as a 3-dimensional array of numbers A_{ijk}. It defines a multi-linear map assigning to every triplet of vectors u, v, w the number $\sum_{i,j,k} A_{ijk} u^i v^j v^k$. [2] A k-form on a manifold attaches a $(0, k)$ tensor at every point.

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[1]There is a "tensor flow" library for example.

[2]**Albert Einstein** would just write $A_{ijk} u^i v^j v^k$ and not bother about the summation symbol.

35.6. $E = \mathbb{R}^n = M(n, 1)$ is the space of **column vectors**. Its **dual** $E^* = M(1, n)$ is the space of **row vectors**. To get more general objects we treat vectors as **maps**. A row vector is a linear map $F : E \to \mathbb{R}$ defined by $F(u) = Fu$ and a column vector defines a linear map $F : E^* \to \mathbb{R}$ by $F(u) = uF$. A map $F(x_1, \ldots, x_n)$ of several variables is called **multi-linear**, if it is linear in each coordinate. The set $T_q^p(E)$ of all **multi-linear maps** $F : (E^*)^p \times E^q \to \mathbb{R}$ is the space of **tensors of type** (p, q). We have $T_0^1(E) = E$ and $T_1^0(E) = E^*$. The space $T_1^1(E)$ can naturally be identified with the space $M(n, n)$ of $n \times n$ matrices. Indeed, given a matrix A, a column vector $v \in E$ and a row vector $w \in E^*$, we get the bi-linear map $F(v, w) = wAv$. It is linear in v and in w. In other words, it is a tensor of type $(1, 1)$.

35.7. Let $\Lambda^q(E)$ be the subspace of $T_q^0(E)$ which consists of tensors F of type $(0, q)$ such that $F(x_1, \ldots x_q)$ is anti-symmetric in $x_1, \ldots, x_q \in E$: this means $F(x_{\sigma(1)}, \ldots, x_{\sigma(q)}) = (-1)^\sigma f(x_1, \ldots, x_q)$ for all $i, j = 1, \ldots, q$, where $(-1)^\sigma$ is the **sign** of the permutation σ of $\{1, \ldots, n\}$. If the **Binomial coefficient** $B(n, q) = n!/(q!(n-q)!)$ counts the number of subsets with q elements $i_1 < \cdots < i_q$ of $\{1, \ldots, n\}$ and E has dimension n, then $\Lambda^q(E)$ has dimension $B(n, q)$. A map $F : E \to T_q^p(E)$ is called a (p, q)-**tensor field**. The set $T_0^1(E)$ is the space of **vector fields**. If $g : \mathbb{R}^m \to \mathbb{R}^n$ is a smooth map, then $F = d^k g$ is a tensor field of type $(0, k)$. A k-**form** is a $(0, k)$-tensor field F with $F(x) \in \Lambda^k(E)$. A 2-form in $\mathbb{R}^3$ for example attaches to $x \in \mathbb{R}^3$ a bi-linear, anti-symmetric map $F(x)(u, v) = -F(x)(v, u)$. One writes $P dy dz + Q dx dz + R dx dy$ where $dydz(u, v) = u_2 v_3 - u_3 v_2$, $dxdz(u, v) = u_1 v_3 - u_3 v_1$, $dxdy(u, v) = u_1 v_2 - v_1 u_2$.

35.8. The **exterior derivative** $d : \Lambda^p \to \Lambda^{p+1}$ is defined for $f \in \Lambda^0$ as $df = f_{x_1} dx_1 + \cdots + f_{x_n} dx_n$ and $d(f dx_{i_1} \cdots dx_{i_p}) = \sum_i f_{x_i} dx_i dx_{i_1} \cdots dx_{i_p}$. For $F = P dx + Q dy$ for example, it is $(P_x dx + P_y dy) dx + (Q_x dx + Q_y dy) dy = (Q_x - P_y) dx dy$ which is the **curl** of F. If $r : G \subset \mathbb{R}^m \to \mathbb{R}^n$ is a parametrization, then $S = r(G)$ is a m-**surface** and $\delta S = r(\delta G)$ is its **boundary** in $\mathbb{R}^n$. If $F \in \Lambda^p(\mathbb{R}^n)$ is a p-**form** on $\mathbb{R}^n$, then $r^* F(x)(u_1, \ldots, u_p) = F(r(x))(dr(x)(u_1), dr(x)(u_2), \ldots, dr(x)(u_p))$ is a p-form in $\mathbb{R}^m$ called the **pull-back** of r. Given a p-form F and an p-surface $S = r(G)$, define the integral $\int_S F = \int_G r^* F$. The **general Stokes theorem** is

> **Theorem:** $\int_S dF = \int_{\delta S} F$ for a $(m-1)$-form F and m surface S in E.

35.9. Proof. As in the proof of the divergence theorem, we can assume that the region G is simultaneously of the form $g_j(x_1, \ldots, \hat{x}_j, \ldots x_m) \le x_j \le h_j(x_1, \ldots, \hat{x}_j, \ldots x_m)$, where $1 \le j \le n$ and that $F = [0, \ldots, 0, F_j, 0, \ldots, 0]$. The coordinate independent definition of dF reduces the result to the divergence theorem in G. QED

EXAMPLES

35.10. For $n = 1$, there are only 0-forms and 1-forms. Both are scalar functions. We write f for a 0-form and $F = f dx$ for a 1-form. The symbol dx abbreviates the linear map $dx(u) = u$. The 1-form assigns to every point the linear map $f(x) dx(u) = f(x) u$. The exterior derivative $d : \Lambda^0 \to \Lambda^1$ is given by $df(x) u = f'(x) u$. Stokes theorem is the **fundamental theorem of calculus** $\int_a^b f'(x) dx = f(b) - f(a)$.

35.11. For $n = 2$, there are 0-forms, 1-forms and 2-forms. It is custom to write $F = Pdx + Qdy$ rather than $F = [P, Q]$ which is thought of as a linear map $F(x, y)(u) = P(x, y)u_1 + Q(x, y)u_2$. A 2-form is also written as $F = fdxdy$ or $F = fdx \wedge dy$. Here $dxdy$ means the bi-linear map $dxdy(u, v) = (u_1v_2 - u_2v_1)$. The 2-form defines such a bi-linear map at every point (x, y). The exterior derivative $d\Lambda^0 \to \Lambda^1$ is $df(x, y)(u_1, u_2) = f_x(x, y)u_1 + f_y(x, y)u_2$ which encodes the Jacobian $df = [f_x, f_y]$, a row vector. The exterior derivative of a 1-form $F = Pdx + Qdy$ is $dF(x, y)(u, v) = (-1)^1 P_y(x, y) \det([u, v]) + (-1)^2 Q_x(x, y) \det([u, v])$ which is $(Q_x - P_y)dxdy$. Using coordinates is convenient as $dF = P_y dydx + Q_x dxdy = (Q_x - P_y)dxdy$ using now that $dydx = -dxdy$.

35.12. For $n = 3$, we write $F = Pdx + Qdy + Rdz$ for a 1-form, and $F = Pdydz + Qdzdx + Rdxdy$ for a 2-form. Here $dydz = dy \wedge dz$ are symbols representing bi-linear maps like $dydz(u, v) = u_2v_3 - v_3u_2$. As a 2-form has 3 components, it can be visualized as vector field. A 3-form $fdxdydz$ defines a scalar function f. The symbol $dxdydz = dx \wedge dy \wedge dz$ represents the map $dxdydz(u, v, w) = \det([uvw])$. The exterior derivative of a 1-form gives the curl because $d(Pdx + Qdy + Rdz) = P_y dydx + P_z dzdx + Q_x dxdy + Q_z dzdy + R_x dxdz + R_y dydz$ which is $(R_y - Q_z)dydz + (P_z - R_x)dzdx + (Q_x - P_y)dxdy$. The exterior derivative of a 2-form $Pdydz + Qdzdx + Rdxdy$ is $P_x dxdydz + Q_y dydzdx + R_z dzdxdy = (P_x + Q_y + R_z)dxdydz$. To integrate a 2-form $F = x^2 yzdxdy + yzdydz + xzdxdz$ over a surface $r(u, v) = [x, y, z] = [uv, u - v, u + v]$ with $G = \{u^2 + v^2 \le 1\}$ we end up with integrating $F(r(u, v)) \cdot r_u \times r_v$. In order to integrate dF for a 1-form $F = Pdx + Qdy + Rdz$ we can also pull back F and get $\iint_G F_v(r(u, v))r_u - F_u(r(u, v)r_v \, dudv$.

35.13. For $n = 4$, where we have 0-forms f, 1-forms $F = Pdx + Qdy + Rdz + Sdw$ and 2-forms $F = F_{12}dxdy + F_{13}dxdz + F_{14}dxdw + F_{23}dydz + F_{24}dydw + F_{34}dzdw$ which are objects with 6 components. Then 3-forms $F = Pdydzdw + Qdxdzdw + Rdxdydw + Sdxdydz$ and finally 4-forms $fdxdydzdw$.

Remarks

35.14. Historically, differential forms emerged in 1922 with Élie Cartan. Most textbooks introduce the Grassmannian algebra early and use the language of "chains" for example which is the language used in algebraic topology. I myself taught the subject in this old-fashioned way too, back in 1995. [3] It was Jean Dieudonné in 1972 who freed the general Stokes theorem from chains and used first the coordinate free pull back idea. This allowed us in this lecture to formulate the general Stokes theorem from scratch **on a single page** with all definitions.

35.15. What is a differential form? We have seen a mathematically precise definition: a differential form is a **kind of field**: it defines a multi-linear anti-symmetric function that is attached to each point of space. But what is the intuition and what are ways to "visualize" and "see" and "understand" such an object? Here are four paths. Maybe one of them helps:

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[3]Caltech notes: https://people.math.harvard.edu/knill/teaching/math109_1995/geometry.pdf

A) Using **Stokes** one can see a form as a functional F, which assigns to a m-dimensional oriented surface S a number $\int_S F \cdot dS$ such that $\int_{-S} F \cdot dS = \int_S (-F) \cdot dS = - \int_S F \cdot dS$. [4] This way of thinking about forms matches what we do in the discrete. If we have a k-form on a graph, then this is a function on k-dimensional oriented complete subgraphs. Given a graph S we have $\int_S F \cdot dS = \sum_{x \in S} F(x)$, where the sum is over all k-dimensional simplices in S.

B) One can understand differential forms better using arithmetic, the **Grassmannian algebra**. This is done with the help of the **tensor product**, which induces an **exterior product** $F \wedge G$ on $\Lambda^p \times \Lambda^q \to \Lambda^{p+q}$. This product generalizes the cross product $\Lambda^1 \times \Lambda^1 \to \Lambda^2$ which works for $n = 3$ as there, the space of 1-forms Λ^1 and 2-forms Λ^2 can be identified. The exterior algebra structure helps to understand k-forms. We can for example see a 2-form as an exterior product $F \wedge G$ of two 1-forms. We can think of a 2-form for example as attaching two vectors at a point and identify two such frames if their orientation and parallelogram areas match.

C) A third way comes through **physics**. We are familiar with manifestations of electomagnetism: we see light, we use magnets to attach papers to the fridge or have magnetic forces keep the laptop lid closed. Electric fields are felt when combing the hair, as we see sparks generated by the high electric field obtained by stripping away the electrons from the head. We use magnetic fields to store information on hard drives and electric fields to store information on a SSD hard drive. Non-visible electro-magnetic fields are used when communicating using cell phones or connecting through blue-tooth or wireless network connections. The electro-magnetic field E, B is actually a 2-form in 4-dimensions. The $B(4, 2) = 6$ components are $(E_1, E_2, E_3, B_1, B_2, B_3)$.

D) A fourth way comes through **discretization**. When formulating Stokes on a discrete network, everything is much easier: a k-form is just a function on oriented k-dimensional complete subgraphs of a network. Start with a graph $G = (V, E)$ and orient the complete subgraphs arbitrarily. Given a k-form F, a function on k-simplices has an exterior derivative at a $k + 1$ dimensional simplex x is defined as $dF(x) = \sum_{y \subset x} \sigma(y, x) F(y)$, where the sum is over all k-dimensional sub-simplices of x and $\sigma(y, x) = 1$ if the orientation of y matches the orientation of x or -1 else. We have for example seen that for a 1-form F, a function on edges, the exterior derivative at a triangle x is the sum over the F values of the edges, where we add up the value negatively if the arrow of the edge does not match the orientation of the triangle.

<div align="center">APPLICATIONS</div>

35.16. An **electromagnetic field** is determined by a 1-form A in 4-dimensional space time. The electromagnetic field is $F = dA$. The Maxwell equations are $dF = 0$ (the relation $d \circ d = 0$ is seen in the homework). The second part of the Maxwell equations are $d^* F = j$, where $d^* : \Lambda^p \to \Lambda^{p-1}$ is the adjoint and j is a 1-form encoding both the electric charge and the electric current. We can always gauge with a gradient $A + df$ so that $d^*(A + df) = 0$ (Coulomb gauge). Using $d^* A = 0$, the Maxwell equations reduced to the Poisson equation $LA = (dd^* + d^* d)A = j$, where L is the **Laplacian** on

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[4]David Bachman's text on differential forms: "it is a thing which can be integrated".

1-forms. The electric current j defines the electromagnetic field F simply by inverting the Laplacian. This is a bit tricky in the continuum, as the inverse is an integral operator. [5] In the discrete it is just the inverse of the matrix L, which by the way is always an invertible $|E| \times |E|$ matrix if the graph $G = (V, E)$ is simply connected. And there was light!

Homework

Problem 35.1: Given the 1-form $F(x, y, z, w) = [x^3, y^5, z^5, w^2] = x^3 dx + y^5 dy + z^5 dz + w^2 dw$ and the curve $C : r(t) = [\cos(t), \sin(t), \cos(t), \sin(t)]$ with $0 \leq t \leq \pi$. Find the line integral $\int_C F(r(t)) \cdot dr$.

Problem 35.2: Given the 1-form $F = [xyz, xy, wx, wxy] = xyz\,dx + xy\,dy + wx\,dz + wxy\,dw$, find the curl dF. Now find $\iint_S dF$ over the 2-dimensional surface $S : x^2 + y^2 \leq 1, z = 1, w = 1$ which has as a boundary the curve $C : r(t) = [\cos(t), \sin(t), 1, 1]^T$, $0 \leq t \leq 2\pi$. You certainly can use the Stokes theorem. If you like to compute both sides of the theorem you can see how the theorem works. The 2-manifold S is parametrized by $r(t, s) = [s, t, 1, 1]^T$. The $(r_s \wedge r_t)_{ij}$ has 6 components, where only one component $(r_s \wedge r_t)_{12}$ is nonzero. This will match with the $dF_{12} = P\,dxdy$ part of the 6-component 2-form dF building the curl. We will have to integrate then over $G = s^2 + t^2 \leq 1$.

Problem 35.3: Given the 2-form $F = z^4 x\,dxdz + xyzw^2\,dydw$ and the 3-sphere $x^2 + y^2 + z^2 + w^2 = 1$ oriented outwards. What is the integral $\iiint_S dF$? To compute this 3D integral, you can use the general integral theorem.

Problem 35.4: Given the 3-form $F = xyz\,dxdydz + y^2 z\,dydzdw$, find the divergence dF. Now find the flux of F through the unit sphere $x^2 + y^2 + z^2 + w^2 = 1$ oriented outwards.

Problem 35.5:
a) Take $f(x, y, z, w)$. Check that $F = df$ satisfies $dF = 0$.
b) Take $F = F_1 dx + F_2 dy + F_3 dz + F_4 dw$. Compute the curl $G = dF$ and check that $dG = 0$.
c) Take the 2-form $F = F_{12} dxdy + F_{13} dxdz + F_{14} dxdw + F_{23} dydz + F_{24} dydw + F_{34} dzdw$. Write down the 3-form $G = dF$ and check $dG = 0$.
d) Take the 3-form $F = F_1 dydzdw + F_2 dxdzdw + F_3 dxdydw + F_4 dxdydz$ and compute the 4-form $G = dF$. Check that $dG = 0$.

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[5]There are thick books about this like Jackson's Electromagnetism, the bible of the topic.

Unit 36: A Discrete World

INTRODUCTION

36.1. The **Maxwell equations** $\boxed{dF = 0, d^*F = j}$ allow to get the electro-magnetic field F from the **current** j. The gravitational field is determined via the **Gauss law** $\boxed{d^*F = \rho}$ from the mass density ρ. With $-\Delta = d^*d$, the **Schrödinger equation** $\boxed{ihu' = -\Delta u + Vu}$ describes the motion of a quantum particles. On a space with a derivative d, there is light, matter and a periodic system of atoms.

FIGURE 1. The **periodic systems of elements** reflects the structure of the eigenvalues and eigenvectors of the **Hydrogen matrix** $L = K + V = -\frac{\hbar^2}{2m}\Delta - \frac{e^2}{4\pi\epsilon_0\rho}$, where $\Delta = $ divgrad, where m, e are mass and charge of the electron and $\hbar, \epsilon_0$ are constants. The eigenvalues of L are $\frac{\hbar^2}{2an^2}$ with Bohr radius $a = \frac{4\pi\epsilon_0\hbar^2}{me^2}$ and eigenvectors $\psi_{nlm} = R_{nl}(\rho)Y_l^m(\theta, \phi)$.

36.2. An important object in calculus is the **Laplacian** $\Delta f = \text{div}(\text{grad}(f))$ which is $\Delta f = f_{xx} + f_{yy} + f_{zz}$. For graphs, it is the **Kirchhoff matrix** $K = d^*d$, where d^* is the transpose matrix of d. The matrix $K = d^*d$ is a square matrix with non-negative eigenvalues. Construct each column with Ke_v where e_v is a basis vector. The 1-form de_v attaches to an edge connected to v the value -1. Then d^*de_v is the function on vertices which attaches to a vertex itself the negative of the **vertex degree** and to each attached node value 1. The Schrödinger equation $i\hbar u' = Ku$ is solved by $u(t) = U(t)u(0) = e^{-itK/\hbar h}u(0)$. You can watch it with:

K=KirchhoffMatrix[WheelGraph[9]];U[t_]:=**MatrixExp**[−**K∗I**∗t];
Animate[MatrixPlot[**Re**[U[t∗.01]]],{t,0,20}]

LECTURE

36.3. A 0-**form** f on a graph $G = (V, E)$ is a function on the vertices V. We also call it a **scalar function**. A 1-**form** is a function on the oriented edges E satisfying $F(a, b) = -F(b, a)$. Informally, as in the continuum, we think of a 1-form as a **vector field**. The **gradient** $F(a, b) = df(a, b) = f(b) - f(a)$ of a 0-form f is a 1 form F. The **curl** of a vector field F is a 2-**form**. It is a function on triangles (a, b, c) given by $dF(a, b, c) = F(a, b) + F(b, c) + F(c, a)$ which can be seen as the line integral along the boundary of the triangle. When describing p-forms for $p > 0$, orientation matters. To fix it, just enumerate the vertices V and then choose the orientation of an edge (a, b) with $a < b$ or the orientation of a triangle (a, b, c) if $a < b < c$. The discrete **Stokes theorem** $\iint_S \text{curl}(F) \cdot dS = \int_C F \cdot dr$ told us that that the sum of the curls of F on triangles of a surface S is equal to the line integral of F along the boundary C of S.
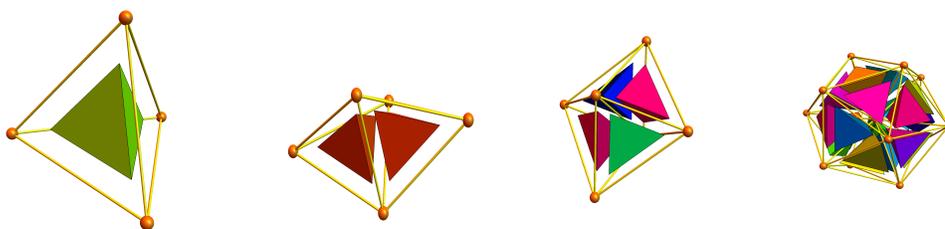
FIGURE 2.   Examples of three dimensional graphs with 1,2,4 and 12 tetrahedra. The divergence $dF(x)$ of a 2-form F is the sum over all values $F(y)$, where $y \subset x$ runs over the triangular faces of x. The sum of all divergences is the flux of F through the boundary because in the inside the fluxes cancel. This is the divergence theorem for solids.

36.4. A **tetrahedral graph** is a collection of four nodes which all are connected to each other. A 3-**form** on a graph G is a function on tetrahedral sub-graphs x of G. An example is the **divergence** $dF(x)$ **of a 2-form** F which is defined as the sum of the $F(y)$ values of the triangles $y \subset x$ enclosing the tetrahedron x. As in the continuum, the orientation plays a role. Here is the **discrete divergence theorem** for a solid G is built by tetrahedra x and where the boundary surface S consists of triangles:

> **Problem A:** Check that $\sum_{x \in G} \text{div}(F)(x) = \sum_{y \in S} F(y)$.

Hint: prove by induction with respect to the number of tetrahedra. First check that if G is a single tetrahedron, this is the definition of the divergence. Then see what happens if a new tetrahedron is added.

36.5. We also have seen that the divergence of the curl of a vector field F is zero: we had $\text{curl}(F) = [R_y - Q_z, P_z - R_x, Q_x - P_y]$ and taking the x derivative of $R_y - Q_z$ is $R_{yx} - Q_{zx}$, the y derivative of $P_z - R_x$ is $P_{zy} - R_{xy}$ and the z-derivative of $Q_x - P_y$ is $Q_{xz} - P_{yz}$. Adding them all up gives 0. In the discrete it is even simpler. Start with a 1-form F on the edges of a graph. Then form the curls, which are functions on the triangles, then add up all these curls. You check:

> **Problem B:** Check: $\text{div}(\text{curl}(F))(x) = 0$ of every F and tetrahedron x.

36.6. The general Stokes theorem is not much different. A p-**simplex** in G in a complete sub-graph with $p+1$ nodes. This means we have all connected to each other. A p-**form** is a function on the set of p-simplices x in G. The function value changes if two elements switch. For example, $F(x_0, x_1, x_2) = F(x_1, x_2, x_0) = F(x_2, x_0, x_1) = -F(x_1, x_0, x_2) = -F(x_0, x_2, x_1) = -F(x_2, x_1, x_0)$.

36.7. The **exterior derivative** of p-form F is the $(p+1)$-form

$$dF(x_0, \ldots, x_{p+1}) = \sum_{j=0}^{p+1} (-1)^j F(x_0, \ldots, \hat{x}_j, \ldots x_{p+1}) \; .$$

> **Problem C:** Check in general that $ddF = 0$.

36.8. The general Stokes theorem tells that for a m-dimensional graph G with boundary S and a $(m-1)$-form F we have

> **Theorem:** $\sum_{x \in G} dF(x) = \sum_{y \in S} F(y)$

GRAVITY

36.9. The **Newton equations** $\frac{d^2}{dt^2} x_k = -\sum_j Gm_j/|x_k - x_j|^2$ with gravitational constant G describe the motion of finitely many mass points with positions $x_k(t) \in \mathbb{R}^3$ and mass m_k. These classical laws govern the motion of **planets** in our solar system, **stars** in a galaxy or **galaxies** in a **galaxy cluster**. While relativity modifies this Newtonian picture slightly and produces corrections which for example manifest in the perihelion advancement of Mercury, the Newtonian theory is amazingly accurate. Gauss derived the gravitational inverse square force F from $\text{div}(F) = 4\pi\sigma$, where σ is the mass density. While divergence usually maps a 2-form to a 3-form, it is the adjoint d^* of the gradient d. In $\mathbb{R}^3$ it is equivalent. Now, $L = \text{div} \circ \text{grad} = d^*d : \Lambda^0 \to \Lambda^0$ is called the **Kirchhoff Laplacian**. The Gauss law of gravity therefore is the **Poisson equation** $\boxed{LV = 4\pi\sigma}$, where V is the gravitational potential, a $0-$form. Since $d^* = 0$ on 0-forms, we can also write $L = dd^* + d^*d$. Classical gravity gets from a mass density σ the gravitational potential V and so the gravitational field as a gradient $F = dV$:

> $(d^*d + dd^*)V = 4\pi\sigma$ defines the gravitational 1-form $F = dV$.

ELECTROMAGNETISM

36.10. The **Maxwell equations** $\text{div}(E) = 4\pi\sigma$, $\text{div}(B) = 0$, $\text{curl}(E) = -B_t$, $\text{curl}(B) = E_t + 4\pi i$ become more elegant when written in four-dimensional **space-time** $\mathbb{R}^4$. There are then two equations only. The first is $dF = 0$ which is evident from $F = dA$ and $d^2 = 0$. The second is $d^*F = 4\pi j$, where j is the 4-**current** encoding both the charge density σ as well as the electric current i. Now $dF = 0$ implies in a simply connected region that $F = dA$, where A is an **electro-magnetic potential**. If $d^*A = 0$ (which can always be achieved by adding a gradient to A) we get the Poisson equation $LA = (dd^* + d^*d)A = 4\pi j$. This completely encodes the Maxwell equations; we can look at it also in a discrete network. Classical electromagnetism in a world with charge and current density j is the field $F = dA$, where A is obtained from

$(d^*d + dd^*)A = 4\pi j$ defines the electromagnetic 2-form $F = dA$.

Quantum mechanics

36.11. In this last homework we deal with a small universe G. We call it **Gaia**, the primordial deity of earth. In Greek mythology, Gaia was the daughter of **Aether** the god of air and **Hemera** the goddess of light. We only create the gravitational field, the electromagnetic field on G and some quanta, so there will be matter and light in this world. But that mathematics is exactly as in the universe we live in: the classical gravitational field is described with the language of Gauss which we have seen to imply the Newton law of gravity. The electromagnetic field is formulated according to Maxwell, but directly in space-time. We also look a bit at quantum mechanics as the eigenvalues and eigenvectors of the Laplacian L play a role when looking at the **Wheeler De Witt** equation, a time-independent Schrödinger equation in space time

$(d^*d + dd^*)F = \lambda F$ defines a wave function F on p-forms.

36.12. The time dependent Schrödinger equation, as mentioned in the introduction can also be studied. For graphs, it is an ordinary differential equation.

Beyond

36.13. The rest will be up to you: it remains to include the Fermionic constituents of matter (quarks (building mesons and baryons) as well as leptons) and bosons (photons, gluons, vector bosons and the Higgs) as well as a few other details called the **Standard model**. Don't complain about the homework, a former 22-student has solved a 10^{222} node homework assignment in less than 7 days ...

FIGURE 3. The Greek godess Gaia, seen in a Roman relief sculpture from the "Ara Pacis Augustae" in Rome. (Image by Dr. Sarah E. Bond.)

Problem 36.1: Given the 1-form F in Figure 4b, find the 0-form $f(x) = d^*F(x) = \sum_{e,e \to x} F(e)$. Check that $\sum_{x \in V} d^*F(x) = 0$. (This conservation law is a variant of the divergence theorem. (In the continuum, where 2-forms and 1-forms are identified and 3-forms are equated with 0-forms, this so called **Kirchhoff law** corresponds to the usual divergence theorem).

Problem 36.2: a) Given the 0-form f in Figure 4a, find $F = df$, then compute $d^*F = d^*df = Lf$.
b) Given the 1-form F in Figure 4b, compute the 2-form dF.
c) Given the 2-form H in Figure 4c, find a 1-form F such that $dF = H$. In classical terms, we look for a vector field F such that $\mathrm{curl}(F)$ is a given scalar field f (Classically this is done by solving $Q_x - P_y = f$ with $F = [0, \int_0^x f(t)\, dt]^T$ for example.)

Problem 36.3: Given the 0-form f in Figure 4d, check that this f satisfies $Kf = \lambda f$ for some constant λ. This is called an eigenvalue of K.

Problem 36.4: Write down the 4×4 Kirchhoff matrix K for the Gaia world. What are the eigenvalues and eigenvectors of K?

Problem 36.5: a) The complete graph with 4 elements is the smallest 3-dimensional "world". Find the Kirchhoff Matrix K of this graph and compute its eigenvalues and eigenvectors. You can use the first line of the Mathematica code below, which computes the Kirchhoff matrix of an other graph and then its Schroedinger evolution.
b) If ψ is an eigenvector of K satisfying $K\psi = \lambda\psi$. Verify that $\psi(t) = e^{-it\lambda/\hbar}\psi$ solves the **Schrödinger equation** $i\hbar\psi' = K\psi$. Use a formula seen earlier in this course to explain why quantum mechanics is called "wave mechanics".

```
K=Normal[KirchhoffMatrix[GridGraph[{10,10}]]];
U[t_]:=MatrixExp[-K*I*t]; MatrixPlot[Abs[U[10.0]]]
```
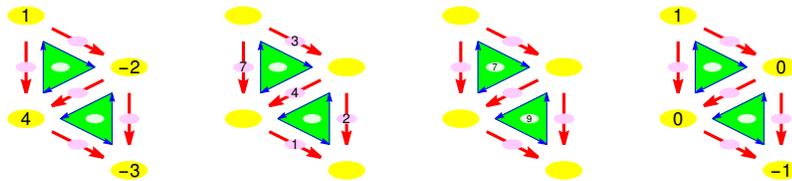

FIGURE 4. a) 0-form, b) 1-form, c) 2-form, d) eigenvector.

SOME LINEAR ALGEBRA

36.14. On Gaia, the space of 0-forms is 4-dimensional, the space of 1-forms is 5 dimensional and the space of 2-forms is 2 dimensional.
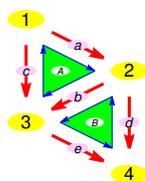
FIGURE 5.

36.15. We see the Dirac matrix $D = d + d^*$ to the left and the Laplacian $L = D^2 = dd^* + d^*d$ to the right. The first 4 columns contain the block $d_0 : \Lambda^0 \to \Lambda^1$ which is the gradient, the top block in the middle 5 columns is d_0^*, the divergence. The bottom block is $d_1 : \Lambda_1 \to \Lambda_2$ the curl. The block in the last 2 columns is $d_1^* : \Lambda^2 \to \Lambda^1$ each of the triangles affects 3 adjacent edges. The number b_0 is the dimension of the kernel of L_0. It is called the 0-th Betti number and counts the number of connectivity components of G (we don't have a multi-verse), the number b_1 is the number of "holes", there are none. Gaia is simply connected.
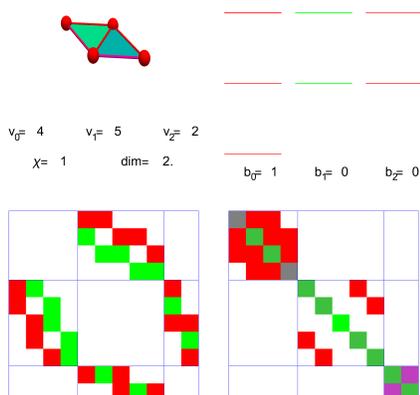
FIGURE 6.

36.16. The gradient d is a matrix which maps a function on vertices to a function on edges. It is a $|E| \times |V|$ matrix. In Mathematica, you can get d^* with "Incidence matrix". Note that Mathematica distinguishes between directed and undirected graphs and that the gradient is the transpose of d. To compute the Kirchhoffmatrix K, you have to use the undirected graph. Then $K = d^*d$. Here is an example verifying $K = d^*d = \text{divgrad}$.

G=Graph[{1−>2,2−>3,3−>4}]; d=**Transpose**[IncidenceMatrix[G]];
K1=KirchhoffMatrix[UndirectedGraph[G]]; K2=**Transpose**[d].d;
Print[K1==K2]; **Print**[**MatrixForm**[K1]]

Unit 37: Review: Geometries and Fields

37.1. The Integral theorems deal with **geometries** G and **fields** F. **Integration** pairs them in the form of **Stokes theorem**

$$\int_G dF = \int_{dG} F$$

which involves the **boundary** dG of G and the **exterior derivative** dF of F. One can classify the theorems by looking at the dimension n of the underlying space and the dimension m of the object G. In dimension n, there are n theorems:

$$
\begin{array}{ccccc}
& & 1 & & \\
& & \downarrow{\scriptstyle \frac{d}{dx}} & & \\
& 1 & \xrightarrow{} & 1 & \\
1 & \xrightarrow{\text{grad}} & 2 & \xrightarrow{\text{curl}} & 1 \\
1 \xrightarrow{\text{grad}} & 3 & \xrightarrow{\text{curl}} & 3 & \xrightarrow{\text{div}} 1
\end{array}
\qquad
\begin{array}{ccccc}
& & 1 & & \\
& & \downarrow & & \\
& 1 & \xrightarrow{\text{FTC}} & 1 & \\
1 & \xrightarrow{\text{FTL}} & 2 & \xrightarrow{\text{Green}} & 1 \\
1 \xrightarrow{\text{FTL}} & 3 & \xrightarrow{\text{Stokes}} & 3 & \xrightarrow{\text{Gauss}} 1
\end{array}
$$
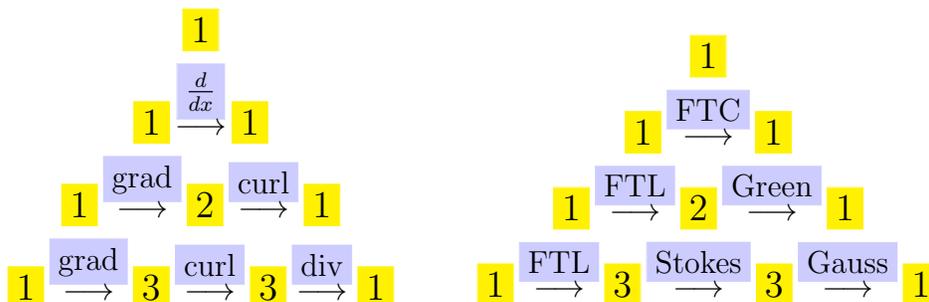
37.2. The **Fundamental theorem of line integrals** is a theorem about the gradient ∇f. It tells that if C is a curve going from A to B and f is a function (that is a 0-form), then

> **Theorem:** $\int_C \nabla f \cdot dr = f(B) - f(A)$

In calculus we write the 1-form as a column vector field ∇f. It actually is a 1-form $F = df$, a field which attaches a row vector to every point. If the 1-form is evaluated at $r'(t)$ one gets $df(r(t))(r'(t))$ which is the matrix product. We integrate then the **pull back** of the 1-form on the interval $[a, b]$. It is the switch from row vectors to column vectors which leads to the **dot product** $\nabla f(r(t)) \cdot r'(t)$. For closed curves, the line integral is zero. It follows also that integration is **path independent**.

37.3. Green's theorem tells that if $G \subset \mathbb{R}^2$ is a region bound by a curve C having G to the left, then

> **Theorem:** $\iint_G \text{curl}(F) \, dxdy = \int_C F \cdot dr$
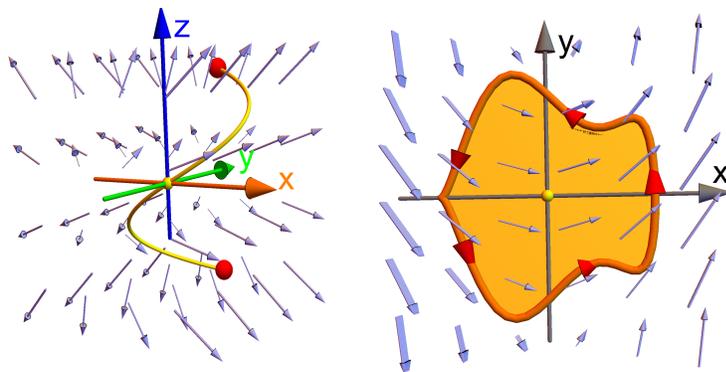
FIGURE 1.   Fundamental theorem of line integrals and Green's theorem.

In the language of forms, $F = Pdx + Qdy$ is a 1-form and $dF = (P_x dx + P_y dy)dx + (Q_x dx + Q_y dy)dy = (Q_x - P_y)dxdy$ is a 2-form. We write this 2-form dF as $Q_x - P_y$ and treat it as a scalar function even so this is not the same as a 0-form, which is a scalar function. If $\mathrm{curl}(F) = 0$ everywhere in $\mathbb{R}^2$ then F is a gradient field.

37.4. Stokes theorem tells that if S is a surface with boundary C oriented to have S to the left and F is a vector field, then

Theorem: $\iint_S \mathrm{curl}(F) \cdot dS = \int_C F \cdot dr$
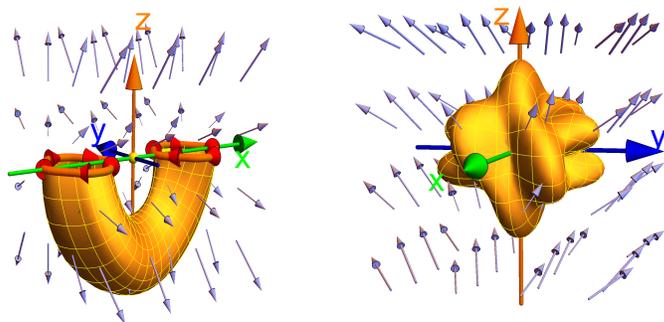
FIGURE 2.   Stokes theorem and the Gauss theorem.

In the general frame work, the field $F = Pdx + Qdy + Rdz$ is a 1-form and the 2-form $dF = (P_x dx + P_y dy + P_z dz)dx + (Q_x dx + Q_y dy + Q_z dz)dy + (R_x dx + R_y dy + R_z dz)dz = (Q_x - P_y)dxdy + (R_y - Q_z)dydz + (P_z - R_x)dzdx$ is written as a column vector field $\mathrm{curl}(F) = [R_y - Q_z, P_z - R_x, Q_x - P_y]^T$. To understand the flux integral, we need to see what a bilinear form like $dxdy$ does on the pair of vectors r_u, r_v. In the case $dxdy$ we have $dxdy(r_u, r_v) = x_u y_v - y_u x_v$ which is the third component of the cross product $r_u \times r_v$ with $r_u = [x_u, y_u, z_u]^T$. Integrating dF over S is the same as integrating the dot product of $\mathrm{curl}(F) \cdot r_u \times r_v$. Stokes theorem implies that the flux of the curl of F only depends on the boundary of S. In particular, the flux of the curl through a closed surface is zero because the boundary is empty.

37.5. Gauss theorem: if the surface S bounds a solid E in space, is oriented outwards, and F is a vector field, then

Theorem: $\iiint_E \text{div}(F) \, dV = \iint_S F \cdot dS$

Gauss theorem deals with a 2-form $F = Pdydz + Qdzdx + Rdxdy$, but because a 2-form has three components, we can write it as a **vector field** $F = [P, Q, R]^T$. We have computed $dF = (P_x dx + P_y dy + Q_z dz)dydz + (Q_x dx + Q_y dy + Q_z dz)dzdx + (R_x dx + R_y dy + R_z dz)dxdy$, where only the terms $P_x dxdydz + Q_y dydzdx + R_z dzdxdy = (P_x + Q_y + R_z)dxdydz$ survive which we associate again with the scalar function $\text{div}(F) = P_x + Q_y + R_z$. The integral of a 3-form over a 3-solid is the usual triple integral. For a divergence free vector field F, the flux through a closed surface is zero. Divergence-free fields are also called **incompressible** or **source free**.

<center>REMARKS</center>

37.6. We see why the 3 dimensional case looks confusing at first. We have three theorems which look very different. This type of confusion is common in science: we put things in the same bucket which actually are different: it is only in 3 dimensions that 1-forms and 2-forms can be identified. Actually, more is mixed up: not only are 1-forms and 2-forms identified, they are also written as vector fields which are T_0^1 tensor fields. From the tensor calculus point of view, we identify the three spaces $T_0^1(E) = E, T_1^0(E) = \Lambda^1(E) = E^*$ and $\Lambda^2(E) \subset T_2^0$. While we can still always identify vector fields with 1-forms, this identification in a general non-flat space will depend on the metric. In $\mathbb{R}^4$, the 2-forms have dimension 6 and can no more be written as a vector. One still does. The electro-magnetic F is a 2-form in $\mathbb{R}^4$ which we write as a pair of two time-dependent vector fields, the electric field E and the magnetic field B.

37.7. Geometries and fields are remarkably similar. On geometries, the **boundary operation** d satisfies $d \circ d = 0$. On fields the **derivative operation** d satisfies $d \circ d = 0$. 'Geometries" as well as "fields" come with an **orientation**: $r_u \times r_v = -r_v \times r_u$, $dxdy = -dydx$. The operations d and d look different because calculus deals with smooth things like curves or surfaces leading to generalized functions. In **quantum calculus** they are thickened up and d, d defined without limit. Fields and geometries then become indistinguishable elements in a Hilbert space. The exterior derivative d has as an adjoint $d = d^*$ which is the boundary operator. It is a kind of quantum field theory as d generates while d^* destroys a "particle". $d^2 = d^2 = 0$ is a "Pauli exclusion".

37.8. We can spin this further: a m**-manifold** S is the image of a parametrization $r : G \subset \mathbb{R}^m \to \mathbb{R}^n$. The Jacobian dr is a **dual** m**-form**, the exterior product of the m vectors dr_{u_1} up to dr_{u_m} (think of m column vectors attached to $r(u) \in S$). If we take a map $s : S \subset \mathbb{R}^n \to \mathbb{R}^m$ and look at $F = ds$, we can think of it as a m-form F (think of m row vectors attached to each point x in $\mathbb{R}^n$). The map s defines $m \times n$ Jacobian $ds(x)$, while the **Jacobian** $dr(u)$ is the $n \times m$ matrix. Cauchy-Binet shows that the flux of $F = ds$ through $r(G) = S$ is the integral $\int_G F = \int_G \det(ds(r(u))dr(u)) \, du$ $= \int_S \det(ds(x)dr(s(x)))$. If $s(r(u)) = u$, then this is a geometric functional. So: **geometries** G can come from maps from a space A to a space B, while **fields** F can come from maps from B to A. The **action integral** $\int_G F$ generalizes the Polyakov action $\int_G \det(dr^T dr) = \int_G |dr|^2$, a case where F and G are dual meaning $s(r(u)) = u$.

Prototype examples

Problem: Compute the line integral of $F(x, y, z) = [5x^4 + zy, 6y^5 + xz, 7z^6 + xy]$ along the path $r(t) = [\sin(5t), \sin(2t), t^2/\pi^2]$ from $t = 0$ to $t = 2\pi$.

Solution: The field is a gradient field df with $f = x^5 + y^6 + z^7 + xyz$. We have $A = r(0) = (0, 0, 0)$ and $B = r(2\pi) = (0, 0, 4)$ and $f(A) = 1$ and $f(B) = 4^7$. The **fundamental theorem of line integrals** gives $\int_C \nabla f \, dr = f(B) - f(A) = 4^7$.

Problem: Find the line integral of the vector field $F(x, y) = [x^4 + \sin(x) + y + 5xy, 4x + y^3]$ along the cardioid $r(t) = (1 + \sin(t))[\cos(t), \sin(t)]$, where t runs from $t = 0$ to $t = 2\pi$.

Solution: We use Green's theorem. Since $\text{curl}(F) = 3 - 5x$, the line integral is the double integral $\iint_G 3 - 5x \, dxdy$. We integrate in polar coordinates and get $\int_0^{2\pi} \int_0^{1+\sin(t)} (3 - 5r\cos(t)) r \, dr \, dt$ which is $9\pi/2$. One can short cut by noticing that by symmetry $\iint_G (-5x) \, dxdy = 0$, so that the integral is 3 times the area $\int_0^{2\pi} (1+\sin(t))^2/2 \, dt = 3\pi/2$ of the cardioid.

Problem: Compute the line integral of $F(x, y, z) = [x^3 + xy, y, z]$ along the polygonal path C connecting the points $(0, 0, 0), (2, 0, 0), (2, 1, 0), (0, 1, 0)$.

Solution: The path C bounds a surface $S : r(u, v) = [u, v, 0]$ parameterized on $G = \{(x, y) \mid x \in [0, 2], y \in [0, 1]\}$. By Stokes theorem, the line integral is equal to the flux of $\text{curl}(F)(x, y, z) = [0, 0, -x]$ through S. The normal vector of S is $r_u \times r_v = [1, 0, 0] \times [0, 1, 0] = [0, 0, 1]$ so that $\int \int_S \text{curl}(F) \cdot dS = \int_0^2 \int_0^1 [0, 0, -u] \cdot [0, 0, 1] \, dvdu = \int_0^2 \int_0^1 -u \, dvdu = -2$.

Problem: Compute the flux of the vector field $F(x, y, z) = [-x, y, z^2]$ through the boundary S of the rectangular box $G = [0, 3] \times [-1, 2] \times [1, 2]$.

Solution: By the **Gauss theorem**, the flux is equal to the triple integral of $\text{div}(F) = 2z$ over the box: $\int_0^3 \int_{-1}^2 \int_1^2 2z \, dzdydx = (3 - 0)(2 - (-1))(4 - 1) = 27$.

Unit 38: Review: Calculus in Hyperspace

GEOMETRIES

38.1. The **four dimensional Euclidean space** $\mathbb{R}^4 = M(4,1)$ is the space of column vectors with four real components $X = [x, y, z, w]^T$. If we think of such a vector as a **point**, we also write $X = (x, y, z, w)$. The **dot product** = **inner product** allows as usual to define **length** $|X| = \sqrt{X \cdot X}$, the **distance** $|X - Y|$ and the **angles** $\cos(\alpha) = (X \cdot Y)/(|X||Y|)$ between vectors. The **Cartesian coordinate system** has now four axes which are perpendicular to each other. Historically, as $\mathbb{R}^4$ is also the space of **quaternions**, it is custom to label the coordinate directions as $1 = [1, 0, 0, 0], i = [0, 1, 0, 0], j = [0, 0, 1, 0], k = [0, 0, 0, 1]$. A vector $[3, 4, 5, 1]$ for example is then written also as $3 + 4i + 5j + k$. We will however keep the vector-form. We will come back in the last section of this document about why quaternions are natural.

38.2. The kernel of the 1×4 matrix $A = [a, b, c, d]$ defines the **linear hyperplane** $ax + by + cz + dw = 0$. It is a 3-dimensional linear space. An example is the **coordinate hyperplane** $x = 0$, which consists of all points $\{(0, y, z, w) \;,\; y, z, w \in \mathbb{R}\}$. More generally, the solution space $ax + by + dz + dw = e$ is an **affine hyperplane**. The kernel of a 2×4 matrix is in general, as an intersection of two hyperplanes, a 2-dimensional plane, which we just call a **plane**. The kernel of a 3×4 matrix A is in general a **line**. Geometrically, it is the intersection of three hyperplanes.

38.3. A symmetric 4×4 matrix B, a row vector $A \in M(1, 4)$ and a constant e define the **hyper quadric** $X \cdot BX + AX = e$. For a diagonal matrix $B = \text{Diag}(a, b, c, d)$, this gives the quadric $ax^2 + by^2 + cz^2 + dw^2 = e$. Examples are the **3-sphere** $x^2 + y^2 + z^2 + w^2 = 1$, the **hyper paraboloid** $x^2 + y^2 + z^2 = w$, the 3-cylinder $x^2 + y^2 + z^2 = 1$ which is the product of a 2-sphere and a line. Or the **cylinder-plane** $x^2 + y^2 = 1$ which can be seen as the product of the 1-sphere with a 2-plane. There are three types of hyperboloids like $x^2 + y^2 + z^2 - w^2 = 1$ $x^2 + y^2 - y^2 - z^2 = 1$ or $x^2 - y^2 - z^2 - w^2 = 1$. One could call them **1-hyper-hyperboloids**, **2-hyper-hyperboloids** and **3-hyper-hyperboloids**, using the Morse index as a label. There is still 1-**hyperbolic-paraboloid** $x^2 + y^2 - z^2 = w$ but there are more degenerate surfaces like $x^2 - y^2 = w$. The two-dimensional torus $\mathbb{T}^2$ can be realized here as a quadratic surface. It is the intersection of $x^2 + y^2 = 1, z^2 + w^2 = 1$. This is the **flat torus**. We can not realize the two-dimensional torus in a flat way in our three dimensional space $\mathbb{R}^3$. In hyper-space, it can. There is also a **three dimensional torus** $\mathbb{T}^3$. To get a parametrization, start with the 2-torus parametrization $r(\phi, \theta) = [(3 + \cos(\phi)) \cos(\theta), (3 + \cos(\phi)) \sin(\theta), \sin(\phi)]$ then expand

the circle to get a **hyper-torus** $r(\phi, \theta, \psi) = [(3 + \cos(\phi))\cos(\theta),\ (3 + \cos(\phi))\sin(\theta),$ $(3 + \sin(\phi))\cos(\psi),\ (3 + \sin(\phi))\sin(\psi)]^T$, You see that for every fixed ψ we have a 2-torus. We can compute $4|dr| = 18 + 6\cos(\phi) + 6\sin(\phi) + \sin(2\phi)$ which is always positive and so verifies that the map from $\mathbb{T}^3$ to $\mathbb{R}^4$ is locally injective. We can also easily check that if ψ or θ is fixed we get a translated scaled version of the 2-torus. If ϕ is fixed, we get the flat 2-torus mentioned above.

38.4. In single variable calculus, one looks at graphs $\{(x, y) \mid y = f(x)\}$ of functions of one variable. In multi-variable, one adds graphs $\{(x, y, z) \mid z = f(x, y)\}$ of functions of two variables. The **graph of a function** $w = f(x, y, z)$ is now a 3-dimensional space. Paraboloids like $w = x^2 + y^2 + z^2$ or $w = x^2 + y^2 - z^2$ are graphs. An other example is the **three dimensional bell hyper-surface** $w = f(x, y, z) = \pi^{-3/2}e^{-x^2-y^2+z^2}$, where the constant has been chosen so that the **hyper-volume** $0 \le w \le f(x, y, z)$ is equal to 1. For obvious reasons, we usually do not draw the graph of a function of three variables as we would have to draw in 4 dimensions. Now, in hyperspace, we can do that.

38.5. Spaces can be parametrized in the same way as we parametrized curves or surfaces in three dimensions. A **curve** is defined by four real functions $x(t), y(t), z(t), w(t)$ of one variables and written as $r(t) = [x(t), y(t), z(t), w(t)]^T$. A **surface** is parametrized by $r(u, v) = [(x(u, v), y(u, v), z(u, v), w(u, v)]$. A **hypersurface** is now defined by $r(u, v, t) = [x(u, v, t), y(u, v, t), z(u, v, t), w(u, v, t)]$.

38.6. A **coordinate change** is defined by a map from $\mathbb{R}^4$ to $\mathbb{R}^4$ given by four differentiable functions: $r(u, v, s, t) = [x(u, v, s, t), y(u, v, s, t), z(u, v, s, t), w(u, v, s, t)]$. We have seen already the parametrization $r(\phi, \theta_1, \theta_0) = [\cos(\phi)\ \cos(\theta_1),\ \cos(\phi)\ \sin(\theta_1),$ $\sin(\phi)\ \cos(\theta_2),\ \sin(\phi)\ \sin(\theta_2)]$ of the **unit 3-sphere**= **hyper-sphere** $x^2 + y^2 + z^2 + w^2 = 1$. Because $z = x^2 + y^2 + z^2$ is a cylinder, there is also a natural **cylindrical coordinate system** in four dimensions. It is given by $r(\rho, \phi, \theta, w) = [\rho\sin(\phi)\cos(\theta),\ \rho\sin(\phi)\sin(\theta),$ $\rho\cos(\phi), w]$. If we write down the Jacobian matrix and compute the determinant we get $\rho^2 \sin(\phi)$ as in spherical coordinates.

FIELDS

38.7. A **scalar function** $f(x, y, z, w)$ is also called a 0-form. A **vector field** is denoted by $F = [P, Q, R, S]^T$ and a **1-form** $F = [P, Q, R, S]$ is written as $F = Pdx + Qdy + Rdz + Sdw$. A **2-form** F has 6 components: $F = Adxdy + Bdxdz + Cdxdw + Pdydz + Qdydz + Rdzdw$. A **3-form** again has four components $Pdydzdw + Qdxdzdw + Rdxdydw + Sdxdydz$ and a **4-form** is again completely determined by a scalar function f because $F = fdxdydzdw$.

38.8. The **exterior derivatives** are computed by using the anti-commutation rule like $dxdy = -dydx$ and $df = f_x dx + f_y dy + f_z dz + f_w dw$ and extending this to terms like $Pdydz = dPdydz = (P_x dx + P_y dy + P_z dz + P_w dw)dydz = P_x dxdydz + P_w dwdydz$. For a 1 form $F = Pdx + Qdy + Rdz + Sdw$ we have
$dF = P_x dxdx + P_y dydx + P_z dzdx + P_w dwdx\ +Q_x dxdy + Q_y dydy + Q_z dzdy + Q_w dwdy$
$+R_x dxdz + R_y dydz + R_z dzdz + R_w dwdz\ +S_x dxdw + S_y dydw + S_z dzdw + S_w dwdw$
which simplifies to expression with 6 terms. We have $\boxed{ddF = 0}$ because every term like $P_{yz} dzdydx$ is paired with a term like $P_{zy} dydzdx$ which cancel. For a 2-form

$F = Adxdy + Bdxdz + Cdwdx + Pdydz + Qdydw + Rdzdw$, we have $dF = (A_z dz + A_w dw)dxdy + (B_y dy + B_w dw)dxdz + (C_y dy + C_z dz)dwdx + (P_x dx + P_w dw)dydz + (Q_x dx + Q_z dz)dydw + (R_x dx + R_y dy)dzdw$ which simplifies to $(Q_z + P_w + R_y)dydzdw + (B_w + C_z + R_x)dxdzdw + (A_w + Q_x + C_y)dxdydw + (A_z + B_y + P_x)dxdydz$. For a 3-form $F = Pdydzdw + Qdzdwdx + Rdwdxdy + Sdxdydz$ we have $dF = (P_x - Q_y + R_z - S_w)dxdydzdw$.

38.9. The **gradient** of a function $f(x, y, z, w)$ is defined as $\nabla f(x, y, z, w) = df^T = [f_x, f_y, f_z, f_w]^T$ The **curl** of a vector field $F(x, y, z, w) = [F_1, F_2, F_3, F_4]^T$ is the hyper-field $dF = [F_{12}, F_{13}, F_{14}, F_{23}, F_{24}, F_{34}]^T$, where we have just chosen a lexigographic order and where $F_{ij} = \partial_{x_j} F_i - \partial_{x_i} F_j$. The **hypercurl** of a hyper vector field $F(x, y, z, w) = \langle F_{12}, F_{13}, F_{14}, F_{21}, F_{23}, F_{34}]$ is a 3-form but can again be associated with a vector field $dF = [F_{234}, F_{134}, F_{124}, F_{123}]^T$. The **divergence** of a vector field $F = [P, Q, R, S]$ is a 4-form $(P_x + Q_y + R_z + S_w)dxdydzdw$ but can again be associated with a scalar field.

38.10. Here are some properties which we have seen already. The gradient $\nabla f = df^T$ is perpendicular to the level surface $f(x, y, z, w) = c$. The curl of the gradient is zero. The hypercurl of the curl is zero. The divergence of the hypercurl is zero. The divergence of the gradient is the Laplacian (using the identifications, the divergence map can be identified with the adjoint $-d^*$). The **chain rule** is $d/dt f(r(t)) = \nabla f(r(t)) \cdot r'(t)$.

38.11. The **line integral** of a vector field F along a curve C is $\int_C F(r(t)) \cdot r'(t)\, dt$. The **flux integral** of a vector field F along a 2-dimensional surface is a **flux integral**. The **hyper flux integral** of a hyper-field F along a surface . The **hyper volume integral** of a function f on a solid G is $\iiiint_G f(x, y, z, w)\, dxdydzdw$.

<div align="center">THEOREMS</div>

38.12. The **fundamental theorem of line integrals** is

> **Theorem:** $\int \nabla f(r(t)) \cdot r'(t)\, dt = f(r(b)) - f(r(a))$.

38.13. The **Stokes theorem** tells that for a surface S and 1 form F

> **Theorem:** $\iint_S \text{curl}(F) \cdot dS = \int_C F \cdot dr$

38.14. The **Hyper Stokes theorem** assures that for a hypersurface S and a 2-form F, the flux of the hypercurl of F through G (a 3D-integral) is the flux of F through the boundary surface S (a 2D-integral)

> **Theorem:** $\iiint_G \text{hypercurl}(F) \cdot dG = \iint_S F \cdot dS$

38.15. The **divergence theorem** assures that for a 3-form (identified as a vector field F) and a solid G with boundary hyper-surface S, we have

> **Theorem:** $\iiiint_G \text{div}(F)\, dV = \iiint_S F \cdot dS$.

Quaternions

38.16. Hyperspace $\mathbb{R}^4$ is special: it is the only Euclidean space for which the unit sphere is a **non-Abelian Lie group**. A **Lie group** G is a manifold $r(\mathbb{R}^m) \subset \mathbb{R}^n$ [1] on which one has a **group operation** $x * y$ which has the property that for every y, the maps $x \to x * y$ and $x \to y * x$ are smooth maps on G. To have a **group** $(G, *)$ we must have the property that $(x * y) * z = x * (y * z)$ and that there is a 1-**element** $1*x = x*1 = x$ such that every element x has an inverse x^{-1} satisfying $x*x^{-1} = 1$. The circle $\{x^2 + y^2 = 1\} = \{z \in \mathbb{C} | |z| = 1\}$ is an example of a group. This multiplication is **Abelian** if $x * y = y * x$ for all $x, y \in G$. The complex plane $\mathbb{C} = \mathbb{R}^2$ is characterized as the only Euclidean space $\mathbb{R}^n$ in which the unit sphere $\mathbb{T}^1 = \{|x| = 1\}$ is an Abelian Lie group. Why Lie groups? They are the dough, **elementary particles** are baked from! Electromagnetism is built from $\mathbb{T}^1$ for example.

38.17. One can write a vector in $\mathbb{R}^4$ also as $v = a+ib+jc+kd$ where i, j, k are symbols. Hamilton noticed that when defining $i^2 = j^2 = k^2 = ijk = -1$, the 4-dimensional space becomes an algebra. An algebra is a linear space which also features a multiplication. Now one has already $M(2,2)$, the space of 2×2 matrices, which is a 4-dimensional algebra, but the algebra which Hamilton found is a **division algebra**: every non-zero element can be inverted. This is not the case for $M(2,2)$. The matrix in which all elements are 1 for example is non zero but it is also not invertible.

38.18. The algebra which Hamilton defined through the relations $i^2 = j^2 = k^2 = ijk = -1$ is called the **quaternion algebra** $\mathbb{H}$. If $\bar{v} = a - ib - jc - kd$, then $|v|^2 = v \cdot v = v\bar{v}$, where the right hand side is a quaternion multiplication. One can readily check that $|vw| = |v||w|$. The reason is that quaternions v can be realized as complex 2×2-matrices: if $A(v) = \begin{bmatrix} a + ib & c + id \\ -c + id & a - ib \end{bmatrix}$, then $|v| = \det(A(v))$ and $A(v)A(w) = A(vw)$. Your favorite AI helps to check this last identity quickly.

```
Import["Quaternions`"];
A[{x_,y_,z_,w_}]:={{x+I*y,z+I*w},{-z+I*w,x-I*y}};
Q=Quaternion[a,b,c,d]**Quaternion[p,q,r,s];
Simplify[A[{a,b,c,d}].A[{p,q,r,s}]==A[Table[Q[[k]],{k,4}]]]
```

38.19. An algebra with the property $|v * w| = |v||w|$ is a **normed division algebra**. By theorems of Hurwitz and Frobenius, there are only four: the reals $\mathbb{R}$, the complex $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. For an associative division algebra, the unit sphere is a Lie group. Because the unit sphere of $\mathbb{R}$ has only two points, the 1-circle $\{|z| = 1\} \subset \mathbb{C}$ and the unit 3-sphere $\{|z| = 1\} \subset \mathbb{H}$ are the only spheres that are Lie groups. There is a unique non-commutative one, the 3-sphere and a unique commutative connected one, the 1-sphere.

Theorem: $\mathbb{H}$ is the only non-Abelian associative normed division algebra.

Oliver Knill, knill@math.harvard.edu, Math 22b, Harvard College, Spring 2022

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[1] Manifolds can be described abstractly, but a theorem of John Nash assures that every manifold can be embedded in some $\mathbb{R}^n$. So, looking at images of maps r is no loss of generality!

Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total:

Unit 39: Final Exam Practice A

PROBLEMS

Problem 39A.1) (10 points):
On the graph G in Figure 1 we are given a 1-form F on a graph
$G = (V, E)$.
a) (3 points) Write the values of the curl dF. As a 2-form it is a function
on the set T of triangles.
b) (3 points) Compute the "discrete divergence" d^*F, which is a 0-form,
a function on the vertices.
c) (4 points) Find the value of the Laplacian $d^*dF + dd^*F$ and enter the
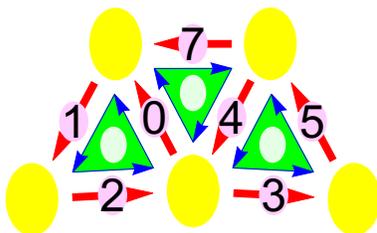values near the edges in Figure 2.

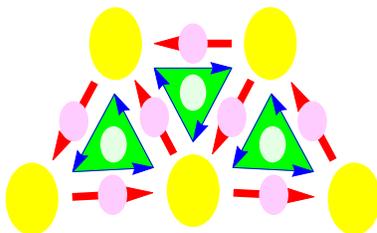FIGURE 1. A graph with a 1-Form F. Enter here the result for a) and b).

FIGURE 2. Enter here the result for c).

Problem 39A.2) (10 points) Each question is one point:
a) Who formulated the law of gravity in the form the partial differential equation $\text{div}(F) = 4\pi\sigma$?
b) The expression $5xdxdzdx + 77dydzdy + 3dxdy + 6dydx$ simplifies to ....
c) What value is $\iint_S [x, y, z] \cdot dS$ if S is the unit sphere oriented outwards?
d) What is the distance between the point $(0, 0, 3)$ and the xy-plane?
e) Is it true that if $|r'(t)| = 1$ everywhere, then $r''(t)$ is perpendicular to the velocity $r'(t)$?
f) What is the distortion factor $|dr|$ for the change of coordinates $r(u, v) = [-2v, 3u]$?
g) If $r(u, v)$ parametrizes a surface in $\mathbb{R}^3$, is it true that $r_u \times (r_u \times r_v)$ tangent to the surface?
h) Yes or no: if $(0, 0, 0)$ is a maximum of $f(x, y, z)$ then $f_{xx}(0, 0, 0) < 0$.
i) Write down the quadratic approximation of $1 + x + y + \sin(x^2 - y^2)$?
j) If $S : f(x, y, z) = x^2 + y^2 + z^2 = 1$ is oriented outwards, then the flux of ∇f through S is either negative, zero or positive. Which of the three cases is it?

Problem 39A.3) (10 points) Each problem is 1 point:
a) Which of the triangles in Figure 3 is integrated over in $\int_0^1 \int_y^1 f(x, y)\, dxdy$?
b) We have seen a counter example for Clairaut's theorem. This function $f(x, y)$ was in C^k but not in C^{k+1}. The integer k indicated how many times we could differentiate f continuously. What was the k?
c) To what group of partial differential equations belongs $\text{div}(E) = 4\pi j + E_t$?
d) Write down the Cauchy-Schwarz inequality.
e) Let G be the first stage of the Menger sponge (with 20 cubes from 27 cubes present). Is it simply connected?
f) Take a exterior derivative of the differential form $F = \sin(xz)dxdy$.
g) Parametrize the surface $x = z^2 - y^3$.
h) Parametrize the curve obtained by intersecting of the ellipsoid $x^2/4 + y^2 + z^2/9 = 1$ with the plane $y = 0$.
i) What surface is given in spherical coordinates as $\sin(\phi)\cos(\theta) = \cos(\phi)$?
j) Write down the general formula for the area of a triangle with vertices $(0, 0, 0), (a, b, c), (u, v, w)$.

Problem 39A.4) (10 points):
a) (6 points) Find the equation of the plane which contains the line $r(t) = [1+t, 2+t, 3-t]$ and which is perpendicular to the plane $\Sigma : x+2y-z = 4$.
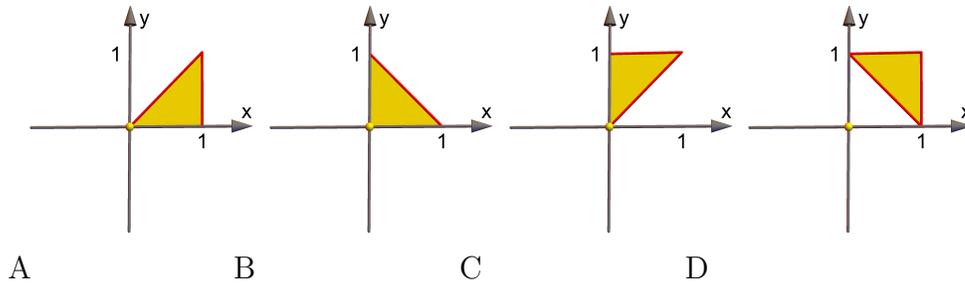b) (4 points) What is the angle between the normal vectors of Σ and the plane you just found?

FIGURE 3.   Four triangles

Problem 39A.5) (10 points):
a) (8 points) Find the critical points of the function $f(x, y) = \cos(x) + y^5 - 5y$ and classify them using the second derivative test. You can assume that $0 \le x < 2\pi$.
b) (2 points) Does the function f have a global maximum or a global minimum?

Problem 39A.6) (10 points):
a) (5 points) Use the Lagrange method to find the maximum of $f(x, y) = y^2 - x$ under the constraint $g(x, y) = x + x^3 - y^2 = 2$.
b) (5 points) The Lagrange equations fail to find the maximum of $f(x, y) = y^2 - x$ under the constraint $g(x, y) = x^3 - y^2 = 0$. Still, the Lagrange theorem still allows you to find the maximum. How?

Problem 39A.7) (10 points):
a) (6 points) Find the tangent plane at the point $P = (4, 2, 1, 1)$ of the surface $x^2 - 2y^2 + z^3 + w^2 = 2$.
b) (4 points) Parametrize the line $r(t)$ which passes through P which is perpendicular to the hyper surface at that point. Then find $(r(1) + r(-1))/2$.

Problem 39A.8) (10 points):
a) Estimate $f(0.012, 0.023)$ for $f(x, y) = \log(1 + x + 3xy)$ using linear approximation.
b) Estimate $f(0.012, 0.023)$ for $f(x, y) = \log(1 + x + 3xy)$ using quadratic approximation.

Problem 39A.9) (10 points):
a) Lets look at the curve which satisfies the acceleration $r''(t) = [-2\cos(t), -2\sin(t), -2\cos(t), -2\sin(t)]$, has the initial position $[2, 0, 2, 0]$ and initial velocity $[0, 2, 0, 2]$. Find $r(t)$.
b) What is the curvature $|T'(t)|/|r'(t)|$ of $r(t)$ at $t = 0$?

Problem 39A.10) (10 points):
a) Integrate the function $f(x,y) = x + x^2 - y^2$ over the region $1 < x^2 + y^2 < 4, xy > 0$.
b) Find the **surface area** of

$$r(t,s) = [\cos(t)\sin(s), \sin(t)\sin(s), \cos(s)]$$

$0 \le t \le 2\pi, 0 \le s \le t/2$.

Problem 39A.11) (10 points):
Let E be the solid

$$x^2 + y^2 \ge z^2, x^2 + y^2 + z^2 \le 9, y \ge |x|.$$

a) (7 points) Integrate

$$\iiint_E x^2 + y^2 + z^2 \; dxdydz.$$

b) (3 points) Let F be a vector field

$$F = [x^3, y^3, z^3]$$

Find the flux of F through the boundary surface of E, oriented outwards.
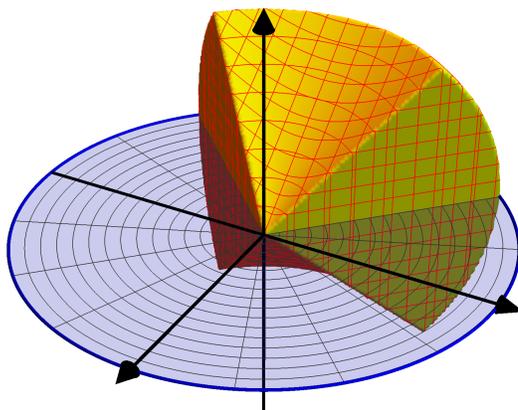
FIGURE 4. The solid in Problem 10.

Problem 39A.12) (10 points):
What is the line integral of the force field $F(x,y,z,w) = [1, 5y^4 + z, 6z^5 + y, 7w^6]^T + [y - w, 0, 0, 0]^T$ along the path $r(t) = [t^3, \sin(6t), \cos(8t), \sin(6t)]$ from $t = 0$ to $t = 2\pi$. Hint. We have written the field by purpose as the sum of two vector fields.

Problem 39A.13) (10 points):
Find the area of the region $|x|^{2/5} + |y|^{2/5} \le 1$. Use an integral theorem.

Problem 39A.14) (10 points):
What is the flux of the vector field $F(x, y, z, w) = [x+\cos(y), y+z^2, 2z, 3w]$ through the boundary of the solid $E : 1 \leq x \leq 3, 3 \leq y \leq 5, 0 \leq z \leq 1, 4 \leq w \leq 8$ oriented outwards?

Problem 39A.15) (10 points):
Find the **flux** of the curl of the vector field

$$F(x, y, z) = [-z, z + \sin(xyz), x - 3]^T$$

through the **twisted surface** seen in Figure 3 is oriented inwards and parametrized by

$$r(t, s) = [(3 + 2\cos(t))\cos(s), (3 + 2\cos(t))\sin(s), s + 2\sin(t)] ,$$
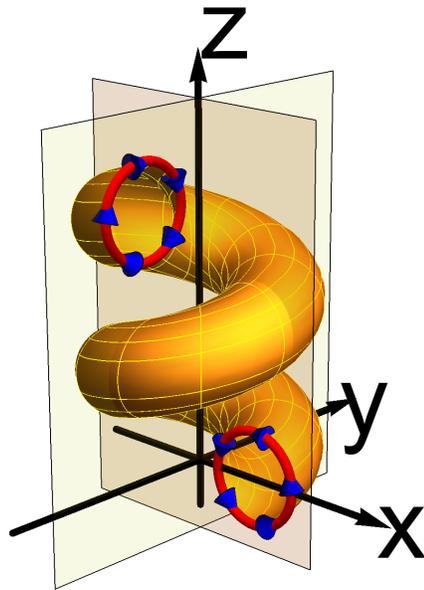
where $0 \leq s \leq 7\pi/2$ and $0 \leq t \leq 2\pi$.

FIGURE 5. The boundary of the surface is made of two circles $r(t, 0)$ and $r(t, 7\pi/2)$. The picture gives the direction of the velocity vectors of these curves (which in each case might or might not be compatible with the orientation of the surface).

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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total :

Welcome to the final exam. Please don't get started yet. We start all together at 9:00 AM after getting reminded about some formalities. You can fill out the attendance slip already. Also, you can already enter your name into the larger box above.

- You only need this booklet and something to write. Please stow away any other material and any electronic devices. Remember the honor code.
- Please write neatly and give details. Except for problems 2 and 3 we want to see details, even if the answer should be obvious to you.
- Try to answer the question on the same page. There is additional space on the back of each page. If you must, use additional scratch paper at the end.
- If you finish a problem somewhere else, please indicate on the problem page where we can find it.
- You have 180 minutes for this 3-hourly.

Unit 39: Final Practice B

Problem 39B.1) (10 points):
The graph $G = (V, E)$ in Figure 1 represents a discrete surface in which all triangles are oriented counterclockwise. The values of a 1-form = vector field F are given.

a) (2 points) Find the line integral of F along the boundary curve oriented counter clockwise.

b) (2 points) Compute the curl $H = dF$ and write its values into the triangles.

c) (2 points) What is the sum of all curl values? Why does it agree with the result in a)?

d) (2 points) Find also $g = d^*F$ and enter it near the vertices.

e) (1 point) True or False: $\sum_{x \in V} g(x) = 0$.

f) (1 point) True of False: we called $L = dd^*$ the Laplacian of G.

FIGURE 1. A discrete 2-dimensional region on which a 1-form F models a vector field. You compute the curl dF and divergence d^*F of F.

Problem 39B.2) (10 points) Each question is one point:

a) Name the 3-dimensional analogue of the Mandelbrot set.

b) If A is a 5×4 matrix, then A^T is a $m \times n$ matrix. What is m and n?

c) Write down the general formula for the arc length of a curve
$$r(t) = [x(t), y(t), z(t)]^T$$
with $a \leq t \leq b$.

d) Write down one possible formula for the curvature of a curve
$$r(t) = [x(t), y(t), z(t)]^T .$$

e) We have seen a parametrization of the 3-sphere invoking three angles ϕ, θ_1, θ_2. Either write down the parametrization or recall the name of the mathematician after whom it this parametrization is named.

f) The general change of variable formula for $\Phi : R \to G$ is $\iiint_R f(u, v, w) \boxed{} du\,dv\,dw = \iiint_G f(x, y, z) \; dx\,dy\,dz.$ Fill in the blank part of the formula.

g) What is the numerical value of $\log(-i)$?

h) We have used the Fubini theorem to prove that C^2 functions $f(x, y)$ satisfy a partial differential equation. Please write down this important partial differential equation as well as its name. (It was used much later in the course.)

i) What is the integration factor $|dr|$ for the parametrization
$$r(u, v) = [a \cos(u) \sin(v), b \sin(u) \sin(v), c \cos(v)]^T ?$$

j) In the first lecture, we have defined $\sqrt{\operatorname{tr}(A^T A)}$ as the length of a matrix. What is the length of the 3×3 matrix which contains 1 everywhere?

Problem 39B.3) (10 points) Each problem is 1 point:

a) Assume that for a Morse function $f(x, y)$ the discriminant D at a critical point (x_0, y_0) is positive and that $f_{yy}(x_0, y_0) < 0$. What can you say about $f_{xx}(x_0, y_0)$?

b) We have proven the identity $|dr| = |r_u \times r_v|$, where r was a map from $\mathbb{R}^m$ to $\mathbb{R}^n$. For which m and n was this identity defined?

c) Which of the following is the correct integration factor when using spherical coordinates in 4 dimensions?
$|d\Phi| = r$
$|d\Phi| = (3 + \cos(\phi))$
$|d\Phi| = \rho^2 \sin(\phi)$
$|d\Phi| = \rho^3 \sin(2\phi)/2$

d) Which of the following vector fields are gradient fields? (It could be none, one, two, three or all.)
$F = [x, 0]^T$
$F = [0, x]^T$
$F = [x, y]^T$
$F = [y, x]^T$

e) Which of the following four surfaces is a one-sheeted hyperboloid? (It could be none, one, two, three or all.)
$x^2 + y^2 = z^2 - 1$
$x^2 - y^2 = 1 - z^2$
$x^2 + y^2 = 1 - z^2$
$x^2 - y^2 = z^2 + 1$

f) Parametrize the surface $x^2 + y^2 - z^2 = 1$ as
$$r(\theta, z) = [............, ..., ...]^T .$$
g) Who was the creative person who discovered dark matter and proposed the mechanism of gravitational lensing?

h) What is the cosine of the angle between the matrices $A, B \in M(2, 2)$, where A is the identity matrix and B is the matrix which has 1 everywhere? You should get a concrete number.

i) We have seen the identity $|v|^2 + |w|^2 = |v - w|^2$, where v, w are vectors in $\mathbb{R}^n$. What conditions do v and w have to satisfy so that the identity holds?

j) Compute the exterior derivative dF of the differential form
$$F = e^x \sin(y)dxdy + \cos(xyz)dydz .$$

Problem 39B.4) (10 points):
a) (4 points) Find the plane Σ which contains the three points

$$A = (3, 2, 1), \quad B = (3, 3, 2), \quad C = (4, 3, 1) .$$

b) (3 points) What is the area of the triangle ABC?

c) (3 points) Find the distance of the origin $O = (0, 0, 0)$ to the plane Σ.

Problem 39B.5) (10 points):
a) (8 points) Find all the critical points of the function
$$f(x, y) = x^5 - 5x + y^3 - 3y$$
and classify these points using the second derivative test.

b) (2 points) Is any of these points a global maximum or global minimum of f?

Problem 39B.6) (10 points):
a) (8 points) Use the Lagrange method to find **all the maxima and all the minima** of
$$f(x, y) = x^2 + y^2$$
under the constraint
$$g(x, y) = x^4 + y^4 = 16 .$$

b) (2 points) In our formulation of Lagrange theorem, we also mentioned the case, where $\nabla g(x, y) = [0, 0]^T$. Why does this case not lead to a critical point here?

Problem 39B.7) (10 points):
a) (5 points) The hyper surface
$$S = \{f(x, y, z, w) = x^2 + y^2 + z^2 - w = 5\}$$
defines a three-dimensional manifold in $\mathbb{R}^4$. It is poetically called a **hyper-paraboloid**. Find the tangent plane to S at the point $(1, 2, 1, 1)$.

b) (5 points) What is the linear approximation $L(x, y, z, w)$ of $f(x, y, z, w)$ at this point $(1, 2, 1, 1)$?

Problem 39B.8) (10 points):
Estimate the value $f(0.1, -0.02)$ for
$$f(x, y) = 3 + x^2 + y + \cos(x + y) + \sin(xy)$$
using quadratic approximation.

Problem 39B.9) (10 points):
a) (8 points) We vacation in the **5-star hotel** called **MOTEL 22** in 5-dimensional space and play there ping-pong. The ball is accelerated by gravity $r''(t) = [x(t), y(t), z(w), v(t), w(t)] = [0, 0, 0, 0, -10]^T$. We hit the ball at $r(0) = [4, 3, 2, 1, 2]^T$ and give it an initial velocity $r'(0) = [5, 6, 0, 0, 3]^T$. Find the trajectory $r(t)$.

b) (2 points) At which positive time $t > 0$ does the ping-pong ball hit the **hyper ping-pong table** $w = 0$? (The points in this space are labeled $[x, y, z, v, w]$.)

Problem 39B.10) (10 points):
a) (5 points) Integrate the function $f(x, y) = (x^2 + y^2)^{22}$ over the region $G = \{1 < x^2 + y^2 < 4, y > 0\}$.

b) (5 points) Find the area of the region enclosed by the curve
$$r(t) = [\cos(t), \sin(t) + \cos(2t)]^T,$$
with $0 \le t \le 2\pi$.

Problem 39B.11) (10 points):
a) (7 points) Integrate
$$f(x, y, z) = x^2 + y^2 + z^2$$
over the solid
$$G = \{x^2 + y^2 + z^2 \le 4, z^2 < 1\}.$$

b) (3 points) What is the volume of the same solid G?

Problem 39B.12) (10 points):
a) (8 points) Compute the line integral of the vector field
$$F = [yzw + x^6, xzw + y^9, xyw - z^3, xyz + w^4]^T$$
along the path
$$r(t) = [t + \sin(t), \cos(2t), \sin(4t), \cos(7t)]^T$$
from $t = 0$ to $t = 2\pi$.

b) (2 points) What is $\int_0^{2\pi} r'(t)\, dt$?

Problem 39B.13) (10 points):

a) (8 points) Find the line integral of the vector field
$$F(x, y) = [3x - y, 7y + \sin(y^4)]^T$$
along the polygon $ABCDE$ with $A = (0,0), B = (2,0), C = (2,4), D = (2,6), E = (0,4)$. The path is closed. It starts at A, then reaches B, C, D, E until returning to A again.

b) (2 points) What is line integral if the curve is traced in the opposite direction?

Problem 39B.14) (10 points):

a) (8 points) What is the flux of the vector field
$$F(x, y, z) = [y + x^3, z + y^3, x + z^3]^T$$
through the sphere $S = \{x^2 + y^2 + z^2 = 9\}$ oriented outwards?

b) (2 points) What is the flux of the same vector field F through the same sphere S but where S is oriented inwards?

Problem 39B.15) (10 points):

a) (7 points) What is the flux of the curl of the vector field
$$F(x, y, z) = [-y, x + z(x^2 + y^5), z]^T$$
through the surface
$$S = \{x^2 + y^2 + z^2 + z(x^4 + y^4 + 2\sin(x - y^2 z)) = 1, z > 0\}$$
oriented upwards?

b) (3 points) The surface in a) was not closed, it did not include the bottom part
$$D = \{z = 0, x^2 + y^2 \leq 1\}.$$
Assume now that we close the bottom and orient the bottom disc D downwards. What is the flux of the curl of the same vector field F through this closed surface obtained by taking the union of S and D?

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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total :

Welcome to the final exam. Please don't get started yet. We start all together at 9:00 AM after getting reminded about some formalities. You can fill out the attendance slip already. Also, you can already enter your name into the larger box above.

- You only need this booklet and something to write. Please stow away any other material and any electronic devices. Remember the honor code.
- Please write neatly and give details. Except for problems 2 and 3 we want to see details, even if the answer should be obvious to you.
- Try to answer the question on the same page. There is additional space on the back of each page. If you must, use additional scratch paper at the end. But put your final result near the question and box the final result.
- If you finish a problem somewhere else, please indicate on the problem page where we can find it.
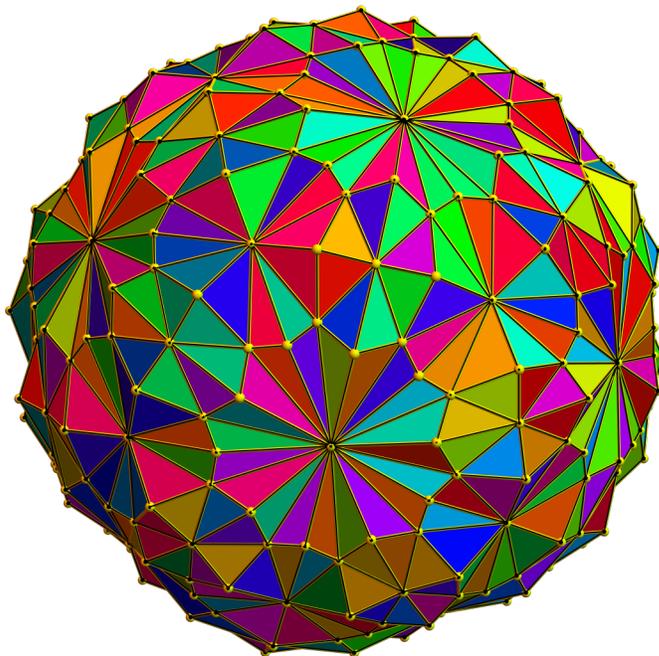- You have 180 minutes for this final exam.

FIGURE 1. A two dimensional discrete sphere S.

Unit 39: Final Exam

Problem 39.1) (10 points):
In Figure 2 (see the next page for a larger version) you see a discrete two dimensional region G in which all triangles are oriented counter clockwise. The one-form F as a function on oriented edges is given in the picture. Answer the following questions and give reasons:

a) (2 points) The curl dF of F is a function on oriented triangles. What can you say about the sum over all the curl values dF in the graph G of Figure 2?

b) (2 points) Is F a gradient field $F = df$ for some function f on vertices?

c) (2 points) What is the sum of the natural divergence values d^*F on vertices?

d) (2 points) What was the name of the matrix $K = d^*d$ that acts on 0-forms. It has been defined more than 150 years ago.

e) (2 points) In Figure 1 on the front page, you saw a two-dimensional discrete sphere S. which plays the role of a closed surface $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$. Given a 1-form F, a function on oriented edges of S, what is the sum over all curls on S? The answer is a number but you have to justify the answer.
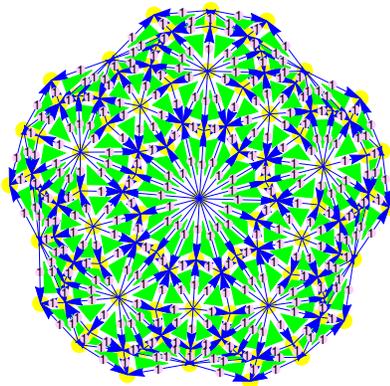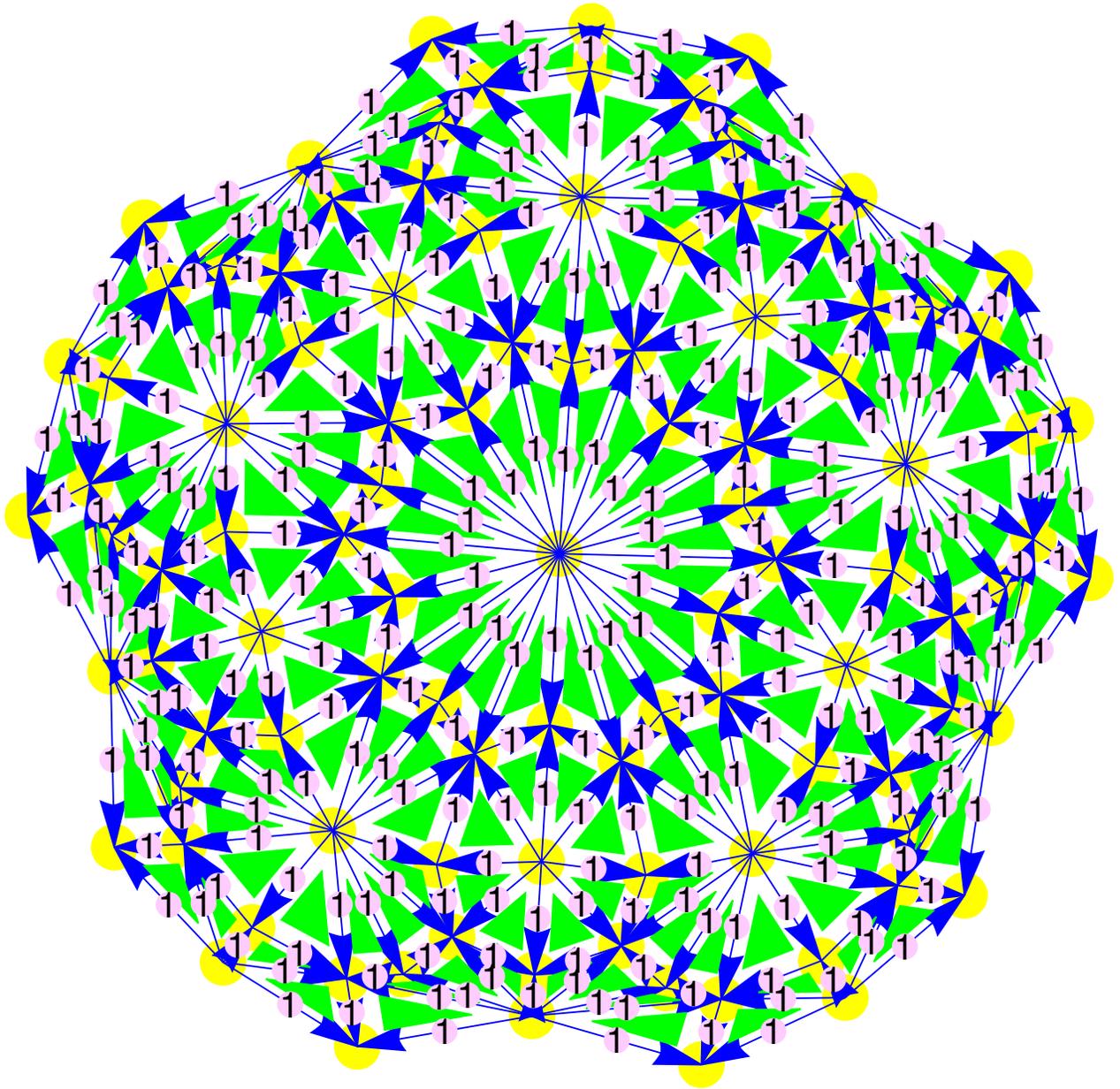

FIGURE 2.

An enlargement of Figure 2, showing the region G from problem 39.1.

Problem 39.2) (10 points) Each question is one point:

a) Albert Einstein used the notation $v_k w^k$ for two vectors v, w. It is today called "Einstein notation". What did Einstein mean, when he wrote $v_k w^k$?

b) If $S = r(R)$ is a two-dimensional surface parametrized by $r(u, v) = [x(u, v), y(u, v), z(u, v)]^T$, what is the relation between $|r_u \times r_v|$ and $\sqrt{\det(dr^T dr)}$?.

c) What is the Newton method used for? We have seen this numerical tool in a proof seminar.

d) What is the curvature of a circle with radius 20?

e) Define the 1×5 matrix $A = [1, 1, 1, 1, 1]$. One of the two matrices $A, B = A^T$ is row reduced. Which one?

f) What is the distortion factor of the coordinate change $\Phi(x, y) = (3x + y, x + y)$?

g) What is the numerical value of i^{22}, if $i = \sqrt{-1}$ is the imaginary unit?

h) What is the name of the differential equation $i\hbar \frac{d}{dt}\psi = K\psi$, where K is a matrix? It appears in a theory which also is called "matrix mechanics".

i) Why is the distance between two lines $r_1(t) = Q + tv$ and $r_2(t) = P + tw$ given by the formula $|(v \times w) \cdot PQ|/|v \times w|$?

j) You are given a Morse function f on a 2-torus and you count that f has 11 maxima and 11 minima. How many saddle points are there?

Herr Einstein wishes you good luck!

Problem 39.3) (10 points) Each question is one point:
In this problem, we work in hyperspace $\mathbb{R}^4$, where points have coordinates (x, y, z, w).

a) Write down the exterior derivative dF of the 2-form
$$F = x^2 y^2 z^2 w^2 \, dydz \ .$$

b) Write down the exterior derivative of the 3-form
$$F = x^2 y^2 z^2 w^2 \, dxdzdw \ .$$

c) Let G be the two-dimensional torus $x^2 + y^2 = 1, z^2 + w^2 = 1$ embedded in $\mathbb{R}^4$. What does the general Stokes theorem tell about $\iint_G F \, dS$, where F is the 2-form from a)?

d) What is $d^2 F = ddF$, where F is the 2-form given in a)?

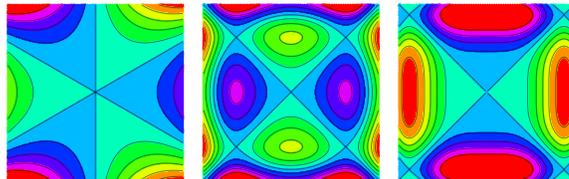e) What is $d^2 F = ddF$, where F is the 3-form given in b)?

f) A $(1, 1)$ tensor on $\mathbb{R}^4$ can be interpreted as a 4×4 ....

g) A $(0, 1)$ tensor on $\mathbb{R}^4$ can also be interpreted as a ....

h) Is $\operatorname{grad}(\operatorname{grad}(f))$ defined if f is a function?

i) Does $\operatorname{div}(\operatorname{div}(F))$ make sense for any field F?

j) You see 3 contour maps of functions f, g and h of two variables. One of them is not Morse. Which one? The first the second or the third?

Problem 39.4) (10 points):
a) (3 points) Parametrize the line L which contains the points

$$A = (3, 2, 1), \quad B = (3, 3, 2) .$$

b) (3 points) Given the additional point $P = (3, 3, 3)$, find the distance between P and L.

c) (4 points) Write down the equation $ax + by + cz = d$ of the plane containing L and P.

Problem 39.5) (10 points):
a) (6 points) Find all the critical points of the function
$$f(x, y) = x^7 - 7x + xy - y$$
and classify them using the second derivative test.

b) (2 points) The island theorem told us that the number of maxima plus the number of minima minus the number of saddle points of f is 1 on an island. In the current case this fails. Why does this not contradict the island theorem?

c) (2 points) Does the function f have a global maximum or a global minimum?

Problem 39.6) (10 points):
a) (7 points) Use the Lagrange method to find the minimum of the function
$$f(x, y, z, w) = x^2 + 2y^2 + 3z^2 + w^2$$
under the constraint
$$g(x, y, z, w) = x + y + z + w = 17.$$

b) (3 points) You saw in a) that in this case, the Lagrange equations are a system of linear equations for a couple of unknown. This can be written in matrix form as $AX = b$, where the vector X encodes the unknown quantities and b is a constant vector. What is the size of the matrix A?

Problem 39.7) (10 points):
a) (5 points) Find the tangent plane at the point $P = (3, 1, 3, -1)$ of the **hyper cone**
$$S = \{f(x, y, z, w) = x^2 + y^2 - z^2 - w^2 = 0\}$$
in $\mathbb{R}^4$.

b) (5 points) Write down the linearization $L(x, y, z, w)$ of $f(x, y, z, w)$ at $(3, 1, 3, -1)$.

Problem 39.8) (10 points):
Estimate the value $f(0.1, -0.02)$ for $f(x, y) = e^{x+y}$ using quadratic approximation $Q(x, y)$ at $(x_0, y_0) = (0, 0)$.

Problem 39.9) (10 points):

a) (6 points) Find the curve $r(t)$ which satisfies $r(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and

$r'(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $r''(t) = \begin{bmatrix} 1 - \sin(t) \\ -4\sin(2t) \\ -9\sin(3t) \end{bmatrix}$.

b) (4 points) What is the curvature of the curve at the point $r(0)$?

Problem 39.10) (10 points):
Find the area of the region enclosed by the curve
$$r(t) = \begin{bmatrix} 3\cos(t) \\ 2\sin(t) + \cos(7t) \end{bmatrix},$$
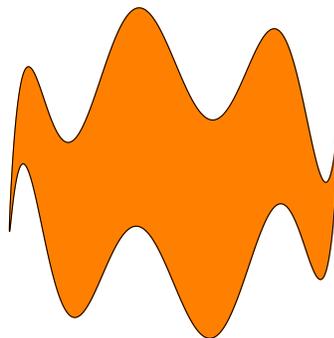where $0 \leq t \leq 2\pi$.

FIGURE 3. The region in problem 39.10.

Problem 39.11) (10 points):

Integrate

$$f(x, y, z) = \frac{e^{x^2+y^2+z^2}}{\sqrt{x^2 + y^2 + z^2}}$$

over the half avocado

$$E = \{4 \le x^2 + y^2 + z^2 \le 16, z \le 0\} \ .$$

In other words, compute $\iiint_E f \, dV$.
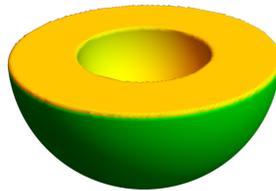
FIGURE 4.   The avocado in problem 39.11.

Problem 39.12) (10 points):

Compute the line integral

$$\int_C F \cdot dr = \int_0^1 F(r(t)) \cdot r'(t) \, dt$$

of the vector field

$$F = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} 3x^2 + yz \\ 3y^2 + xz \\ 3z^2 + xy \end{bmatrix}$$

along the path C parametrized by

$$r(t) = \begin{bmatrix} \cos(7\pi t)e^{t(1-t)} \\ \sin(11\pi t) \\ e^{t(1-t)} \end{bmatrix}$$

from $t = 0$ to $t = 1$.

Problem 39.13) (10 points):
Find the line integral $\int_C F \cdot dr$ of the vector field
$$F(x, y) = \left[\begin{array}{c} y + x^4 \\ y^3 + y^4 \end{array} \right]$$
along the boundary C of the hexagon region shown in the picture. The curve C is a closed polygon going counter clockwise from $(2, 0)$ over $(1, 2), (-1, 2), (-2, 0), (-1, -2), (1, -2)$ back to $(2, 0)$.
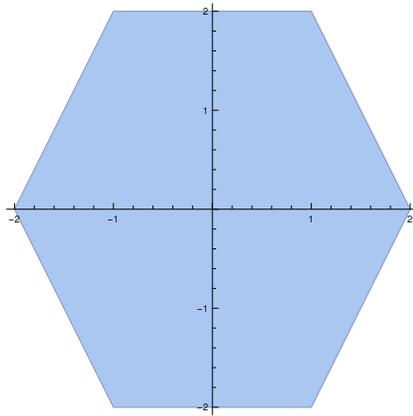

FIGURE 5.   The hexagon in Problem 39.13.

Problem 39.14) (10 points):
Find the flux $\iint_S \mathrm{curl}(F) \cdot dS$ of the curl of the vector field
$$F = \left[\begin{array}{c} x^7 \\ -x \\ \sin(z^2) + z^3 x \end{array} \right]$$
through the surface S parametrized by
$$r(s, t) = \left[\begin{array}{c} (6 + 2\cos^2(s/2)\cos(t))\cos(2s) \\ 2\cos^2(s/2)\sin(t) + 2s \\ (6 + 2\cos^2(s/2)\cos(t))\sin(2s) \end{array} \right]$$
with $0 \le s \le 7\pi/2$ and $0 \le t < 2\pi$. **Hint:** The surface has two boundary curves obtained by looking at $s = 0$ or $s = 7\pi/2$. We don't tell you the orientation of the larger curve
$$r_1(t) = r(0, t) = [6 + 2\cos(t), 2\sin(t), 0]^T$$
is but you should know that the smaller curve
$$r_2(t) = r(7\pi/2, t) = [-6 - \cos(t), \sin(t) + 7\pi, 0]^T$$
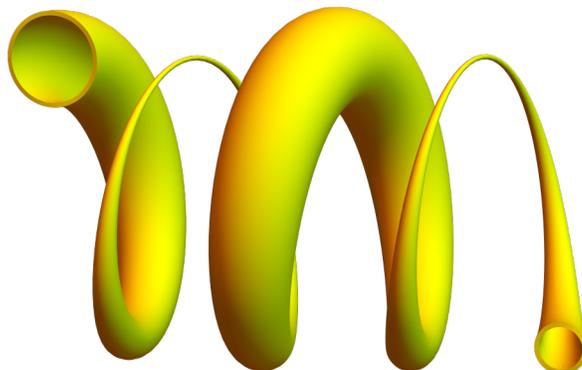is correctly oriented.

FIGURE 6. The surface S with two boundary circles in Problem 39.14.

Problem 39.15) (10 points):
Find the flux

$$\iint_S F \cdot dS$$

of the vector field

$$F = \begin{bmatrix} \sin(z) + y^3 + x \\ \sin(x) + z^3 + y \\ \sin(y) + x^3 + z \end{bmatrix}$$

through the boundary surface S of the solid E given in the picture. The solid is obtained by sculpuring a cube $-1 \leq x \leq 1, -1 \leq y \leq 1, -1 \leq z \leq 1$ of side length 2, by cutting away at each corner the points in distance less than 1 from that corner. In other words, we look at the points in the cube which have distance larger than 1 from any of the 8 corners. The surface S bounding the solid E is oriented outwards.
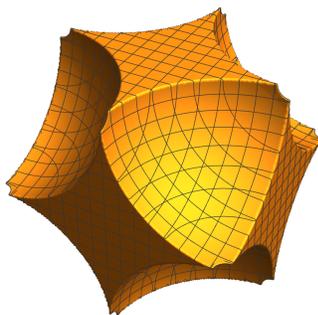

FIGURE 7. The solid given in Problem 39.15.

Unit 39: Keywords for the Final (see also Units 14+28)

Discrete Calculus

- $G = (V, E)$ graph with vertex set V and edge set E.
- 0-form: function on V. Discrete scalar function
- 1-form: function on oriented E. Discrete vector field
- 2-form: function on oriented triangles T.
- $d(f) = \text{grad}(f)$ is a function on edges $a -> b$ defined by $f(b) - f(a)$.
- $H = dF = \text{curl}(F)$ is a function on triangles obtained by summing F along the triangle.
- For a 1-form F, d^*F is a function on vertices. Add up the attached edge values.
- For a 2-form H, d^*H is a function on edges. Add up the attached triangle values.

New People

- Mentioned: Cartan, Maxwell, Stokes, Green, Gauss, Newton, Einstein, Kirchhoff, Menger, Koch, Escher, Peirce

Partial Derivatives

- $L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ linear approximation
- $Q(x, y) = L(x_0, y_0) + f_{xx}(x - x_0)^2/2 + f_{yy}(y - y_0)^2/2 + f_{xy}(x - x_0)(y - y_0)$.
- use $L(x, y)$ to estimate $f(x, y)$ near $f(x_0, y_0)$. The result is $f(x_0, y_0) + a(x - x_0) + b(y - y_0)$
- tangent plane: $ax + by + cz = d$ with $a = f_x, b = f_y, c = f_z, d = ax_0 + by_0 + cz_0$
- estimate $f(x, y)$ by $L(x, y)$ or $Q(x, y)$ near (x_0, y_0)
- $f_{xy} = f_{yx}$ Clairaut's theorem for functions which are in C^2.
- $r_u(u, v), r_v(u, v)$ tangent to surface parameterized by $r(u, v)$

Parametrization

- $r : G \subset \mathbb{R}^m \to \mathbb{R}^n$, dr Jacobian
- $g = dr^T dr$ first fundamental form, $|dr| = \sqrt{g}$ distortion factor.
- $\text{curl}(F)(r(u, v)) \cdot (r_u \times r_v) = F_u \cdot r_v - F_v \cdot r_u$ important formula

Partial Differential Equations

- $f_{xy} = f_{yx}$ Clairaut
- $f_t = f_{xx}$ heat equation
- $f_{tt} - f_{xx} = 0$ wave equation
- $f_x - f_t = 0$ transport equation
- $f_{xx} + f_{yy} = 0$ Laplace equation

- [ ] $f_t + f f_x = f_{xx}$ Burgers equation
- [ ] $dF^* = j, dF = 0$, Maxwell equations
- [ ] $\text{div}(F) = 4\pi\sigma$, Gravity equation

Gradient

- [ ] $\nabla f(x, y) = [f_x, f_y]^T$, $\nabla f(x, y, z) = [f_x, f_y, f_z]^T$, gradient
- [ ] $D_v f = \nabla f \cdot v$ directional derivative
- [ ] $\frac{d}{dt} f(r(t)) = \nabla f(r(t)) \cdot r'(t)$ chain rule
- [ ] $\nabla f(x_0, y_0)$ is orthogonal to the level curve $f(x, y) = c$ containing (x_0, y_0)
- [ ] $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface $f(x, y, z) = c$ containing (x_0, y_0, z_0)
- [ ] $\frac{d}{dt} f(x + tv) = D_v f$ by chain rule
- [ ] $(x - x_0) f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0) = 0$ tangent line
- [ ] $(x - x_0) f_x(x_0, y_0, z_0) + (y - y_0) f_y(x_0, y_0, z_0) + (z - z_0) f_z(x_0, y_0, z_0) = 0$ tangent plane
- [ ] $D_v f(x_0, y_0)$ is maximal in the $v = \nabla f(x_0, y_0)/|\nabla f(x_0, y_0)|$ direction
- [ ] $f(x, y)$ increases in the $\nabla f/|\nabla f|$ direction at points which are not critical points
- [ ] if $D_v f(x) = 0$ for all v, then $\nabla f(x) = 0$
- [ ] $f(x, y, z) = c$ defines $y = g(x, y)$, and $g_x(x, y) = -f_x(x, y, z)/f_z(x, y, z)$ implicit diff

Extrema

- [ ] $\nabla f(x, y) = [0, 0]^T$, critical point
- [ ] $D = \det(d^2 f) = f_{xx} f_{yy} - f_{xy}^2$ discriminant.
- [ ] Morse: critical point and $D \neq 0$, in 2D looks like $x^2 + y^2, x^2 - y^2, -x^2 - y^2$
- [ ] $f(x_0, y_0) \geq f(x, y)$ in a neighborhood of (x_0, y_0) local maximum
- [ ] $f(x_0, y_0) \leq f(x, y)$ in a neighborhood of (x_0, y_0) local minimum
- [ ] $\nabla f(x, y) = \lambda \nabla g(x, y), g(x, y) = c$, λ Lagrange equations
- [ ] $\nabla f(x, y, z) = \lambda \nabla g(x, y, z), g(x, y, z) = c$, λ Lagrange equations
- [ ] second derivative test: $\nabla f = (0, 0), D > 0, f_{xx} < 0$ **local max**, $\nabla f = (0, 0), D > 0, f_{xx} > 0$ **local min**, $\nabla f = (0, 0), D < 0$ **saddle point**
- [ ] $f(x_0, y_0) \geq f(x, y)$ everywhere, global maximum
- [ ] $f(x_0, y_0) \leq f(x, y)$ everywhere, global minimum

Double Integrals

- [ ] $\iint_R f(x, y) \, dy dx$ double integral
- [ ] $\int_a^b \int_c^d f(x, y) \, dy dx$ integral over rectangle
- [ ] $\int_a^b \int_{c(x)}^{d(x)} f(x, y) \, dy dx$ bottom-top region
- [ ] $\int_c^d \int_{a(y)}^{b(y)} f(x, y) \, dx dy$ left-right region
- [ ] $\iint_R f(r, \theta) \boxed{r} \, dr d\theta$ polar coordinates
- [ ] $\iint_R |r_u \times r_v| \, du dv$ surface area
- [ ] $\int_a^b \int_c^d f(x, y) \, dy dx = \int_c^d \int_a^b f(x, y) \, dx dy$ Fubini
- [ ] $\iint_R \boxed{1} \, dx dy$ area of region R
- [ ] $\iint_R f(x, y) \, dx dy$ signed volume of solid bound by graph of f and xy-plane

Triple Integrals

- $\iiint_R f(x, y, z)\ dzdydx$ triple integral
- $\int_a^b \int_c^d \int_u^v f(x, y, z)\ dzdydx$ integral over rectangular box
- $\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x, y)\ dzdydx$ type I region
- $\iiint_R f(r, \theta, z)\ \boxed{r}\ dzdrd\theta$ integral in cylindrical coordinates
- $\iiint_R f(\rho, \theta, \phi)\ \boxed{\rho^2 \sin(\phi)}\ d\rho d\phi d\theta$ integral in spherical coordinates
- $\int_a^b \int_c^d \int_u^v f(x, y, z)\ dzdydx = \int_u^v \int_c^d \int_a^b f(x, y, z)\ dxdydz$ Fubini
- $V = \iiint_E \boxed{1}\ dzdydx$ volume of solid E
- $M = \iiint_E \sigma(x, y, z)\ dxdydz$ mass of solid E with density σ

Line Integrals

- $F(x, y) = [P(x, y), Q(x, y)]^T$ vector field in the plane
- $F(x, y, z) = [P(x, y, z), Q(x, y, z), R(x, y, z)]^T$ vector field in space
- $\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r\,'(t)\ dt$ line integral
- $F(x, y) = \nabla f(x, y)$ gradient field = potential field = conservative field

Fundamental theorem of line integrals

- FTLI: $F(x, y) = \nabla f(x, y)$, $\int_a^b F(r(t)) \cdot r\,'(t))\ dt = f(r(b)) - f(r(a))$
- closed loop property $\int_C F\ dr = 0$, for all closed curves C
- always equivalent: closed loop property, path independence and gradient field
- mixed derivative test $\mathrm{curl}(F) \neq 0$ assures F is not a gradient field
- in simply connected regions: $\mathrm{curl}(F) = 0$ implies that field F is conservative
- Conservative field: can not be used for perpetual motion.

Green's Theorem

- $F(x, y) = [P, Q]^T$, curl in two dimensions: $\mathrm{curl}(F) = Q_x - P_y$
- Green's theorem: C boundary of R, then $\int_C F \cdot dr = \iint_R \mathrm{curl}(F)\ dxdy$
- Area computation: Take F with $\mathrm{curl}(F) = Q_x - P_y = 1$ like $F = [-y, 0]^T$ or $F = [0, x]^T$
- Green's theorem is useful to compute difficult line integrals or difficult 2D integrals

Flux integrals

- $F(x, y, z)$ vector field, $S = r(R)$ parametrized surface
- $r_u \times r_v\ dudv = dS$ is a 2-form on surface
- $\int \int_S F \cdot dS = \int \int_S F(r(u, v)) \cdot (r_u \times r_v)\ dudv$ flux integral

Stokes Theorem

- $F(x, y, z) = [P, Q, R]^T$, $\mathrm{curl}([P, Q, R]^T) = [R_y - Q_z, P_z - R_x, Q_x - P_y]^T = \nabla \times F$
- Stokes's theorem: C boundary of surface S, then $\int_C F \cdot dr = \iint_S \mathrm{curl}(F) \cdot dS$
- Stokes theorem allows to compute difficult flux integrals or difficult line integrals

Grad Curl Div

- $\nabla = [\partial_x, \partial_y, \partial_z]^T$, $F = \nabla f$, $\mathrm{curl}(F) = \nabla \times F$, $\mathrm{div}(F) = \nabla \cdot F$
- $\mathrm{div}(\mathrm{curl}(F)) = 0$ and $\mathrm{curl}(\mathrm{grad}(f)) = 0$

☐ $\operatorname{div}(\operatorname{grad}(f)) = \Delta f$ Laplacian

☐   incompressible = divergence free field: $\operatorname{div}(F) = 0$ everywhere. Implies $F = \operatorname{curl}(H)$

☐   irrotational = $\operatorname{curl}(F) = 0$ everywhere. Implies $F = \operatorname{grad}(f)$

Divergence Theorem

☐ $\operatorname{div}([P, Q, R]^T) = P_x + Q_y + R_z = \nabla \cdot F$

☐   divergence theorem: solid E, boundary S then $\iint_S F \cdot dS = \iiint_E \operatorname{div}(F)\ dV$

☐   the divergence theorem allows to compute difficult flux integrals or difficult 3D integrals

Some topology

☐   simply connected region D: can deform any closed curve within D to a point

☐   interior of a region D: points in D for which small neighborhood is still in D

☐   boundary of curve C: the end points of the curve

☐   boundary of S points on surface not in the interior of the parameter domain

☐   boundary of solid G: points in G which are not in the interior of D

☐   closed surface: a surface without boundary like a sphere

☐   closed curve: a curve with no boundary like a knot

Some surface parameterizations

☐   sphere of radius ρ: $r(u, v) = [\rho \cos(u) \sin(v), \rho \sin(u) \sin(v), \rho \cos(v)]^T$

☐   graph of function $f(x, y)$: $= r(u, v) = [u, v, f(u, v)]^T$

☐   example: Paraboloid: $r(u, v) = [u, v, u^2 + v^2]^T$.

☐   plane containing P and vectors u, v: $r(s, t) = P + su + tv$

☐   surface of revolution: distance $g(z)$ of $z - axis$: $r(u, v) = [g(v) \cos(u), g(v) \sin(u), v]^T$

☐   example: Cylinder: $r(u, v) = [\cos(u), \sin(u), v]^T$

☐   example: Cone: $r(u, v) = [v \cos(u), v \sin(u), v]^T$

☐   example: Paraboloid: $r(u, v) = [\sqrt{v} \cos(u), \sqrt{v} \sin(u), v]^T$

Integration for integral theorems

☐   Double and triple integral: $\iint_G f(x, y) dA, \iiint_G f(x, y, z) dV$.

☐   Line integral: $\int_a^b F(r(t)) \cdot r\,'(t)\ dt$

☐   Flux integral: $\int \int_S F(r(u, v)) \cdot (r_u \times r_v)\ dudv$

Differential forms

☐   A **tensor** of type (p, q) is a multi-linear map $(E^*)^p \times E^q \to \mathbb{R}$.

☐   A **k-form** is a field, which attaches at every point a multi-linear anti-symmetric map of k variables.

☐ $F = 5x^3 dydz + 7 \sin(y) x dxdz + 3 \cos(xy) dxdy$ is an example of a 2-form. In calculus this is identified with a vector field $F = [5x^3, 7 \sin(y) x, 3 \cos(xy)]$.

☐   The exterior derivative of a term like $F = Pdxdy$ is $dF = (P_x dx + P_y dy + P_z dz) dxdy = P_z dzdxdy = P_z dxdydz$.

☐   The **General Stokes theorem** tells $\int_G dF = \int_{dG} F$, where dG is the **boundary** of G.

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