Lecture 2: Arithmetic

The first mathematical steps were done by **Babylonian**, **Egyptian**, **Chinese**, **Indian** and **Greek** thinkers. The oldest mathematical discipline is **Arithmetic**, the theory of manipulating numbers. Everything starts with the class of **natural numbers** $1, 2, 3, 4...$ where one can add and multiply. While addition is natural like when adding 3 sticks to 5 sticks to get 8 sticks, the multiplicative operation $*$ is more subtle: $3*4$ can be read that we take 3 copies of 4 and get $4+4+4 = 12$. And $4*3$ means we take 4 copies of 3 to get $3+3+3+3 = 12$. The first number counts operations while the second counts objects. To see that $3*4 = 4*3$, spacial insight is needed: we can arrange the 12 objects in a rectangle. Realizing an addition and multiplication structure on the natural numbers is a **great moment** in mathematics. It leads naturally to more general numbers. There are two major motivations to **extend a given number system**: we want to

1. **perform or invert operations** and get results. 2. **solve equations**.

For example, in order to solve $x+3 = 1$ one needs integers, to solve $3x = 4$ one needs fractions, to solve $x^2 = 2$ one needs real numbers, to solve $x^2 = -2$ one needs complex numbers.

Numbers	Operation to complete	Examples of equations to solve
Natural numbers	addition and multiplication	$5+x = 9$
Positive fractions	addition and division	$5x = 8$
Integers	also subtraction	$5+x = 3$
Rational numbers	also division	$3x = 5$
Algebraic numbers	taking positive roots	$x^2 = 2$, $2x+x^2-x^3 = 2$
Real numbers	taking limits	$x = 1-1/3+1/5-+...$,$\cos(x) = x$
Complex numbers	take any roots	$x^2 = -2$
Surreal numbers	transfinite limits	$x^2 = \omega$, $1/x = \omega$
Surreal complex	any operation	$x^2+1 = -\omega$

The development and history of arithmetic follows this principle: humans started with natural numbers, dealt with positive fractions, reluctantly introduced negative numbers and zero to get integers, struggled to "realize" real numbers, were scared to introduce complex numbers, hardly accepted surreal numbers and most do not even know about surreal complex numbers. Ironically, as simple but impossibly difficult questions in number theory show, the modern point of view is the opposite to Kronecker's **"God made the integers; all else is the work of man"**:

The **surreal complex** numbers are the most **natural** numbers;
The **natural** numbers are the most **complex, surreal** numbers.

Natural numbers. Counting can be realized by sticks, bones, knots on a string or pebbles. The **tally stick** concept is still used today when playing card games: where bundles of fives are formed, maybe by crossing 4 "sticks" with a fifth. An old stone age tally stick, the **wolf radius bone** contains 55 notches, with 5 groups of 5. It is probably more than 30'000 years old. An other famous paleolithic tally stick is the **Ishango bone**, the fibula of a baboon. It could be 20'000 - 30'000 years old. Eves dates it to 9000-6500 BC. Earlier counting could have been done by assembling **pebbles**, tying **knots** in a string, making **scratches** in dirt, but no such traces have survived the thousands of years. The **Roman system** improved the tally stick concept by introducing new symbols for larger numbers like $V = 5, X = 10, L = 40, C = 100, D = 500, M = 1000$. in order to avoid bundling too many single sticks. The system is unfit for computations as simple calculations $VIII + VII = XV$ show. **Clay tablets**, some as early as 2000 BC and others from 600 - 300 BC are known. They feature **Akkadian arithmetic** using the base 60. Advantages of 60 over 10 are the presence of more factors. It still survives in time division like 60 minutes per hour. **The Egyptians** used the base 10. The most important source on Egyptian mathematics is the **Rhind Papyrus** of 1650 BC. Hieratic numerals were used to write on papyrus from 2500 BC on. **Egyptian numerals** are hieroglyphics. They were found in carvings on tombs and monuments and are 5000 years old. The modern way to write numbers like the number 2010 is the **Hindu-Arab system** which diffused to the West only during the late Middle ages. It replaced the more primitive **Roman system**. Greek arithmetic used a primitive number system with no place values: 9 Greek letters for $1, 2, \dots 9$, nine for $10, 20, \dots, 90$ and nine for $100, 200, \dots, 900$.

Integers. **Indian Mathematics** morphed the place-value system into a modern method of writing numbers. Hindu astronomers used words to represent digits, but the numbers would be written in the opposite order. Sometimes after 500, the Hindus changed to a digital notation which included the symbol 0. Negative numbers were introduced around 100 BC in the **Chinese** text "Nine Chapters on the Mathematica art". Also the **Bakshali manuscript**, written around 300 AD subtracts numbers carried out additions with negative numbers, where + was used to indicate a negative sign. In Europe, negative numbers were avoided until the 15'th century.

Fractions: **Babylonians** could handle fractions. The **Egyptians** also used fractions, but wrote every fraction a as a sum of fractions with unit numerator and distinct denominators, like $4/5 = 1/2 + 1/4 + 1/20$ or $5/6 = 1/2 + 1/3$. Because of this cumbersome computation, Egyptian mathematics failed to progress beyond a primitive stage. The modern decimal fractions used nowadays for numerical calculations were adopted in Europe only in 1595.

Real numbers: It was the Greeks who realized first that some naturally occurring lengths are irrational: the insight that the diagonal of the square is not a rational number produced a crisis. Only much later, it became clear that "most" numbers are not rational. **Georg Cantor** realized first that the cardinality of all real numbers is much larger than the cardinality of the integers: while one can enumerate all integers and rational numbers, One can not enumerate the real numbers. One consequence is that most real numbers are transcendental: most numbers do not occur as solutions of polynomial equations with integer coefficients. The number π is an example. The concept of real numbers is closely related to the **concept of limit**. Sums like $1 + 1/4 + 1/9 + 1/16 + 1/25 + ...$ approach real numbers which are not rational any more.

Complex numbers: Not every polynomial equation has a real solution. To solve $x^2 = -1$ for example, we need new numbers. One idea is to use pairs of numbers (a, b) where $(a, 0) = a$ are the usual numbers and extend addition and multiplication $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$. With this multiplication, the number $(0, 1)$ has the property that $(0, 1) \cdot (0, 1) = (-1, 0) = -1$. It is more convenient to write $a + ib$ where $i = (0, 1)$ satisfies $i^2 = -1$. One can now use the common rules of addition and multiplication.

Surreal numbers: First introduce the Cantor infinite number ω, the smallest number greater than all finite counting numbers. Similarly as real numbers fill in the gaps between the integers, the surreal numbers fill in the gaps between Cantors ordinal numbers. They are written as $\{a, b, c, ...|d, e, f, ...\}$ meaning that the "simplest" number is larger than $a, b, c...$ and smaller than $d, e, f, ...$ We have $\{|\} = 0, \{0|\} = 1, \{1|\} = 2$ and $\{0|1\} = 1/2$ or $\{|0\} = -1$. Surreals were introduced in the 1970'ies by John Conway. This is no accident and confirms a major pedagogical principle: **late human discovery manifests in increased difficulty to teach it**.