Lecture 2: Remarks

Here are a few remarks and references.

- I myself got acquainted to natural numbers by "Cuisinaire sticks", an educational tool still used in Montesory schools today. I used them until the end of high school, when I worked on problems in additive number theory. My exposure to new number system pretty much followed the historical path. Early in college read Knuth's novel "Surreal" (in German "Insel der Zahlen") in which the name "surreal numbers" was introduced. I also bought Convay's book "Numbers and Games" (in German "Zahlen und Spiele"). I was fascinated by the results in that book: the transfinite numbers form a field which is somehow ultimate; every other ordered field is part of it. But I must admit that I did not have the energy nor time to study the subject in college. And I found the notation still not optimal. It is not so easy to compute with surreal numbers, even so Conway made it a "game".

- The focus on why $n \times m = m \times n$ is a true insight seems silly. Isn't it an assumption or axioms we make The question is why we can make this assumption is less obvious and it is related to our experience. Once the multiplicative structure on positive integers is fixed, the extension steps to larger and larger structures, integers, rational, real, complex, surreal is almost forced. But the start assumption is related to a symmetry we notice when realizing $n \times m$ and $m \times n$ in two different ways. It is related to the isotropy of space. Rotation by 90 degrees preserves the area of a rectangle. Commutativity is not so obvious any more in the microscopic and it seems that on a very small scale, space is noncommutative. Essentially all mathematics has been generalized to noncommutative structures and these structures are very much in line with quantum mechanics. Lets end this discussion with a quote of Landau. It can be found in the book of Hairer and Wanner "Analysis by its history". Landau said in 1930: *Please forget everything you have learned in school; for you haven't learned it. My daughters have been studying for several semesters already, think they have learned differential and integral calculus in school and even today don't know why $x * y = y * x$ is true.*

- The initial mystery that in the multiplication the first entry is behaving differently is in a modern point of view formulated that the Abelian group of integers is also Z module. The integers Z act by scalar multiplication as $nx = x + x + \ldots + x$. Even so the effect is the same as the multiplication $n \cdot x$ in the group one distinguishes between scalar multiplication and multiplication with n.

- Is there anything beyond complex numbers? The number system was extended to complex numbers by Gauss in Germany, to quaternions by Hamilton, which later led to dot and cross product and modern multivariable calculus. One can build a multiplication on quadruples or octuples of numbers, these form **quaternions** and **octonions**. But they do not satisfy anymore all laws we are familiar with. Quaternions are already no more commutative. Octonions are no more associative.

- The class of numbers which were illustrated in this lecture can be extended. There are other number systems which are important. Instead of the complex numbers, one can algebraically complete an other evaluation and get Q-adic numbers for example. Furthermore, more algebraic structures have been built, which generalize numbers. Groups, rings, fields, algebras, modules, vector spaces. But this is no more "Arithmetic", it is considered part of "Algebra".

- Here are some sources, I used: [9], [3],[4], [6], [5], [2], [7], [1].

- Were humans the first species which counted? It seems that some animals can count too: **ants** count steps to measure or rather estimate distance. It seems that **primates** can count too, but only after training, and then only up to 7. [8]

- The history of arithmetic teaches us an important lesson: **Notation** is crucial. The roman system for example was cumbersome to do computations with. The Babylonians had a better system and consequently produced better mathematics. Also the Greek system was not well suited for arithmetics. This explains that while geometry was on a sophisticated level, algebra got stuck. Imagine, the Greeks would have built the bridge from geometry to algebra already as Descartes did 1638. Eves book would have many more volumes.

- Also the Mayan number system dating back to 400 AC used a **vigesimal system** with base 20 (the number of fingers and toes).

- Conway's construction of surreal numbers is motivated by the concept of **Dedekind cut**, which is used to construct the set of real numbers. Like surreal numbers, a Dedekind cut can be written as a pair $\{A, B\}$. But now, A is a set of rational numbers and B is a set of rational numbers and $a < b$ for every $a \in A, b \in B$. Furthermore, there should be no rational number r with $r > a$ for every $a \in A$ and $r < b$ for every $b \in B$. For example $\sqrt{2} = \{A, B\}$, where $A = \{p/q \mid p^2/q^2 < 2\}$ and $B = \{p/q \mid p^2/q^2 \geq 2\}$.

- A modern and elegant way to introduce real numbers is to see them as the **completion** of the rational numbers. One has to be careful however not to use properties of the real line: when invoking a **real-valued** metric $d(x, y)$, one implicitly uses properties of the real line already. The concept of Cauchy sequences is part of calculus. Conway argues in [1] that it is not elegant to invoke advanced tools like Cauchy sequences to introduce something as basic as real numbers. I myself was taught a careful Dedekind cut construction in the first semester of my undergraduate studies.

- There are a few important transitions when constructing new numbers: the **closure of multiplication** leading to fractions, the **closure of addition** leading to negative numbers, **completion** with respect to some valuation which is related to taking **limits**, **algebraic completion** leading to algebraic or complex numbers and finally **transfinite limits**, leading to **hyper-reals** used in nonstandard analysis or surreal numbers. Nonstandard analysis is an elegant way but there are cultural and maybe pedagogical obstacles which prevents that it is more widely used.

Numbers diagram showing: Numbers, ALGEBRAIC COMPLETION, U, COMPLEX, C, LIMITS COMPLETION, SURREAL, S, FRACTIONS, R, REAL, "LARGEST POSSIBLE ORDERED FIELD", RATIONAL, Q, FRACTIONS, Q+, Q_p P-ADIC, H HYPER REAL, N NATURAL, Z INTEGERS, A ALGEBRAIC, NEGATIVE NUMBERS, TRANSFINITE LIMIT, Saturday, January 30, 2010

References

[1] John H Conway. *On Numbers and Games*. A K Peters, Ltd, 2001.

[2] John H. Conway and Richard K. Guy. *The book of numbers*. Copernicus, 1996.

[3] Howard Eves. *Great moments in mathematics (I and II*. The Dolciani Mathematical Expositions. Mathematical Association of America, Washington, D.C., 1981.

[4] Ivor Grattan-Guinness. *The Rainbow of Mathematics*. W. W. Norton, Company, 2000.

[5] Victor J. Katz, editor. *Using history to teach mathematics*, volume 51 of *MAA Notes*. Mathematical Association of America, Washington, DC, 2000. An international perspective.

[6] Morris Kline. *Mathematical Thought from Ancient to Modern Time*. Oxford University Press, 1972.

[7] D. E. Knuth. *Surreal numbers*. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1974.

[8] Clifford A. Pickover. *The Math book, From Pythagoras to the 57th dimension. 250 Milestones in the History of Mathematics*. Sterling, New York, 2009.

[9] Ernst Sondheimer and Alan Rogerson. *Numbers and infinity*. Dover Publications Inc., Mineola, NY, 2006. A historical account of mathematical concepts, Reprint of the 1981 original.