Lecture 6: Calculus

Calculus is the theory of **differentiation** and **integration**. For any function $f(x)$, one can get new functions by taking differences or sums. These procedures D_n and S_n often have limits.

The difference	$D_n f(x) = n(f(x+\frac{1}{n}) - f(x))$	becomes the	**derivative**	$f'(x)$.
The sum	$S_n f(x) = \frac{1}{n}\sum_{0 \leq \frac{k}{n} < x} f(\frac{k}{n})$	becomes the	**integral**	$\int_0^x f(t)\ dt$.

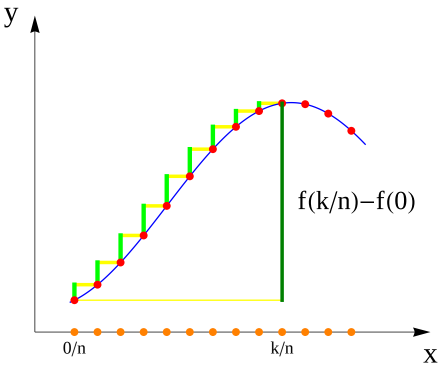
The two limits, provided they exist, have a geometric interpretation:

$D_n f$	means	rise over run	and becomes	the	slope of the	graph of f.
$S_n f$	means	areas of rectangles	and becomes	the	area under the	graph of f.

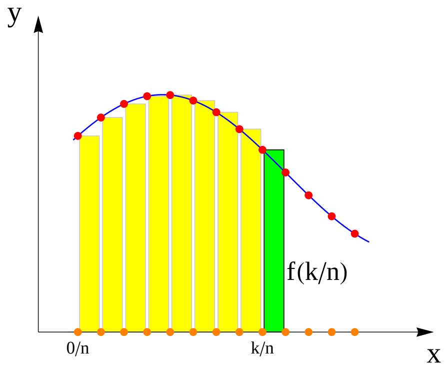
Differentiation and integration are linked by the **fundamental theorem of calculus**:

$S_n D_n f(\frac{k}{n}) = f(\frac{k}{n}) - f(0)$	becomes in the limit	$\int_0^x f'(t)\ dt = f(x) - f(0)$
$D_n S_n f(\frac{k}{n}) = f(\frac{k}{n})$	becomes in the limit	$(\int_0^x f(t)\ dt)' = f(x)$

While the left hand side holds for **all functions** f and all n. the upper right hand side works only for "differentiable functions" and the lower right hand side allows for continuous functions to compute the **anti-derivative** $F(x) = \int_0^x f(x)\ dx$. The two discrete "=quantum" formulas for $D_n S_n$ and S_n, D_n embrace the core of calculus.



Sum the differences and get
$S_n D_n f(\frac{k}{n}) = f(\frac{k}{n}) - f(0)$



Difference the sum and get
$D_n S_n f(\frac{k}{n}) = f(\frac{k}{n})$

Examples. 1) $f(x) = x^k$. With $h = 1/n$, we have $D_n(f) = [(x+h)^k - x^k]/h = (khx^{k-1} + h^2 C(x))/h = kx^{k-1} + hC(x)$, where $C(x)$ stays bounded for $h \to 0$. The expression becomes in the limit $h \to 0$ the function $f'(x) = kx^{k-1}$. From the fundamental theorem of calculus, we see that the anti-derivative of $f(x) = x^k$ is $x^{k+1}/(k+1)$. The linearity of D_n, S_n immediately establishes differentiation and integration formulas for **polynomials**.

2) Define the polynomial $\mathbf{exp}_n(x) = (1 + x/n)^n$. Its derivative is $(1+x/n)^{n-1} = \exp_n(x)/(1 + x/n)$. In the limit $n \to \infty$ we get a function $\exp(x)$ which has the property that $\exp'(x) = \exp(x)$. It is called the **exponential function**. We have $\exp_n(x)\exp_n(y) = \exp_n(x+y)$ plus a term of the order $1/n$. In the limit, $\exp(x)\exp(y) = \exp(x+y)$ relates multiplication and addition.

3) The function $\log_n(y) = n(y^{1/n} - 1)$ for positive y is the inverse of $\exp_n(x)$. We have $\log_n'(x) = 1/x^{(1-1/n)}$. In the limit, it defines the **logarithmic function** $\log(x)$. It is the inverse of $\exp(x)$ and has the derivative $1/x$. We also know $\log(xy) = \log(x)\log(y)$ for positive x, y.

We can also define a^b for positive a with $a^b = e^{b\log(a)}$. We have $a^b * a^c = a^{b+c}$ so that for example $a^{1/2} = \sqrt{a}$.

4) The polynomials $\sin_n(x) = \mathrm{Re}\exp_n(ix)$ and $\cos_n(x) = \mathrm{Im}\exp_n(ix)$ have the property that $\exp_n(ix) = \cos_n(x) + i\sin_n(x)$. We have $\sin_n'(x) = \cos_n(x)/(1+x/n)$ and $\cos_n'(x) = -\sin_n(x)/(1 + x/n)$ so that in the limit the **trigonometric functions** satisfy $\sin'(x) = \cos(x), \cos'(x) = -\sin(x)$ and $\exp(ix) = \cos(ix) + i\sin(ix)$. Since the polynomial $|\exp_n(ix)|^2 = 1 + x^2/n + 3x^4/n^3 + ...$ (which ends with nx^{n+1}/n^n if n is odd) converges to 1 for $n \to \infty$, we know that $\cos(x) + i\sin(x)$ is on the unit circle. The multiplicative property of exp implies that the curve $(\cos(t), \sin(t))$ moves on the unit circle with constant speed. To find T such that $(\cos(T), \sin(T)) = (1, 0)$ for the first time, we note that $\exp_n(0) = 1, \exp_n'(0) = 1$ for all n. The speed is initially 1 and the curve returns to $(1, 0)$ at $t = 2\pi$, the length of the unit circle. $\cos(t)$ and $\sin(t)$ can be seen as the ratios between lengths of the sides in a right triangle. Formulas like $\cos(2x) = \cos(x)^2 - \sin(x)^2, \sin(2x) = 2\cos(x)\sin(x)$ follow by comparing real and imaginary part of $\cos(2x) + i\sin(2x) = \exp(2ix) = (\cos(x) + i\sin(x))^2$.

There are two important **series** in calculus: first, the **geometric series**

$$S = 1 + a + a^2 + a^3 + ...$$

Because $1 = S - aS$ we have $S = 1/(1-a)$. By differentiating $1/(1-a) = 1 + a + a^2 + a^3 + ...$ one can sum other series.

The second is the **zeta series** which is the **zeta function** as a function of s:

$$S = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + ...$$

The series has positive terms and is smaller than $\int_1^\infty x^{-s}\ ds = \frac{x^{-s+1}}{s+1}|_1^\infty = \frac{1}{1-s}$ so that it converges for $s > 1$. It diverges for $s = 1$, where it is the **harmonic series**.

Calculus was invented to **model nature**. Important tools are **differential equations**. The two major examples are the **exponential growth or decay equation** $f' = af$ and the **harmonic oscillator** $f'' = -c^2 f$. The first equation has the solution $f(t) = f(0)e^{at}$ and models **exponential growth** for $a > 0$ or **exponential decay** for negative a. The harmonic oscillator has the solution $f(t) = f(0)\cos(ct) + f'(0)/c\sin(ct)$. It is used to model **waves**.
Continuous functions play an important role in calculus. For them, one has the **intermediate value theorem**: for any c between $f(a)$ and $f(b)$, there exists $a < x < b$ with $f(x) = c$.
Extremization is an other large theme: to minimize a function f, look for places where $f'(x) = 0$. The second derivative test $f'' > 0$ assures to have a minimum there.
Series representations of functions are given by **Taylors formula** $f(x) = \sum_{k=0}^\infty D^k f(0)x^n/n! = \exp(Dx)f(x)$. Our introduction of $\exp, \sin, \cos$ above essentially defines the functions as such.

Calculus was created by Newton and Leibniz in the 17'th century. Multi-variable calculus generalizes the fundamental theorem of calculus with the formula $\int_G F'dx = \int_{G'} F dx$, where G' is the boundary of an object G like a surface and F is the derivative of an object F like a vector field. Development of calculus continues until today. My exposition here is influenced by **quantum calculus**, calculus without limits. Other newer developments are **nonstandard calculus** or **non-commutative flavors** of calculus. Derivatives and integrals can work in more generality.