Lecture 6: Calculus

About the exposition of Calculus

In lecture, I have tried to summarize (and derive!) the most important points of calculus. The handout does this on two pages. I have tried in class to present the core of calculus in a quarter of an hour and we have seen Ed Burger presenting calculus in 20 minutes (18 and 1/2 minutes actually).

My efforts to reduce things to the core is a reaction to an "focus free" attitude towards calculus which manifests in the **insane inflation** of text book sizes. **Eli Maor** expresses it well in his book "The facts on file: Calculus Handbook, 2003":

"Over the past 25 years or so, the typical college calculus textbook has grown from a modest 350-page book to a huge volume of some 1,200 pages, with thousands of exercises, special topics, interviews with career mathematicians, 10 or more appendixes, and much, much more. But as the old adage goes, more is not always better. The enormous size and sheer volume of these monsters (not to mention their weight!) have made their use a daunting task. Both student and instructor are lost in a sea of information, not knowing which material is important and which can be skipped. As if the study of calculus is not a challenge already, these huge texts make the task even more difficult."

I personally find that there truth in that. Textbooks like **Snyder's Elementary Textbook on the Calculus** was 200 pages long. Also Maor's book is a short dictionary or glossary and does not substitute a textbook. But it can help to focus.

The text book issue will change anyway. The scandalous prizes of textbooks which I can only explained by **prize fixing** will change fast. The reason is the appearance of electronic textbook readers which forces publishers to become reasonable with prizes. If buying two textbooks costs more than a new computer, things are out of proportion. If this is not going to change, it will force students to get electronic versions of a text book (I myself can get a 1200 page book into electronic OCR'ed PDF form with a time effort of maybe half an hour. Most of my time for that is spent with cutting off the spine of the book and afterwards flipping through the pages to check after scanning to see that all pages are there. OCR and PDF optimization which can take some time is done by the computer in the background.

The fundamental theorem of calculus

In the handout, I tried to give a **self contained exposition of trigonometric functions** that does not make use of anything we know before. If you read it you will find that it contains a proof of Euler's formula $e^{ix} = \cos(x) + i\sin(x)$ without Taylor series. In reality, one can compute well with the polynomial exponentials and their inverses. In order to verify that $\exp_n(x)\exp_n(y) = \exp_n(x+y)$ plus something small, we verify

$$((1 + x/n)^n * (1 + y/n)^n)/(1 + (x+y)/n)^n = (1 + xy/(n(n+x+y)))^n \sim \exp_n(xy/n) .$$

The two page exposition contained in that handout is the heart of calculus. Without taking limits, it is simple algebra and before taking limits, there are **absolutely no restrictions** on functions. It is the concept of **limit** which gives students the impression that the subject is hard. Here again, history leads the way and shows that the struggle to understand limits is one of the hardest part of calculus. While it is reasonable to ask whether it is worth the trouble, and almost everybody I have talked to in the last couple of years about the importance of limit consider the question **heretic**. But developments like nonstandard calculus 30 years ago or quantum calculus 10 years ago and entire business calculus sequences not mentioning limits with one word have shown that it is possible to avoid limits. While I still think that limits have to be discussed, the question is valid, whether one has to read 300 pages before coming to the essentials of calculus.

About Series

Henri Poincaré mentions in his "New Methods of Celestial Mechanics" the two series $S_n = \sum_n 1000^n/n!$ and $S_n = \sum_n n!/1000^n$. In this lecture, we had a worksheet on this. We know that the first series has a limit and the second not. An astronomer however would rule the first to be divergent and the second to be convergent. Also the harmonic series $\sum_{n=1}^{\infty} 1/n$ which is known to diverge has a limit when we do experiments (it would take times of the age of the universe to reach the value 50 even if we could do trillions of additions every second). The fact that the concept of limits needed virtually 1000 years to develop is no accident. The struggle started with the Zeno paradoxa and continues until today.

Divergent series like $\sum_{n=1}^{\infty} 1/n^s$ for negative s are more important today than convergent series. Both in number theory, where zeta functions are central as well as in physics, where the description of nature uses divergent series in many ways. Series like $1 + 3 + 9 + 27 + 81 + ...$ make sense if we note that it is a geometric series $1 + a + a^2 + a^3 + ...$ with limit $1/(1-a)$ which becomes $-1/2$ for $a = 3$. Again, also here, the concept of limit is bypassed on a fundamental level and that the limit of $1 + 3 + 9 + 27 + 81 + ...$ does not exist is irrelevant. There is much cultural baggage which makes us believe that smoothness is nice, similarly as astronomers before Kepler thought the circle is nice and modeled everything with epicycles.

Are limits at all necessary? As calculus sequences for business students has shown, it can be done without doing big damage. Even nature tries to avoid limits. Our calculus for smooth functions is an idealization which was originally invented to make classical mechanics work. In reality, on the microscopic and fundamental level, nature behaves differently and "calculus without limits" works well. We still do not know how space looks like on a very small scale but we have indications like the Planck constant $\hbar$ telling us that the continuum is an idealization and simplification. Because we like smooth objects, we have to deal with limits. If we relax the class of functions, mathematics becomes easier. Quantum mechanics deals with Hilbert spaces of functions in which smooth functions form a tiny slice only. Nature likes simplicity and quantum mechanics with a linear evolution is on a fundamental level much easier than classical nonlinear mechanics with chaotic motion. This is a well known principle in analysis: solutions to equations are often modeled better with non smooth functions. Basic properties of primes are studied with zeta functions $\sum_n 1/n^s$ for values of s, where the limits do not exist.

Which Series are important?

All important series in calculus are **Dirichlet series** of the form

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \ .$$

For $\lambda_n = \log(n)$ this is an ordinary Dirichlet series, while for $\lambda_n = n$, it is a Taylor series $\sum_n a_n z^n$ with $z = e^{-s}$. Calculus deals mostly with the case, when the coefficients a_n are constant. For Taylor series, one gets the Geometric series. For ordinary series one obtains the so called β-series.

It sounds heretic to many but the convergence of series is actually no so relevant as it appears to the beginner. Most interesting mathematics happens with divergent series. This insight is probably 100 years old, but it was only appreciated about 60 years ago with Hardy's book "divergent series". Mathematicians analyze concrete objects like prime numbers with divergent series, physicists unlock the secrets of the early universe with divergent series and topologists study surfaces with divergent series. This is all rigorous mathematics of course and one has sometimes to work a bit to prove that the series can be made sense also at places where the original series does not exist. Real numbers have appeared to be strange to the Pythagoreans, imaginary numbers to a mathematician in the 16'th century, divergent series like $1 + 4 + 9 + 16 + 25 + ...$ can be made sense of like one can understand $1 + 2 + 4 + 8 + 16 + 32 + ... = 1/(1-2) = -1/2$ with the formula $S = 1/(1-a)$. Again, one has to work sometimes hard to understand the values of divergent series. The zeta function $\sum_n 1/n^s$ for example is not understood on the axes $re(s) = 1/2$. Are all zeros there? This is the famous **Riemann hypothesis**.

The divergence of the harmonic series has first been established by Nioclae Oresme, a French scholar who lived in the 14'th century. The name "harmonic" is due to connections of members of the series and the intervals of the musical scale: 1,3/2, 1+5/6,2+1/12. Source: Eli Maor, To infinity and Beyond, page 26).

Differential equations

The differential equations mentioned on the handout are the most important ones. **Linear algebra** reduces all linear equations to them. Fourier theory reduces many partial differential equations to them.

One important nonlinear amendment to the exponential growth equation is the logistic model, where $f' = af(1 - f/M)$.

Differential equations are a natural way **introduce** basic functions like exponentials or trigonometric functions. Note that the **difference equation** $D_n f = af$ means $f(x + 1/n) = f(x) + D_n f(x)/n = f(x)(1 + a/n)$ so that $f(1) = f(0)\exp_n(a)$.