Lecture 12: Dynamical systems

Dynamical systems theory studies time evolution of systems. If time is **continuous** the evolution is defined by a **differential equation** $\dot{x} = f(x)$. If time is **discrete** we look at the **iteration of a map** $x \to T(x)$. The goal is to **predict the future** of the system when the present state is known. A **differential equation** is an equation of the form $d/dtx(t) = f(x(t))$, where the unknown quantity is a path $x(t)$ in some phase space. We know the **velocity** $d/dtx(t) = \dot{x}(t)$ at all times and the initial configuration $x(0))$, we can to compute the **trajectory** $x(t)$. What happens at a future time? Does $x(t)$ stay in a bounded region or escape to infinity? Which areas of the phase space are visited and how often? Are there periodic, almost periodic or probabilistic subsystems? Some simple dynamical systems in one dimension:

$x'(t) = 3x(t), x(0) = 2$	Solution: $x(t) = 2e^{3t}$	exponential growth
$x'(t) = -2x(t), x(0) = 4$	Solution: $x(t) = 4e^{-2t}$	exponential decay
$x'(t) = x(t)(2 - x(t)), x(0) = 1$	Solution: $x(t) = 2e^t/(1 + e^{2t})$	logistic equation

A **map** is a rule which assigns to a quantity $x(t)$ a new quantity $x(t + 1) = T(x(t))$. The state $x(t)$ of the system determines the situation $x(t + 1)$ at time $t + 1$. Examples:

$T(x) = x + 1, x(0) = 2$	Solution: $x(t) = 2 + t$	Addition map
$T(x) = 2x, x(0) = 3$	Solution: $x(t) = 3 * 2^t$	Scaling map
$T(x) = 2x + 1, x(0) = 0$	Solution: $x(t) = 2^t - 1$	Affine map

These extremely simple examples suggest that one always can give a **formula** for the n'th term and so predict the future trajectory well. We will see in class that for examples which look almost as simple like the **Ulam map** $T(x) = 4x(1 - x)$ on the interval $[0, 1]$, we have absolutely no idea what happens after a few hundred iterates even if we would know the initial position with the accuracy of the Planck scale and even if we would increase the accuracy and assume the Planck scale is our universe and know it to the Planck scale of that mini universe.

Famous maps in one dimensions:

$T(x) = cx(1 - x)$	**Logistic map**
$T(x) = x + \alpha \bmod 1$	**Rotation**
$T(x) = 2x \bmod 1$	**Doubling map**

Celebrities in two dimensions:

$T(x, y) = (x^2 + c - y, x)$	**Henon map**
$T(x, y) = (c\sin(x) + 2x - y, x)$	**Standard map**
$T(x, y) = (2x + y, x + y)$	**Cat map**

Dynamical system theory is interdisciplinary: essentially all fields of mathematics use it:

Systems of arithmetic nature

$T(x) = x - f(x)/f'(x)$	**Newton steps**
$T(x, y) = (\frac{2xy}{x+y}, \frac{x+y}{2})$	**Averaging**

Systems of number theoretical nature:

$T(x) = \frac{x}{2}$ (even x), $3x + 1$ else	**Collatz map**
$T(x) = \sigma(n)$ divisors	**Amicable map**

Systems of geometric nature:

Bounce ball in region	**Billiard map**
Pedal triangle	**Pedal map**

Systems of topological nature

$\dot{x} = -\nabla f(x)$	**Gradient flow**
$\dot{x} = \vec{n}(x)\kappa(x)$	**Curvature flow**

Systems of analytical nature

$T(Y) = f(Y) \cup g(Y)$	**Chaos Game**
$\dot{L} = [L^+ - L^-, L]$	**Isospectral flow**

Systems of probabilistic nature

$T(\omega, x) = (\sigma\omega, A(\omega)x)$	**Markov chain**
$T(\omega, x) = (\sigma\omega, x + X(\omega))$	**Random walk**

About 100 years ago, **Henry Poincaré** was able to deal with **chaos** of low dimensional systems. While **statistical mechanics** had formalized the evolution of large systems with probabilistic methods already, the new insight was that simple systems like a **three body problem** or a **billiard map** can produce very complicated motion. It was Poincaré who saw that even for such low dimensional and completely deterministic systems, random motion can emerge. While physisists have dealt with chaos earlier by assuming it or artificially feeding it into equations like the **Boltzmann equation**, the occurrence of stochastic motion in geodesic flows or billiards or restricted three body problems was a surprise. These findings needed half a century to sink in and only with the emergence of computers in the 1960ies, the awakening happened. Icons like Lorentz helped to popularize the findings and we owe them the **"butterfly effect"** picture: a wing of a butterfly can produce a tornado in Texas in a few weeks. The reason for this statement is that the complicated equations to simulate the weather reduce under extreme simplifications and truncations to a simple differential equation $\dot{x} = \sigma(y - x), \dot{y} = rx - y - xz, \dot{z} = xy - bz$, the **Lorenz system**. For $\sigma = 10, r = 28, b = 8/3$, Ed Lorenz discovered in 1963 an interesting long time behavior and an aperiodic "attractor". Ruelle-Takens called it a **strange attractor**. It is a **great moment** in mathematics to realize that attractors of simple systems can become fractals on which the motion is chaotic. It suggests that such behavior is abundant. What is chaos? If a dynamical system shows **sensitive dependence on initial conditions**, we talk about **chaos**. The following experiment illustrates this. Define two maps $T(x) = 4x(1-x)$ and $S(x) = 4x-4x^2$. Start in both cases with 0.3 and iterate the map. After a few dozen iterations, the two orbits are no more the same. How come? Is not $T(x) = S(x)$?

The sensitive dependence on initial conditions is measured by how fast the derivative dT^n of the n'th iterate grows. The exponential growth rate γ is called the **Lyapunov exponent**. A small error of the size h will be amplified to $he^{\gamma n}$ after n iterates. In the case of the Logistic map with $c = 4$, the Lyapunov exponent is $\log(2)$ and an error of 10^{-16} is amplified to $2^n \cdot 10^{-16}$. For time $n = 53$ already the error is of the order 1. This explains the above experiment with the different maps. The maps $T(x)$ and $S(x)$ round differently on the level 10^{-16}. After 53 iterations, these initial fluctuation errors have grown to a macroscopic size.
Here is a famous open problem which has resisted many attempts to solve it: Show that the map $T(x, y) = (c\sin(2\pi x) + 2x - y, x)$ with $T^n(x, y) = (f_n(x, y), g_n(x, y))$ has sensitive dependence on initial conditions on a set of positive area. More precisely, verify that for $c > 2$ and all n $\frac{1}{n}\int_0^1\int_0^1 \log|\partial_x f_n(x, y)|\, dxdy \geq \log(\frac{c}{2})$. The left hand side converges to the average of the Lyapunov exponents which is in this case also the **entropy** of the map. For some systems, one can compute the entropy. The logistic map with $c = 4$ for example, which is also called the **Ulam map**, has entropy $\log(2)$. The cat map $T(x, y) = (2x + y, x + y) \bmod 1$ has entropy $\log|(\sqrt{5} + 3)/2|$. This is the logarithm of the larger eigenvalue of the matrix.

While questions about simple maps look artificial at first, the mechanisms prevail in other systems: in astronomy, when studying planetary motion or electrons in the van Allen belt, in mechanics when studying coupled penduli or nonlinear oscillators, in fluid dynamics when studying vortex motion or turbulence, in geometry, when studying the evolution of light on a surface, the change of weather or tsunamis in the ocean. Dynamical systems theory started historically with the problem to understand the **motion of planets**. Newton realized that this is governed by a differential equation, the **n-body problem** $x_j''(t) = \sum_{i=1}^n \frac{c_{ij}(x_i - x_j)}{|x_i - x_j|^3}$ where c_{ij} depends on the masses and the gravitational constant. If one body is the sun and no interaction of the planets is assumed and using the common center of gravity as the origin, this reduces to the **Kepler problem** $x''(t) = -Cx/|x|^3$, where planets move on **ellipses**, the radius vector sweeps equal area in each time and the period squared is proportional to the semi-major axes cubed. A great moment in astronomy was when Kepler derived these laws empirically. An other great moment in mathematics is Newton's theoretically derivation from the differential equations.