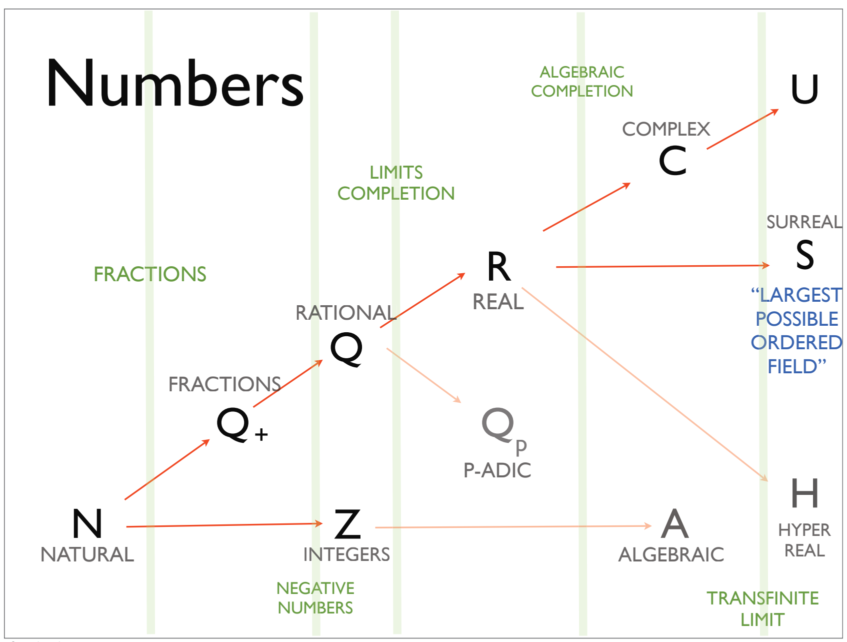
Lecture 2: Arithmetic

Plan

In this second lecture, we learn about the structure of numbers. Here is a plan of the lecture:

1. Presentation: how the system of numbers is built.
2. Presentation: historical development.
3. Worksheet: working with historical number systems. Building a Clay tablet.
4. Worksheet: the crisis of the square root of two.

The structure of numbers



Could we build an other number system?

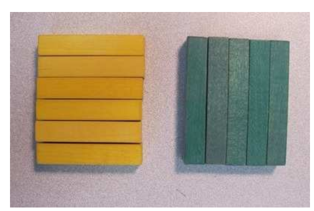
Once we start with the natural numbers and set rules for then, the extension of these rules to larger number system is pretty much determined. The constructions shown in my slide on numbers are all natural and forced when asked to solve equations. There is one step at the beginning which is rather arbitrary: why is multiplication commutative for natural numbers. The answer is, that there is no particular reason. We can do otherwise too. There are "number systems" like with matrices, in which the multiplication is not commutative any more. Nature has chosen this arithmetic in the quantum world. It only appears strange because we do not experience these rules in our daily lives. For addition, the question of commutativity is less disputed. A quote of **Vladimir Arnold** who just died recently:

> "To the question what is "2 + 3" a primary school pupil replied 3+2, since addition is commutative. He did not know what the sum was equal to and could not even understand what he was asked about."

This contemplation on the question why $n \times m = m \times n$ is true seems silly at first. Isn't it an assumption or an axiom? The question why we have this axiom is less obvious. It relies on our spacial experience. Once the multiplicative structure on positive integers is fixed, the extension steps to larger and larger structures, integers, rational, real, complex, surreal is almost forced. But the initial assumption is related to a spacial symmetry. We notice it when realizing the two expressions $n \times m$ and $m \times n$ geometrically in two different ways. Taking n sticks of length m and taking m sticks of length n produce the same area if we assume isotropy of space. Rotation by 90 degrees preserves the area of a rectangle.

Commutativity is not in the microscopic and on a very small scale, space is noncommutative. Fortunately, all essentially mathematics (from algebra to measure theory over topology and differential geometry) has been generalized to noncommutative structures and these structures are very much in line with quantum mechanics. Lets end this discussion with a quote of Landau. It can be found in the book of Hairer and Wanner "Analysis by its history". Landau said in 1930:

> *Please forget everything you have learned in school; for you haven't learned it. My daughters have been studying for several semesters already, think they have learned differential and integral calculus in school and even today don't know why $x * y = y * x$ is true.*



The picture shows my Cuisenaire sticks from my first grade Mathematics class. It illustrates a Cuisenaire proof that $5 \times 6 = 6 \times 5$. It uses the spacial insight that both sides are the same area.

There is absolutely **no algebraic reason**, why we should assume commutativity for multiplication. Indeed, there are independent mathematical constructions, where $p \times q \neq q \times p$ in general. Our insight to assume commutativity relied on spacial symmetry. In other spaces like position-momentum planes in mechanics, the relation $pq = qp$ is **false** in general and the **commutation relation** $p \times q - q \times p = h$ replaces it. While strange at first, it turns out that physics based on non-commutative arithmetic can be much more elegant than our classical physics. To speak with Leibniz, "we live in the best of all possible worlds" and non-commutative structures turn out to have a lot of advantage. Stability of atoms and stars relies on such physics. In a commutative world, we could not exist.