

Lecture 4: Number Theory

In this lecture, I look first at some open problems in number theory then work on two nice theorems in prime numbers, Fermat's theorem and Wilson's theorem. **Fermat's theorem** tells that $a^p - p$ is divisible by p if p is prime and **Wilson's theorem** tells that $(p - 1)! + 1$ is divisible by p if and only if p is prime.

Open problems in mathematics are the fuel for doing mathematics. Fortunately, open problems never will stop to appear, they are the heads of a **hydra**. After solving one, 5 new ones appear.

Well, here are 5 extremely famous problems.

Twin prime conjecture



There are infinitely many prime twins $p, p + 2$.

The largest known twin primes $p, p + 2$ are given by $p = 2003663613 \cdot 2^{2195000} - 1$, a number with almost 60'000 digits. It has been found in 2007. There are analogue problems for cousin primes $p, p + 4$ or sexy primes $p, p + 6$ or Sophie Germaine primes, where $p, 2p + 1$ are prime.

Goldbach conjecture



Every even integer $n > 2$ is a sum of two primes.

The Goldbach conjecture has been numerically verified until $1.6 \cdot 10^{18}$. Mathematically it is known that every sufficiently large odd number is the sum of 3 primes. One believes this "weak Goldbach conjecture" for 3 primes is true for every odd integer larger than 7.

Andrica conjecture



The prime gap estimate $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ holds.

There are other prime gap estimate conjectures like Polignac's conjecture claiming that there are infinitely many prime gaps for every even number n . It is stronger than the twin prime conjecture.

Legendre's conjecture claims that there exists a prime between any two perfect squares.

Odd perfect numbers



Probably the oldest problems in mathematics is the question

There is an odd perfect number.

A perfect number is equal to the sum of all its proper positive divisors. Like $6 = 1 + 2 + 3$. The search for perfect numbers is related to the search of large prime numbers. The largest prime number known today is $p = 2^{43112609} - 1$. It is called a Mersenne prime. Every even perfect number is of the form $2^{n-1}(2^n - 1)$ where $2^n - 1$ is prime.

Diophantine equations



Many problems about Diophantine equations are unsettled, equations with integer solutions. Here is an example:

Solve $x^5 + y^5 + z^5 = w^5$ for $x, y, z, w \in \mathbb{N}$.

Also $x^5 + y^5 = u^5 + v^5$ has no nontrivial solutions yet. Probabilistic considerations suggest that there are no solutions. The analogue equation $x^4 + y^4 + z^4 = w^4$ had been settled by Noam Elkies in 1988 who found $2682440^4 + 15365639^4 + 18796760^4 = 20615673^4$.