

Lecture 6: Calculus

This semester, I experimented with a different approach to calculus. Lets see what you think about it. Differences measure **change**, sums measure how things **accumulate**. The process of taking differences is in a limit called **derivative**. The process of taking sums is in the limit called **integral**. These two processes are related in an intimate way. In this lecture, we want to look at these two processes in a discrete setup where functions are evaluated on integers.

Start with the sequence of integers

$$1, 2, 3, 4, \dots$$

We say $f(1) = 1, f(2) = 2, f(3) = 3$ etc and call f a **function**. It assigns to a number a number. It assigns for example to the number 100 the result $f(100) = 100$. Now we add these numbers up. The sum of the first n numbers is called

$$Sf(n) = f(1) + f(2) + f(3) + \dots + f(n).$$

In our case we get

$$1, 3, 6, 10, 15, \dots$$

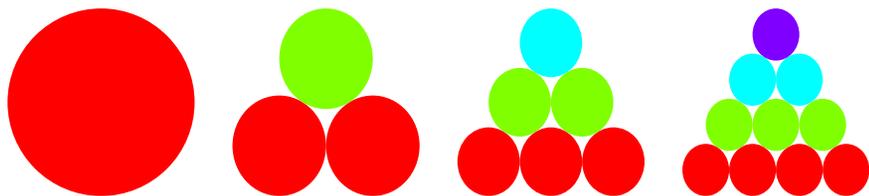
It defines a new function g which satisfies $g(1) = 1, g(2) = 3, g(3) = 6$ etc. The new numbers are known as the **triangular numbers**. From the function g we can get f back by taking difference:

$$Dg(n) = g(n) - g(n-1) = f(n).$$

For example $Dg(5) = g(5) - g(4) = 15 - 10 = 5$ and this is indeed $f(5)$.

Karl-Friedrich Gauss realized as a 7 year old kid when giving the task to sum up the first 100 numbers that it is the same as adding up 50 times 101 which is 5050 by writing it as $(1 + 100) + (2 + 99) + (3 + 98) \dots (100 + 1)$ leading to $n/2$ terms of $(n + 1)$ if n is even. Taking differences verifies $Dg(n) = (n + 1)n/2 - n(n - 1)/2 = n = f(n)$.

$$1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad 28 \quad 36 \quad 45 \quad \dots$$

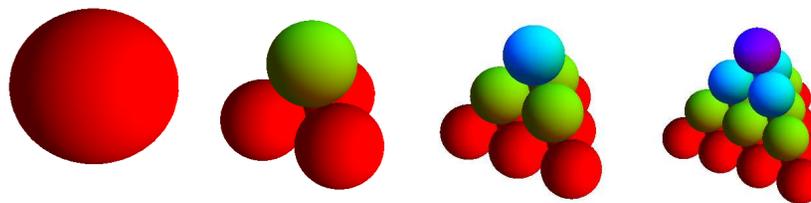


$n=1$ Lets add up the new sequence again and compute $h = Sg$. We get the sequence

$$1, 4, 10, 20, 35, \dots$$

These numbers are called the **tetrahedral numbers** because one use $h(n)$ marbles to build a tetrahedron of side length n . For example, we need $h(4) = 20$ golf balls for example to build a tetrahedron of side length 4. The formula which holds for h is $h(n) = n(n + 1)(n + 2)/6$.

Also this sequence defines a **function**. For example, $g(3) = 10$. But what is $g(100)$? Can we find a formula for $g(n)$?



$n=1$ We can see that summing up the differences gives the function in the same way as differencing the sum:

$$SDf(n) = f(n) - f(0), DSf(n) = f(n)$$

This is an arithmetic version of the **fundamental theorem of calculus**. The process of adding up numbers will lead to the **integral** $\int_0^x f(x) dx$. The process of taking differences will lead to the **derivative** $\frac{d}{dx} f(x)$. One of the high lights of this course is to understand the **fundamental theorem of calculus**:

$$\int_0^x \frac{d}{dt} f(t) dt = f(x) - f(0), \frac{d}{dx} \int_0^x f(t) dt = f(x)$$

and see why it is such a fantastic result. You see formally that it fits the result for difference and sum. A major goal of this course will be to understand the fundamental theorem result and see its use. But we have packed the essence of the theorem in the above version with S and D . It is a version which will lead us.

- 1 **Problem:** Given the sequence $1, 1, 2, 3, 5, 8, 13, 21, \dots$ which satisfies the rule $f(x) = f(x - 1) + f(x - 2)$. It defines a function on the positive integers. For example, $f(6) = 8$. What is the function $g = Df$, if we assume $f(0) = 0$? **Solution:** We take the difference between successive numbers and get the sequence of numbers

$$1, 0, 1, 1, 2, 3, 5, 8, \dots$$

After 2 entries, the same sequence appears again. We can also deduce directly from the above recursion that f has the property that $Df(x) = f(x - 2)$. It is called the **Fibonacci sequence**, a sequence of great fame.

- 2 **Problem:** Take the same function f given by the sequence $1, 1, 2, 3, 5, 8, 13, 21, \dots$ but now compute the function $h(n) = Sf(n)$ obtained by summing the first n numbers up. It gives the sequence $1, 2, 4, 7, 12, 20, 33, \dots$. What sequence is that?

Solution: Because $Df(x) = f(x - 2)$ we have $f(x) - f(0) = SDf(x) = Sf(x - 2)$ so that $Sf(x) = f(x + 2) - f(2)$. Summing the Fibonacci sequence produces the Fibonacci sequence shifted to the left with $f(2) = 1$ is subtracted. It has been relatively easy to find the sum, because we knew what the difference operation did. This example shows:

We can study differences to understand sums.

The next problem illustrates this too:

3 Problem: Find the next term in the sequence

2 6 12 20 30 42 56 72 90 110 132 . **Solution:** Take differences

2	6	12	20	30	42	56	72	90	110	132	
2	4	6	8	10	12	14	16	18	20	22	
2	2	2	2	2	2	2	2	2	2	2	
0	0	0	0	0	0	0	0	0	0	0	

Now we can add an additional number, starting from the bottom and working us up.

2	6	12	20	30	42	56	72	90	110	132	156
2	4	6	8	10	12	14	16	18	20	22	24
2	2	2	2	2	2	2	2	2	2	2	2
0	0	0	0	0	0	0	0	0	0	0	0

4 Find the next term in the sequence 3, 12, 33, 72, 135, 228, 357, 528, 747, 1020, 1353.... To do so, compute successive derivatives $g = Df$ of f , then $h = Dg$ until you see a pattern. Solution:

(3	12	33	72	135	228	357	528	747	1020	1353
	3	9	21	39	63	93	129	171	219	273	333
	0	6	12	18	24	30	36	42	48	54	60
	6	6	6	6	6	6	6	6	6	6	6
	0	0	0	0	0	0	0	0	0	0	0

We see that after taking 4 differences, we reach zero. This implies that the function is a cubic polynomial. We can get the next term by looking at the bottom and adding up.

(3	12	33	72	135	228	357	528	747	1020	1353	1752
	3	9	21	39	63	93	129	171	219	273	333	399
	0	6	12	18	24	30	36	42	48	54	60	66
	6	6	6	6	6	6	6	6	6	6	6	6
	0	0	0	0	0	0	0	0	0	0	0	0