

Lecture 7: Set Theory and Logic

Please look at the 2 page handout of 2010 for a summary of the topics. Here a few addenda for this semester: .

The foundations of mathematics were profoundly altered by two figures **Georg Cantor** and **Kurt Gödel**. We want to focus on those in this lecture.

We will tell the story in the presentation part and there are parallels. Both Mathematicians changed how to think about mathematics. Already in Cantors case, there was resistance and his mathematics needed time to be accepted. Ultimately Hilbert said: "Nobody will drive us from the paradise that Cantor has created for us.

Cantor made clear what "cardinality" is and showed us that different infinities exist and that some infinities are the same even so we might believe they are not.

Counting: Hilbert's Hotel

Hilbert's hotel is located on route 8. It has countably many rooms numbered $1, 2, 3, \dots$. The hotel is fully booked. As a newcomer arrives. David, the hotel manager is mortified. especially because David is a VIP wanting a room with a small number. David has an idea and moves guest in room i to room $i + 1$ and gives the newcomer the first room 1.

An other day, the hotel is empty but a large group arrives. They are the "fractions" on their way to a cardinal match with the "squares". Can David accomodate them? He thinks hard and finally manages.

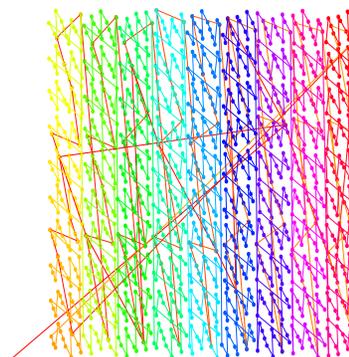
In the summer, the "reals" appear. David is not there but has George, the apprentice is in the office. The group consists of all real numbers between 0 and 1. Can George accomodate them? As much as he tries to shift and renumber, he can not do it.

Counting: the interval

The interval $(-1, 1)$ has the same cardinality than the real line. The function $f(x) = \tan(\pi x/2)$ maps the interval $(-1, 1)$ onto the real line, one to one.

The square $(0, 1) \times (0, 1)$ has the same cardinality than the real line. A bijection can be constructed by $f(0.a_1a_2a_3a_4\dots) = (0.a_1a_3a_5, 0.a_2a_4a_6\dots)$.

Here is an attempt to make a picture of the "curve" which gives this very discontinuous map from the interval $(0, 1)$ to the unit square. We see the image of the points $k/1301$ for $k = 1, \dots, 1300$.



This picture illustrates the crazy function f from the interval to the unit square. There are lots of discontinuities but the function is $1 - 1$.

Arithmetic with sets

One can calculate with sets as with numbers. They form a "Boolean ring".

Addition: $A + B = A \Delta B$ with the zero element \emptyset

Multiplication: $A \cdot B = A \cap B$ with the one element Ω .

All the rules of the real numbers apply but there are additional consequences which appear a bit strange $A + A = 0$ and $A^2 = A \cdot A = A$. This means that A is its own additive inverse.

Paradoxa

We have seen a few paradoxa like the **Liars paradox**, the **barbers paradox**, the surprise exam problem. Here is an other, the **Berry paradox** which comes somehow close to the Goedel numbering:

The smallest integer not definable in less than 11 words.

The problem is that this number is defined with 10 words. This looks like a stupid example but it illustrates that there are properties of numbers like "the shortest way to describe the number" which is not computable.