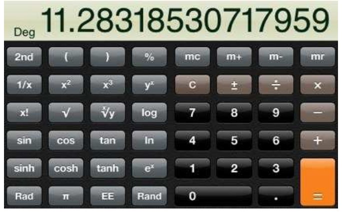
Lecture 12: Dynamical systems

Part I: Chaos

1. Objective

Simple transformations can produce chaotic outcome. Make sure your calculator is in the "Rad" mode. Remember that 2π radians is equal to 360 degrees. You can check whether your calculator is in Radian mode, by computing $\cos(\pi)$ and gt -1. Make sure your calculator is in rad mode. Use a scientific calculator. In the iphone calculator for example, turn the device to get to the scientific mode.



The Scientific Calculator built in by default in the Iphone/Ipod appears when you turn the device.

Order and Chaos with the calculator

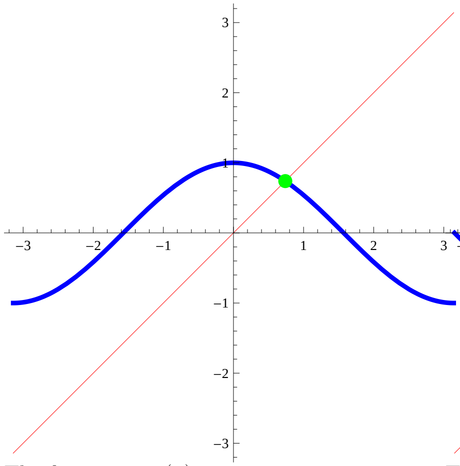
1) Take a calculator, and pushing repetitively the button cos. What do you observe?

2) Now repeat pushing the sin x button. What do you observe?

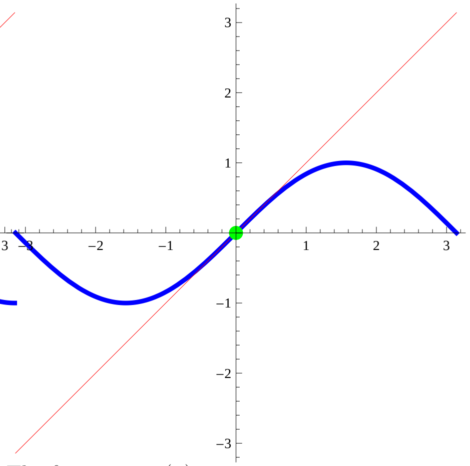
3) Now push x^2 repetitively.

4) What do you see if you push the buttons sin, then type $1/x$ and repeat this process again and again?

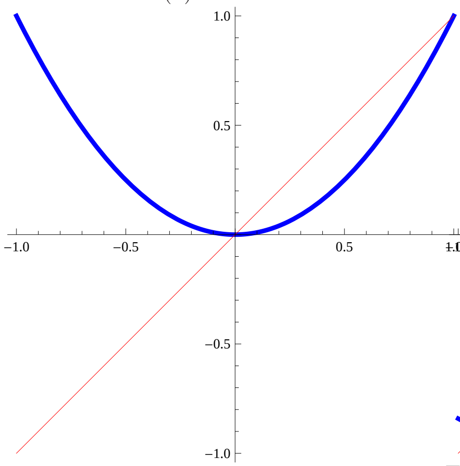
4) Can you find other "chaotic" key combinations? Experiment also with Deg and Rad changes.



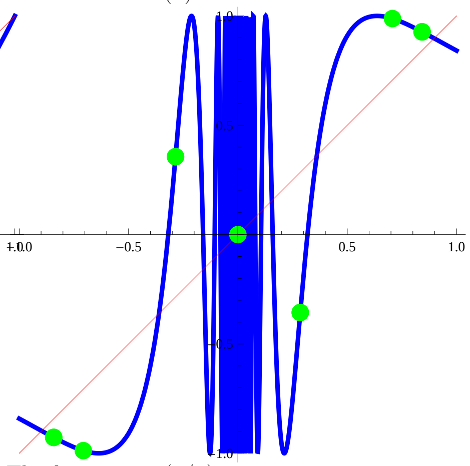
The function $\cos(x)$



The function $\sin(x)$



The function x^2



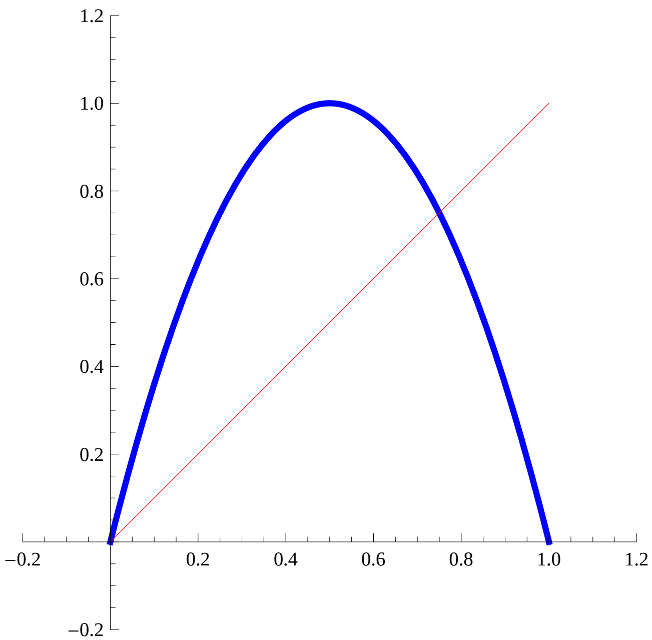
The function $\sin(1/x)$

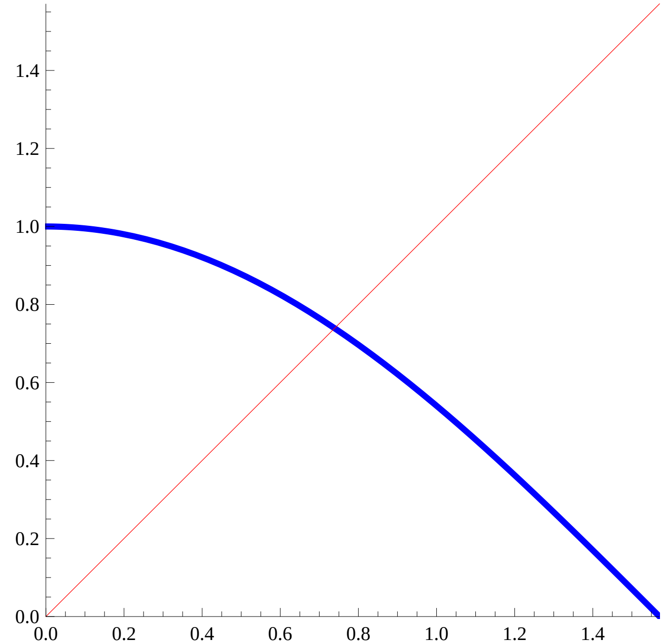
Part II: The Cobweb construction

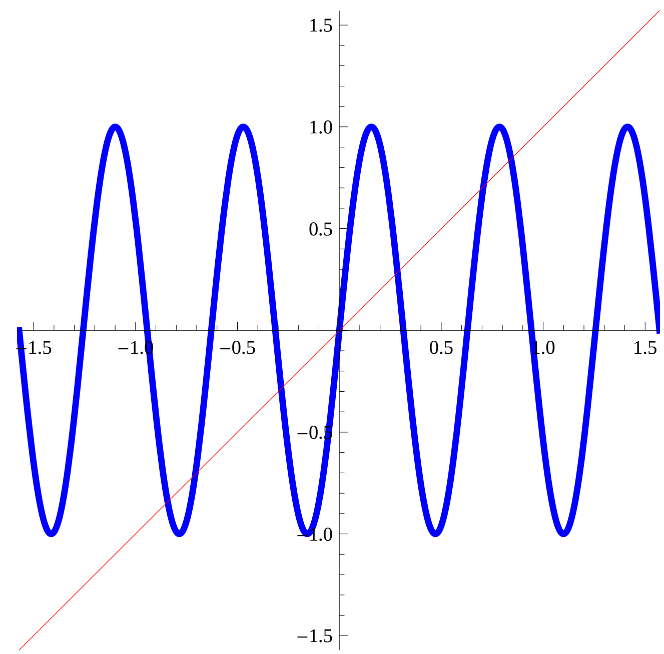
1. Objective

We graphically compute a few iterates of one dimensional maps.

2. Stability







Lecture 12: Part III: The Ulam Collatz system

1. Objective

We look at a dynamical system of number theoretical nature.

2. The Collatz system

In the **Collatz system**, we start with an integer and map it with the following rule:

$$T(x) = \begin{cases} x/2 & x \text{ even} \\ 3x+1 & x \text{ odd} \end{cases}$$

The question is whether the orbit always ends up with 1.

For example: $x = 7$ produces $7, 22, 11, 34, 17$.

3. Experiment

1) Start with the initial condition 26:

2) Start with the initial condition 9:

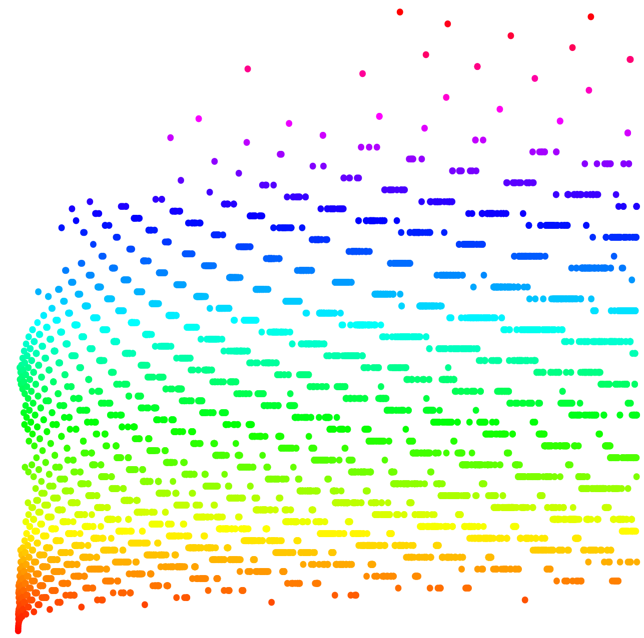
3) Start with the initial condition 2048:

4) What is wrong with the following proof of the Collatz conjecture?

Proof. Consider only the odd numbers in the Collatz sequence. We show that each odd number is in average 3/4 times smaller than the previous one:

With probability 1/2 the number $3x+1$ is divisible by 2 and not 4: this increases x by 3/2
With probability 1/4 the number $3x+1$ is divisible by 4 and not 8: this decreases x by 3/4
With probability 1/8 the number $3x+1$ is divisible by 8 and not 16: this decreases x by 3/8

To compute the probability, we take logarithms and compute $a = \sum_{n=1}^{\infty} \frac{1}{2^n} \log(3/2^n)$. The average decay rate of the size of a number is the factor $e^a = 3/4$.



3) The Collatz system certainly can be modified. Can you find one, for which there is a nontrivial loop?

Lecture 12: Part IV: Cellular automata

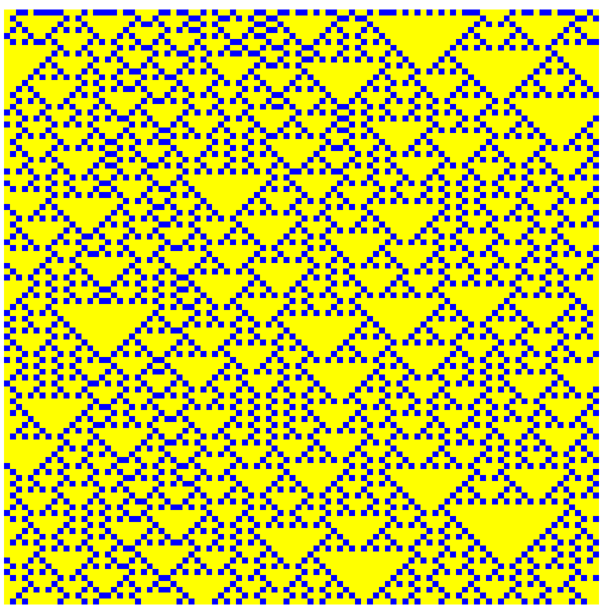
1. Objective

We look at a dynamical systems called Cellular automata. These are continuous maps on sequence spaces in which the evolution rule is translational invariant.

2. The Rule 18 CA

neighborhood	new middle cell
111	0
110	0
101	0
100	1
011	0
010	0
001	1
000	0

Because only cell neighborhoods of the form 100 and 001 lead to an offspring 1, we and $100 = 4, 001 = 1$ in binary, we have $2^4 + 2^1 = 18$.



3. Run it

1	0	1	0	0	0	1	0	0	1	0	1	1	0	1	0

Cellular Automata Offer New Outlook on Life, the Universe, and Everything

What kind of world do we live in? The question has been bandied about for thousands of years by philosophers, theologians, and politicians. More recently, a spectrum of talk show hosts have weighed in on the subject. So far, no one's come up with an answer that everyone can agree on.

Mathematicians have considered the same question. But where others worry over the blurred boundaries of Good and Evil, mathematicians ponder a sharper dichotomy: the Continuous versus the Discrete.

Continuous mathematics, exemplified by calculus and differential equations, has long dominated mathematical descriptions of the world. But discrete mathematics is making a bid for primacy. With modern computers, researchers have discovered astonishingly complex behavior in seemingly simple, finite systems. The results have led some theorists to speculate that discrete models, which lend themselves to digital computation, are the "right" way to study nature.

Erica Jen, a mathematician at Los Alamos National Laboratory in Los Alamos, New Mexico, is one of a growing number of researchers who believe that discrete mathematics can mirror many aspects of physical reality fully as well as the more customary continuous theories. Jen has been studying mathematical properties of discrete systems known as cellular automata. These systems, she says, are useful models for many types of complex physical, chemical, or biological systems. They also have an amazing life of their own.

Cellular automata "exhibit an extremely rich and diverse range of pattern formation," Jen says. Among the most interesting are "self-organizing" patterns: highly structured features that seem to emerge spontaneously from a "primordial soup" of random binary

Erica Jen. *(Photo courtesy of Erica Jen.)*

Figure 2. *A 100 × 100 "majority vote" cellular automaton proceeds from a random initial state (top) to a final state (bottom right, facing page). On each ballot, every cell looks at the cells around it, and changes color value if its current value is in the minority. Most cells have 8 neighbors, but cells on the edges have 5 neighbors, and corner cells only 3. Some features of the final state take shape with the first round of voting (middle).*

digits. Jen and her colleagues hope to understand exactly how these patterns arise and precisely what properties they possess. By studying cellular automata with mathematical tools from areas such as abstract algebra and number theory, Jen hopes to bring theoretical rigor to a subject that is often as much art as science.

Loosely speaking, a cellular automaton is a "pixelization" of space and time: Instead of varying continuously from point to point and moment to moment, cellular automata consist of discrete "cells" with discrete values that change instantaneously at discrete intervals, much like frames in a movie. The crucial feature, moreover, is a *rule* that prescribes exactly how each cell's value changes depending on the values of nearby cells.

One possible rule, for example, is a "majority vote": Each cell in a system of black and white squares could be programmed to switch color if the majority of its immediate neighbors are of the opposite color (see Figure 2). Another rule might specify that the value of each cell change to the sum of the values of the cells surrounding it—or, reducing things to black and white again, to the *parity* of the sum (black could be odd and white even).

"The essential features of cellular automata are that they are deterministic and discrete in space, time, and state values; they evolve according to local interaction rules; and these rules apply

Jen hopes to bring theoretical rigor to a subject that is often as much art as science.

Computer technology was not really up to the job of exploring cellular automata until the 1980s.

synchronously and homogeneously across the system," Jen explains. These features accord well with standard physical assumptions about the uniformity of space and time (the laws of physics are the same everywhere) and the impossibility of instantaneous action at a distance (nothing travels faster than the speed of light). They also lend themselves to modeling complex systems consisting of a large number of simple components that are locally connected. Perhaps most important, these features are tailor-made for digital computation.

Cellular automata were first dreamt of in the early 1950s by John von Neumann and Stanislaw Ulam, as tools for studying biological systems. In the late 1960s, John Conway, then at Cambridge University (now at Princeton), invented rules for a cellular automaton he called the Game of Life, which Martin Gardner popularized in his column for *Scientific American*. But computer technology was not really up to the job of exploring cellular automata until the 1980s, when color graphics workstations replaced the clattering teletype machines that trailed alphanumeric symbols with a room-sized mainframe in another building.

With today's high-speed machines (fated, no doubt, to seem painfully slow in another few years), researchers can glimpse the complex patterns that often arise from the repeated application of the simple rules that define cellular automata. Fast computers allow experiments with relatively large systems: Automata with thousands of cells can be followed for hundreds of time steps on a personal computer; workstations and supercomputers can track systems with millions of cells for thousands of time steps.

Jen's research focuses on a class of one-dimensional systems called "elementary" cellular automata. Each state of such a system is represented by a row of black and white pixels, corresponding to a string of 1's and 0's, and the update rule uses only the value of a given cell and the values of its two adjoining cells. (To simplify the description, researchers often work with a "wrap around" model, in which the two ends are joined, so that all cells are treated alike.) The evolution of a one-dimensional automaton is conveniently displayed in a two-dimensional format, each new row below its predecessor. (Researchers also often "colorize" their elementary systems to highlight key features.) The result can be as richly textured as a Navajo weaving.

In the early 1980s, Stephen Wolfram, then at the Institute for Advanced Study in Princeton, roughed out a classification scheme