

Lecture 13: Experimental mathematics

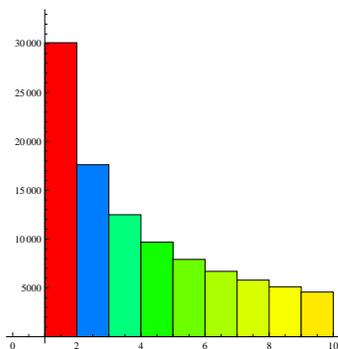
Experimental mathematics brings mathematics close to physics: we do experiments, for example with the help of a computer. Instead of talking about this abstractly, I tell you a concrete story which illustrates this.

Benford's law tells that the first digits of the sequence 2^n has a distribution which satisfies $p_k = \log_{10}(1 + 1/k)$. The digit 1 for example occurs with about 30 percent. During our discussion, we asked ourselves, what happens if we look at the first significant digit of n^2 or the first significant digit of the primes p_n .

First experiment: exponentials

We look at the numbers 2^n for $n = 1$ to $n = 100'000$ and look at the first digit:

```
data = Table[First[IntegerDigits[2^n]], {n, 1, 100000}];
S = Histogram[data, 10, ColorFunction -> Hue]
```



This is a situation one can understand well. If we look at the logarithms, then $\log(2^n) = n \log(2)$. The first digit is 1 if the rest of $[n \log(2)]$ modulo 1 is between 0 and $\log(2)$. The first digit is 2 if it is between $\log(2)$ and $\log(3)$ etc. The probability that the letter is k is $\log_{10}(k+1) - \log_{10}(k)$.

Second experiment: squares

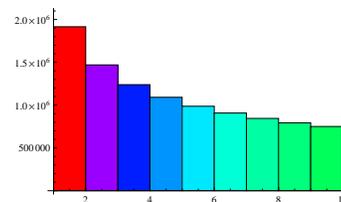
Now let's look at the first significant digit of the squares $1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, 961, 1024, 1089, 1156, 1225, 1296, 1369, 1444, 1521, 1600, 1681, 1764, 1849, 1936, 2025, 2116, 2209, 2304, 2401, 2500, 2601, 2704, 2809, 2916, 3025, 3136, 3249, 3364, 3481, 3600, 3721, 3844, 3969, 4096, 4225, 4356, 4489, 4624, 4761, 4900, 5041, 5184, 5329, 5476, 5625, 5776, 5929, 6084, 6241, 6400, 6561, 6724, 6889, 7056, 7225, 7396, 7569, 7744, 7921, 8100, 8281, 8464, 8649, 8836, 9025, 9216, 9409, 9604, 9801, 10000$. Here is an experiment with 1 Million.

```
data = Table[First[IntegerDigits[n^2]], {n, 1, 1000000}];
S = Histogram[data, 10, ColorFunction -> Hue]
```

It is interesting because we want to see what the distribution of $2 \log(n)$ is modulo 1. It looks as if we have a similar Benford law here. Indeed it is a generalized Benford law with $p_k = \frac{\int_k^{k+1} x^{-\alpha} dx}{\int_1^{10} x^{-\alpha} dx} = \frac{[(k+1)^{1-\alpha} - k^{1-\alpha}]/(10^{1-\alpha} - 1)}{1}$. It interpolates the Benford law $\alpha = 1$ with the uniform distribution $\alpha = 0$.

We have the digit 1, if $\log(n) \in k + [0, \log(2)]$.

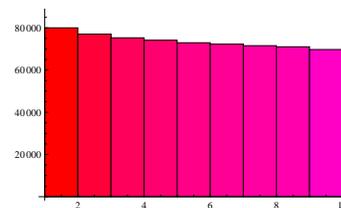
How many cases are in 1000 and 2000. It is $\sqrt{2000} - \sqrt{1000} = \sqrt{1000}(\sqrt{2} - 1)$. How many cases are in 2000 and 3000. It is $\sqrt{3000} - \sqrt{2000} = \sqrt{1000}(\sqrt{3} - \sqrt{2})$.



Third experiment: Primes

What is the first significant digit of the prime numbers?

```
data = Table[First[IntegerDigits[Prime[n]]], {n, 1, 664000}];
S = Histogram[data, 10, ColorFunction -> Hue]
```



How many primes are there in 1000 and 2000. We expect $1000/\text{Log}[1000]$ primes in there and $1000/\text{Log}[2000]$ with first significant digit 2.

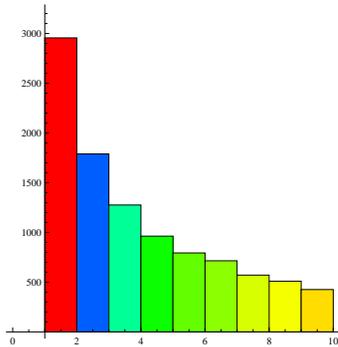
```
S1=ListPlot[Table[PrimePi[k],k,10000]]; S2=ListPlot[Table[k/Log[k],k,10000]]; Show[S1,S2]
```

We expect the distribution to be $a/\log(k)$, where $a = \sum 1/\log(k)$.

Forth experiment: factorials

For factorials, the limiting distribution is known to be the Benford distribution. There is no reason why $\log_{10}(n!) \bmod 1$ should not be uniformly distributed.

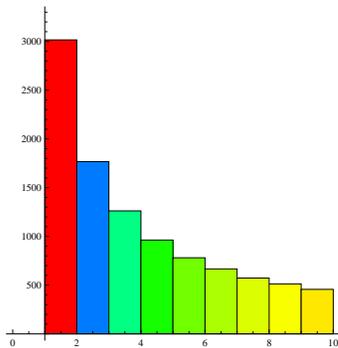
```
data = Table[First[IntegerDigits[n!]], {n, 1, 10000}];
S = Histogram[data, 10, ColorFunction -> Hue]
```



Fifth experiment: partitions

Also for the partition numbers, $p(n)$, which give the number of possibilities in which the number n can be written as a sum of integers, we measure that the Benford distribution takes place. As far as we know this is not known.

```
data = Table[First[IntegerDigits[PartitionsP[n]]], {n, 1, 10000}];
S = Histogram[data, 10, ColorFunction -> Hue]
```



Problems with known answers

- 1 Find the distribution of the first significant digit of $n^2 + 2^n$.
- 2 Find the distribution of the first significant digit of the Fibonacci sequence 1,1,2,3,5,8,13,...
- 3 Find the distribution of the first significant digit of numbers n^{100} .
- 4 Verify that for the factorials $n!$, the first significant digit has the Benford distribution? (Benford)
- 5 Verify that for n^n , the first significant digit has the Benford distribution? ¹

Problems with unknown answers

- 6 Find the distribution of the first significant digit of primes. One finds a n dependent generalized Benford law with $\alpha(n) = 1/(\log(n) - 1.1)$. ²
- 7 Find the distribution of the first significant digit of $[\exp(\sqrt{n})]$ where $[x]$ is the largest integer smaller or equal to x .
- 8 What is the distribution of the first significant digit of $[n \log(n)]$ where $[x]$ is the largest integer smaller or equal to x .
- 9 What is the distribution of the first significant digit of the Partition numbers $p(n)$?
- 10 What is the distribution of the first significant digit of $[n^2 \sin(n)]$ where $[x]$ is the largest integer smaller or equal to x .

¹F. Benford, The law of anomalous numbers. Proc. Am. Phil. Soc. 78, p. 551-572, 1938

²B. Luque and L. Lacas, The first-digit frequencies of prime numbers and Riemann zeta zeros", Proc R. Soc A 2009, 465, 2197-2216, 2009