# Lecture 13: Experimental mathematics

Experimental mathematics brings mathematics close to physics: we do experiments, for example with the help of a computer. Instead of talking about this abstractly, I tell you a concrete story which illustrates this.

Benfords law tells that the first digits of the sequence 2^n has a distribution which satisfies $p_k = \log_{10}(1 + 1/k)$. The digit 1 for example occurs with about 30 percent. During our discussion, we asked ourselves, what happens if we look at the first significant digit of n^2 or the first significant digit of the primes p_n.

First experiment: exponentials

We look at the numbers 2^n for $n = 1$ to $n = 100'000$ and look at the first digit:

```
data = Table[First[IntegerDigits[2^n]], {n, 1, 100000}];
S = Histogram[data, 10, ColorFunction -> Hue]
```
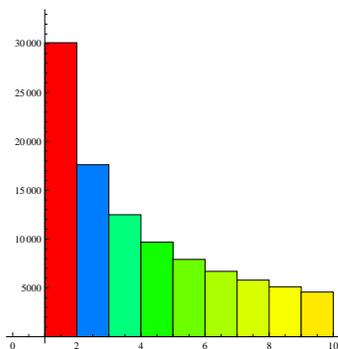

This is a situation one can understand well. If we look at the logarithms, then $\log(2^n) = n\log(2)$. The first digit is 1 if the rest of $[n\log(2)]$ modulo 1 is between 0 and $\log(2)$. The first digit is 2 if it is between $\log(2)$ and $\log(3)$ etc. The probability that the letter is k is $\log_{10}(k + 1) - \log_{10}(k)$.

Second experiment: squares

Now lets look at the first significant digit of the squares $1, 4, 9, 1, 2, 3, 4, 6, 8, 1, 1, 1, 1, 1, 2, 2, 2, 3, 3, 4, 4, 4$ Here is an experiment with 1 Million.

```
data = Table[First[IntegerDigits[n^2]], {n, 1, 10000000}];
S = Histogram[data, 10, ColorFunction -> Hue]
```

It is interesting because we want to see what the distribution of $2\log(n)$ is modulo 1. It looks as if we have a similar Benford law here. Indeed it is a generalized Benford law with $p_k = \frac{\int_k^{k+1} x^{-\alpha}\,dx}{\int_1^{10} x^{-\alpha}\,dx} = [(k+1)^{1-\alpha} - k^{1-\alpha}]/(10^{1-\alpha} - 1)$. It interpolates the Benford law $\alpha = 1$ with the uniform distribution $\alpha = 0$.

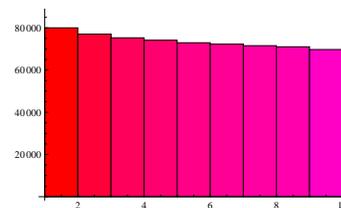We have the digit 1, if $\log(n) \in k + [0, \log(2)]$.
How many cases are in 1000 and 2000. It is $\sqrt{2000} - \sqrt{1000} = \sqrt{1000}(\sqrt{2} - 1)$. How many cases are in 2000 and 3000. It is $\sqrt{3000} - \sqrt{2000} = \sqrt{1000}(\sqrt{3} - \sqrt{2})$.
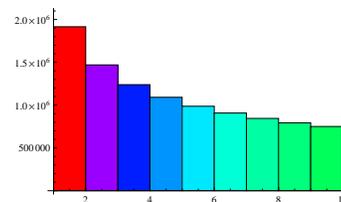
Third experiment: Primes

What is the first significant digit of the prime numbers?

```
data = Table[First[IntegerDigits[Prime[n]]], {n, 1, 664000}];
S = Histogram[data, 10, ColorFunction -> Hue]
```


How many primes are there in 1000 and 2000. We expect $1000/Log[1000]$ primes in there and $1000/Log[2000]$ with first significant digit 2.
S1=ListPlot[Table[PrimePi[k],k,10000]]; S2=ListPlot[Table[k/Log[k],k,10000]]; Show[S1,S2]
We expect the distribution to be $a/\log(k)$, where $a = \sum 1/\log(k)$.

Forth experiment: factorials

For factorials, the limiting distribution is known to be the Bedord distribution. There is no reason why $\log_{10}(n!)\,mod\,1$ should not be uniformly distributed.
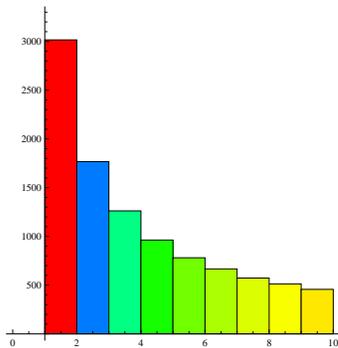
```
data = Table[First[IntegerDigits[n!]], {n, 1, 10000}];
S = Histogram[data, 10, ColorFunction -> Hue]
```
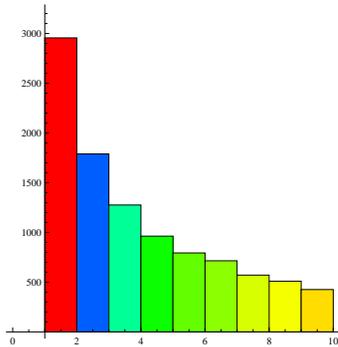
Fifth experiment: partitions

Also for the partition numbers, $p(n)$, which give the number of possibilities in which the number n can be written as a sum of integers, we measure that the Benford distribution takes place. As far as we know this is not known.

```
data = Table[First[IntegerDigits[PartitionsP[n]]], {n, 1, 10000}];
S = Histogram[data, 10, ColorFunction -> Hue]
```

[1]F. Benford, The law of anomalous numbers. Proc. Am. Phil. Soc. 78, p. 551-572, 1938

[2]"B. Luque and L. Lacas, The first-digit frequencies of prime numbers and Riemann zeta zeros", Proc R. Soc A 2009, 465, 2197-2216, 2009