

Top 10 Greatest Moments in (Artistic) Mathematics

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Final Project

For this project, I chose to explore math through the lens of two of my hobbies: Photography and Circus Arts. I have chosen 10 intersections between art and mathematics that revolve around the ideas of creation, expression and transmission. How do we create art?

How do we express our art? How do we transmit our knowledge about art? I have chosen 10 of my favorite representations, artists and instances and described both the art form and the mathematics. They are arranged by category, photography coming first, and then in no particular order.

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1

THE ART: “The Rule of Thirds”

In photography, the rule of thirds is a loose rule describing aesthetics. The rule of thirds a loose rule used by photographers to compose ascetically pleasing shots. It states that if a shot is divided into nine equal parts created by two horizontal lines and two vertical lines. Think of a tic-tac-toe board drawn over a picture. The most aesthetically pleasing pictures depict subjects that fall along the intersections of those lines. Although it is most commonly used in composing photographs, this rule dates back to the 1700s. It was first written by John Thomas Smith in 1797 in his book *Remarks on Rural Scenery*. He writes, "I have found the ratio of about two thirds to one third, or of one to two, a much better and more harmonizing proportion, than the precise formal *half*, the two far extending *four-fifths* -and, in short...is agreed to be the most beautiful, (or, in other words, the most picturesque) medium of curves." ¹

The rule of thirds is thought to be the most ascetically pleasing because it approximates the Golden Ratio.

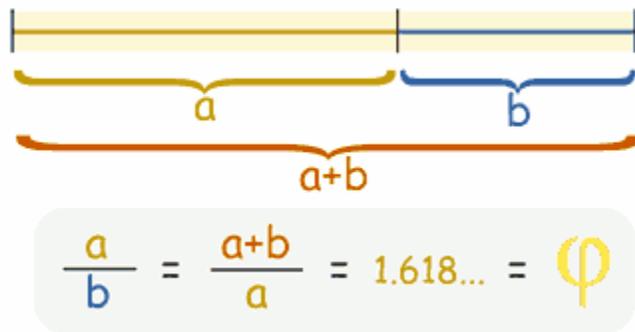


THE MATH: Phi, ϕ

Phi, the Golden Ratio, and the Golden spiral are all names for a proportion first described by Euclid in *The Elements* around the year 300BC. He calls it an "extreme and mean ratio" and describes writes, "a straight line is said to have an extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the lesser."³ Visually:

1 [Rule of Thirds](#). Wikipedia.com. Last accessed, May 2011.

2 Photo Credit: Karen Sokolow, 2008



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This proportion can be found in the natural and man made world. The Parthenon was constructed this ratio, the Mona Lisa expresses this proportion, the chambers of a nautilus are arranged in this manner and the petals of flowers show an overlap with this arrangement.

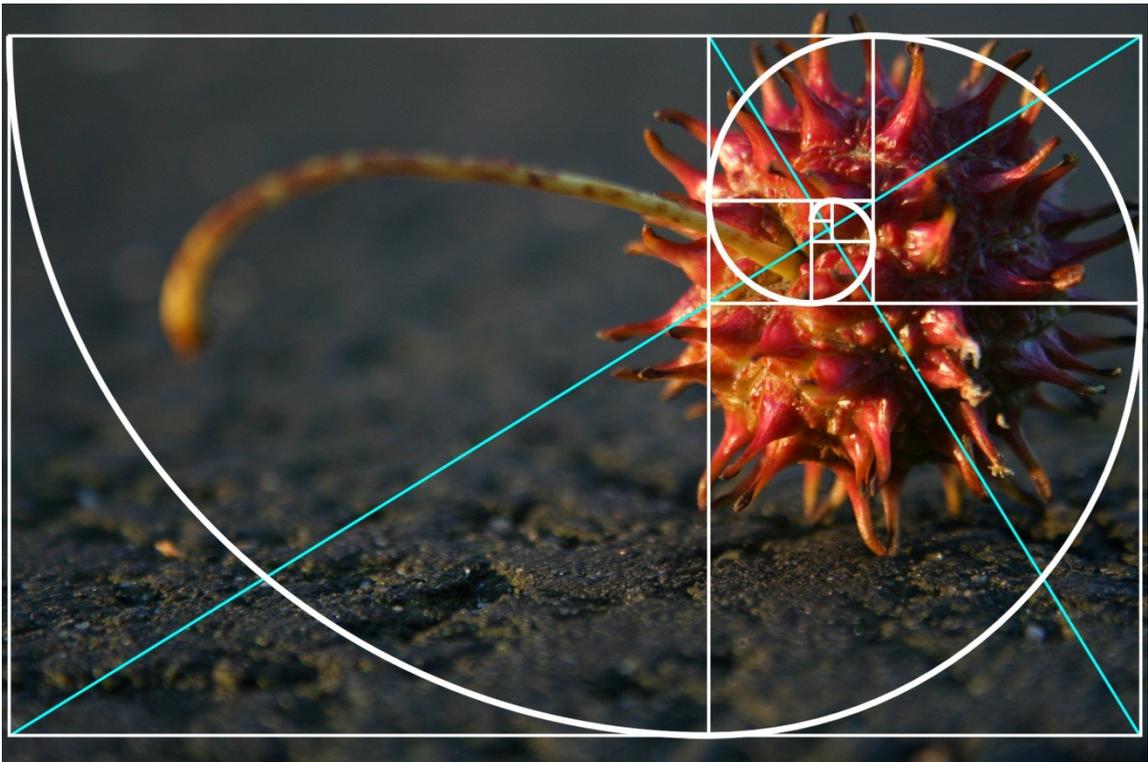
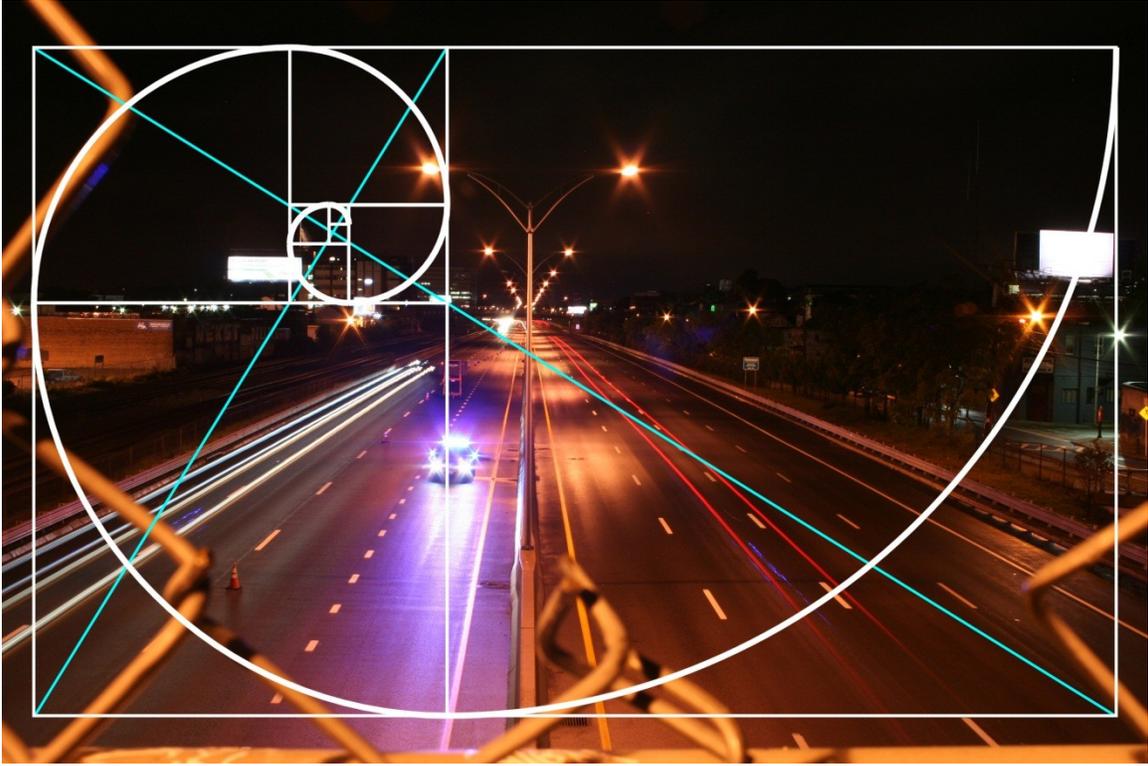
Fra Luca Bartolemo de Pacioli wrote a treatise of ϕ in 1509 called *Divina Proportione*. He was the one who coined the term Golden Ratio.⁵

Although the rule of thirds is not the *same* as the Golden Ratio, it pleasing proportionality may originate from the ubiquitous Golden Ratio.

3 Livio, Mario. *The Golden Ratio: The Story of Phi, the World's Most Astonishing Number*. New York: Broadway Books, 2003. Page 3.

4 Photo Credit: Image from mathisfun.com.

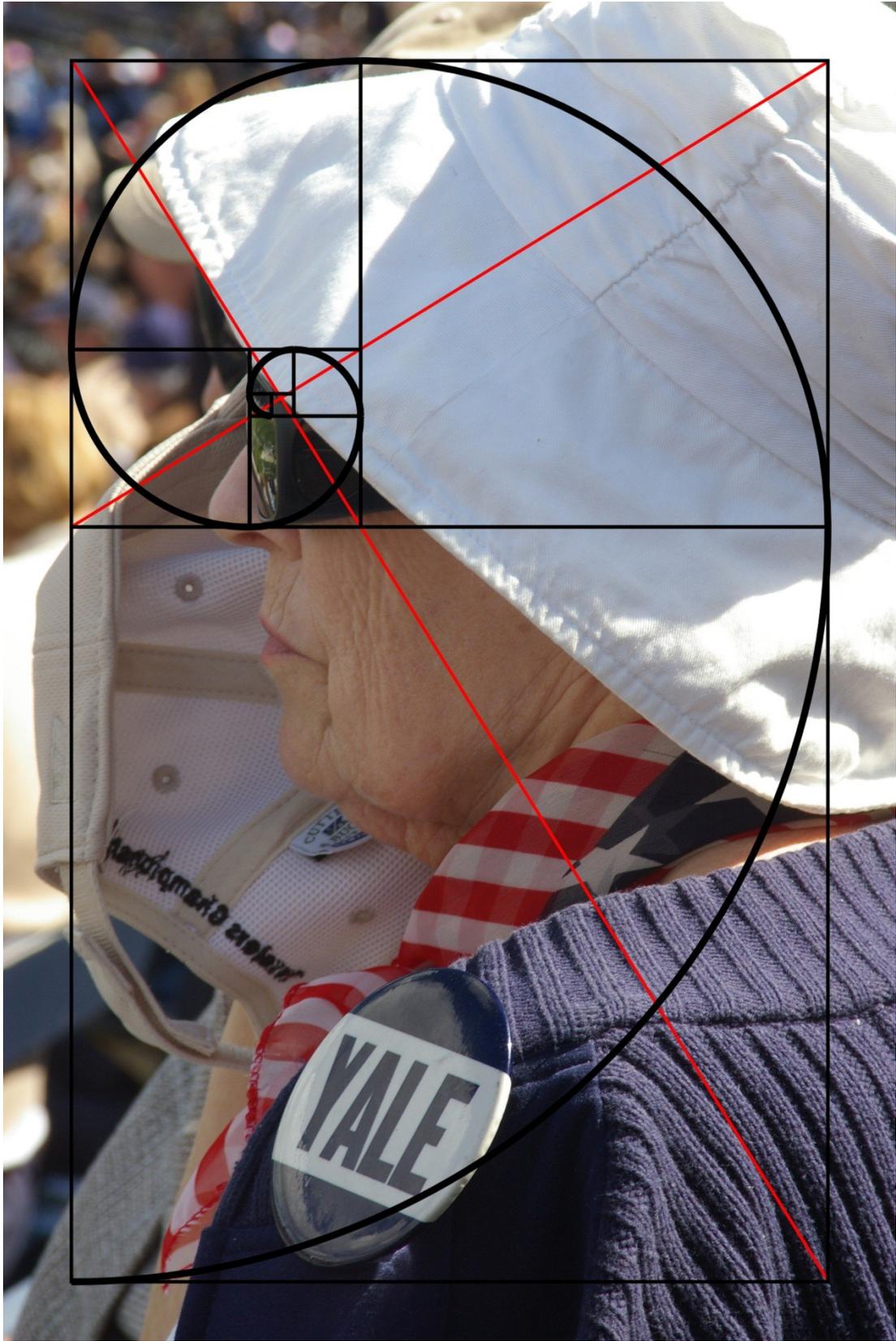
5 Pickover, Clifford A. *The Math Book: From Pythagoras to the 57th Dimension, 250 milestones in the History of Mathematics*. New York: Sterling, 2009. Page 112.



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6 Photo Credit: Karen Sokolow, 2009

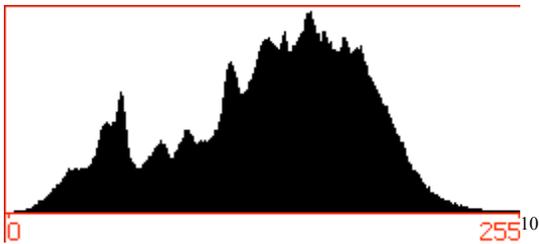
7 Photo Credit: Karen Sokolow, 2009



2

THE ART: Digital Photography and Histograms

The first digital cameras meant for mass consumption entered the market in early 1990.⁹ As the technology grew more advanced, there grew a demand for instantaneous feedback concerning the exposure of an individual photograph. Now, almost all digital cameras make a histogram of each photo plotting the brightness of the light against the number of pixels.

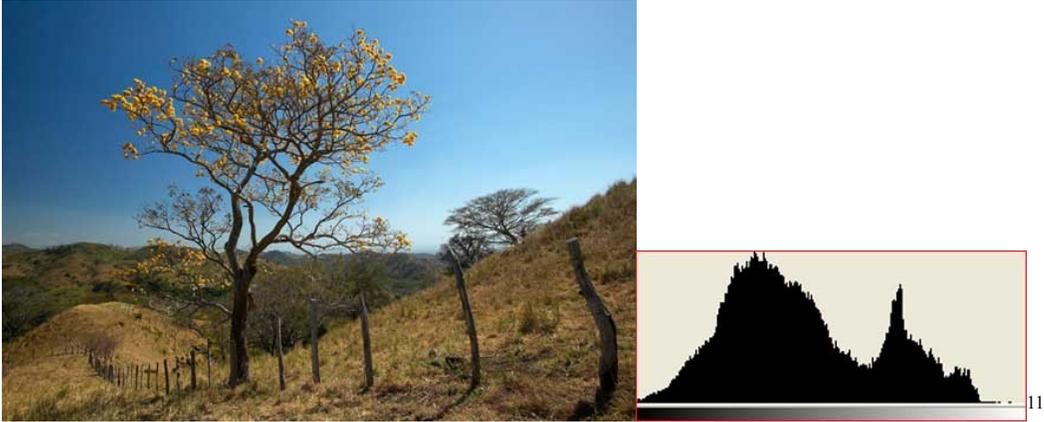


The far left of the graph represent the pixels that have measured no light, the far right with complete light. The goal is to have most of the pixels fall in the middle range, indicating a well exposed picture. Below are a photograph and its corresponding histogram. This picture shows no spikes or outlying regions, and is therefore considered to be well exposed.

⁸ Photo Credit: Karen Sokolow, 2009

⁹ Bellis, Mary. "History of the Digital Camera". Accessed May, 2011.
<http://inventors.about.com/library/inventors/bldigitalcamera.htm>

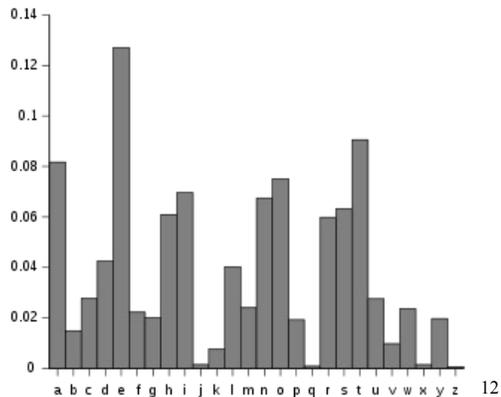
¹⁰Photo Credit: Image taken from A Harvard Extension school class I attended in 2009, CS E-7 taught by Dan Armendariz



As with most aspects of photography, reading a histogram is does not guarantee information about the picture. It is possible to have a well exposed photograph with mostly dark pixels or mostly light pixels. It is, though, a valuable tool for today's digital photographer.

THE MATH: Frequency Analysis of Light

Cryptanalysis, the science of breaking codes, employs a technique called frequency analysis. First used in the 9th century by Al Kindi, it measures the frequency of a certain letter appearing in a general standard text. Once the general frequency is known, the coded text can be analyzed to search for familiar distributions, revealing the substitution code. For example, if the code is sent using the English language, it can be reading according to the following frequency graph:



11 Photo Credit: From <http://www.luminous-landscape.com/tutorials/understanding-series/understanding-histograms.shtml>

12 Photo Credit: Wikipedia.com, article on Frequency Analysis.

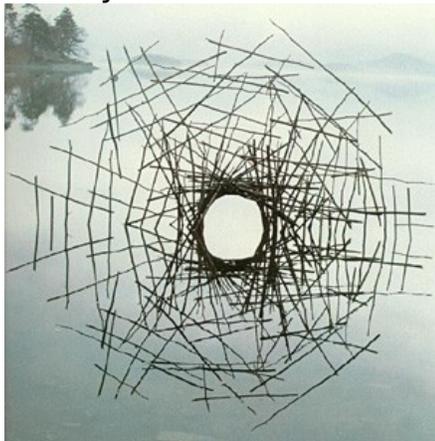
If the letter M, for example appears most often in our code, it is most likely the letter E.

In digital photography, a histogram can be thought of as a specific frequency analysis. If a photograph is a code where the value of the light is not immediately apparent, a histogram gives the frequency at which certain levels of light appear. The standard frequency to which the photograph is compared is a bell curve. In this way, we can use the cryptanalytical concept of frequency analysis to “decode” and find the “meaning” in the exposure of the photograph.

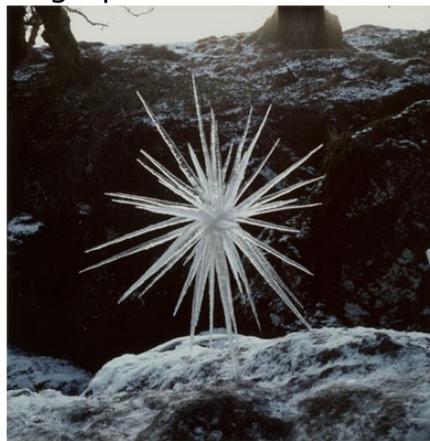
3

THE ART: Andy Goldsworthy

Andy Goldsworthy, a British artist, naturalist and photographer became famous for his ephemeral art. He creates images from leaves, petals, ice, stones and other objects found in nature, photographs his creations and then leaves them to disintegrate in their environment. In his own words, “the ephemeral work is...very, very important. I can’t not make it.”¹³ Goldsworthy, in addition to being an artist, is a former professor of applied mathematics at the University of Leeds. Several of his photographs are shown below.



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13 Brunton, Michael. “Q & A with Andy Goldsworthy.” *Time Magazine* 13 April 2007. Accessed May 2011. <http://www.time.com/time/arts/article/0,8599,1610464,00.html>.

14 Photo Credit: Andy Goldsworthy, found on <http://www.writedesignonline.com/history-culture/AndyGoldsworthy/overview.htm>

15 PhotoCredit: Andy Goldsworthy, found on <http://sixohthree.tumblr.com/post/308724336/andy-goldsworthy-okay-last-one-i-promise>



THE MATH: Polyhedra and Stellated Polyhedra

Many of Andy Goldsworthy's photographs depict polyhedra and stellated polyhedra. The study of these solids dates back to the ancient Greek world. There are five regular polyhedra (the tetrahedron, the square, the dodecahedron, the octahedron and the icosahedron). Each polyhedron is convex and has identical polygon faces.¹⁷ Later, Archimedes compiled a list of semi-regular polyhedra. The shapes are convex whose faces are regular polyhedra. The faces, though, do not need to be identical as long as the same pattern of faces occurs at each vertex. Archimedes recorded thirteen semi-regular polyhedra.¹⁸

Stellated polyhedra are an offshoot of these regular shapes. They are visually striking, seeming to burst like stars, but they are actually larger polygons that intersect each other at regular intervals. In 1619, Johannes Kepler described the process of stellating as extending edges or faces until they meet to form new shapes.¹⁹ Stellations are much more numerous in number than regular or semi-regular polyhedra.

Third Stellation Dodecahedron
Icosahedron:

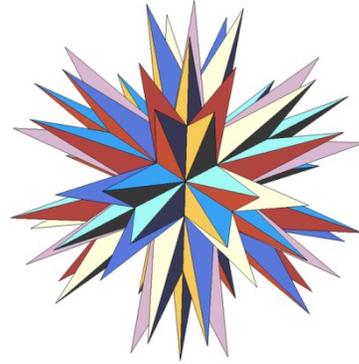
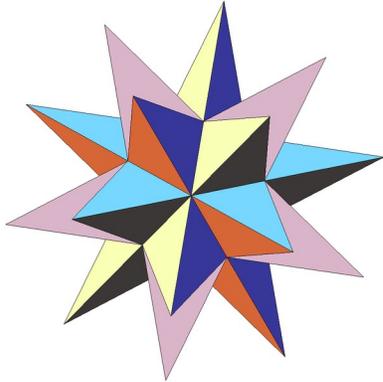
Seventeenth Stellation

16 Photo Credit: Andy Goldsworthy, found on http://stillcoloringoutofthelines.blogspot.com/2010_05_01_archive.html

17 Stillwell, John. *Mathematics and Its History, Third Edition*. New York: Springer, 2010. Page 21.

18 Pickover. Page 64.

19 "Stellated Polyhedra". Accessed May 2011 http://math.arizona.edu/~models/Stellated_Polyhedra/



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4

THE ART: Color in Photography

When taking photographs in color, it is important to remember the complexities of color. Colors are recorded in the camera and then can be altered post production. The alteration of color presents two particular problems. First, color is recorded in the camera using the light spectrum. This is generally measured in three variables, Red Green and Blue which, when combined, make pure white light. Even though light is recorded using the light spectrum, it is printed using a different system. This system, typically called the CMYK system is made of a combination of Cyan, Magenta, Yellow and Black (the K). The Second important issue to photographers is the impact of color on a photograph. Manipulating color can make an ordinary photograph striking.



²¹ Without color editing



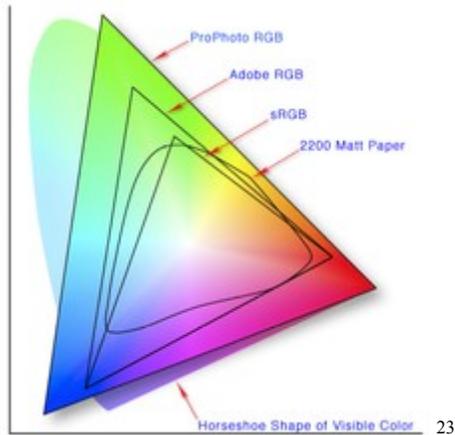
²² With color editing

THE MATH: Graphing Color Space

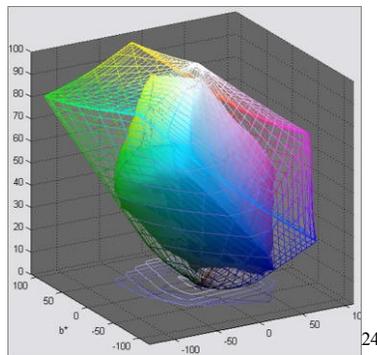
Color space is a mathematical construct used to describe the way that colors interact with one another. It is a way to apply a concrete model to what would otherwise be an arbitrary counting system. The x-axis indicates the chromaticity and the y-axis indicates luminosity. The outline of the graph indicates monochromatic light. Various color editing programs have different abilities within the color space graph called a gamut. The gamut indicates the limits of how an editor can shift the colors of a photograph.

21 Photo Credit: Karen Sokolow, 2009

22 Photo Credit: Karen Sokolow, 2009



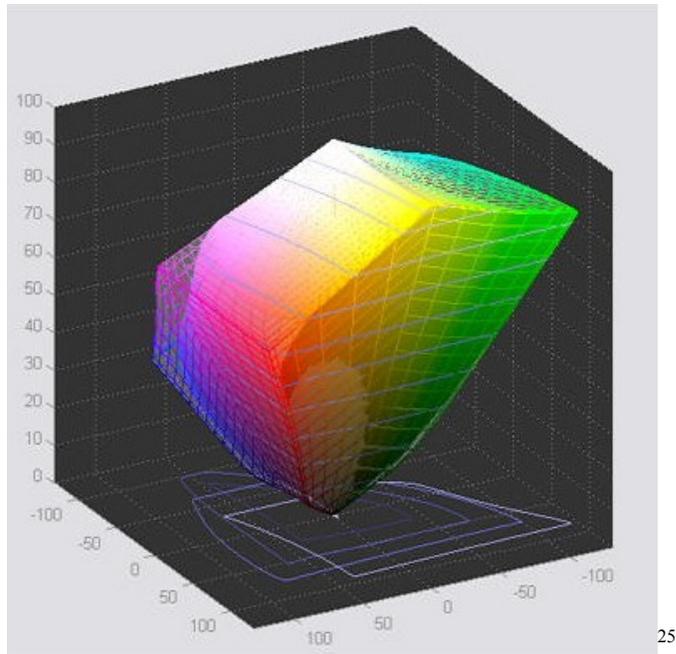
Occasionally, Color space is graphed in three dimensions. There are two general conventions for graphing color space in three dimensions. First, each of the three values of light are given their own axis. These graphs look like this:



The second convention for graphing three dimensional color space is to assign the z-axis a “brightness”. Whiteness is higher on the z-axis and the Blackness is lower on the z-axis.

23 Photo Credit: Wikipedia.com, Article on Color Space

24 Photo Credit: <http://www.gamutvision.com/docs/printest.html>



Graphing color space allows photo editors to analyze the possibilities of color within the photograph as well as the possibilities or liabilities of their individual editing system.²⁶

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THE ART: Edward Burtynsky Photography

²⁵ Photo Credit: http://www.gamutvision.com/docs/camera_scanner.html

²⁶ All information for this section came from personal class notes taken during a Harvard Extension School Class, CS E-7 taught by Dan Armendariz

Edward Burtynsky is a Canadian photographer famous for his bleak pictures of industrial landscapes and cityscapes. His photographs portray a typical landscape presentation but the landscapes that he presents are often marred or ruined by human desecration. Alternatively, he will often take photographs of cityscapes that are quiet, peaceful, abandoned and uninhabited by people. Of his own work, Burtanky says,, "mass consumerism...and the resulting degradation of our environment intrinsic to the process of making things should be of deep concern to all."²⁷



THE MATH: Fractals

27 Burtansky, Edward. *China: The Photographs of Edward Burtansky*. Germany: Steidl, 2005. Page 7

28 Photo Credit: Edward Burtansky, *Monegros*, 2010. Image found in http://www.metiviergallery.com/exhibition.php?exhibition=burtynsky_2011

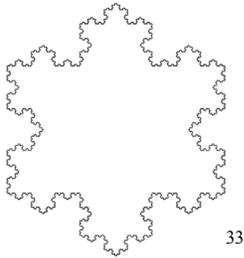
29 Photo Credit: Edward Burkansy. Found in <http://www.thephtoargus.com/photographers/photographer-showcase-edward-burtynsky/>

Edward Burtansky's photographs are interesting from a mathematic point of view because they exhibit strong fractal patterns. A fractal is a set with a non-integer dimension. Among the first of these fractals is the Cantor set, discovered in 1875³⁰. Other famous examples of fractal sets include the Pythagorean tree, Koch Snowflake and the Mandelbrot set. In fact, the term fractal was coined in 1975 by the mathematician Benoit Mandelbrot³¹. Dimensionality can be computed using the following equation: $-\log(n)/\log(r) = \text{dim}(x)$

Where n represents the number of squares of length r that are needed to cover the outer line of the shape.

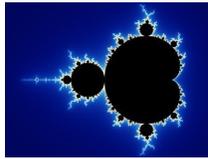
For example: a line is a one dimensional object so if we cover it with 100 squares of length 1/100, we get $-1/-1 = 1$, a single dimension³².

Fractals are those sets that do not have integer answers when solved for dimensionality.



33

Koch Snowflake



34

Mandelbrot Set



35

Pythagorean Tree

30 Information gathered from the Oliver Knill's class handouts, Lecture 10.

31 Pickover, 460.

32 From lecture notes in E-320, Mathematics from a Historical Perspective.

33 Photo Credit: Wikipedia.com, Koch snowflake

34 Photo Credit: Wikipedia.com, Mandelbrot set

35 Photo Credit: Wikipedia.com, Pythagorean Tree.

6

THE ART AND THE MATH: Site Swap and Juggling Patterns

Juggling is the art of throwing objects and catching. The earliest recorded depictions of juggling date back to tomb paintings in Egypt c. 2000 BC. During the Roman Empire, officers used to entertain their troops by juggling. Throughout the middle ages in Europe, juggling was a form of public entertainment. The first modern circus was opened by Philip Astley in 1768³⁶ and today, many jugglers find inspiration in popular circus groups such as Cirque du Soleil, Les Sept Doigts de la Main and Cirque Eloize.



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Juggling patterns are traditionally named by their creators or their visual impact. Cascade is the crossed pattern seen most often and Shower is the name for circular patterns. Other, more complex patterns have names like Mills Mess, The Machine, Tennis, and Burkes Barrage. Boston has its own juggling pattern called The Boston Mess (apparently because it looks like the roads in Boston). These names, while recognized in the United States, are not international and tell very little information about the juggling pattern itself. With the advent of the internet and the possibility to share patterns internationally, using a single language or naming system is no longer practical.

In the early 1990s, a system of juggling counting was pioneered. This system is called Site Swap. It is a counting system using height and time as the base that allows for international communication and almost limitless creation.

36 “History of Juggling” Wikipedia.com, accessed May 2011

37 Photo Credit: Karen Sokolow, 2009



There are 3 basic guidelines for Site Swap:

1. Site Swap assumes that the juggler is alternating hands. It does not matter if the left or right hand starts the pattern.
2. The numbers represent the height each ball achieves during a typical throw. For example, a 3 represents the height the ball attains during a 3 ball traditional pattern. It is possible for 3 balls to be thrown at height 5. Heights above 9 are written alphabetically to avoid confusion. A height of 11, for example, is written as b.
3. Sequences that repeat need not be written. Thus 3333333 can be reduced to 3.

There are several other intricacies of Site Swap, beyond the 3 basic rules.

1. The average of the numbers can tell the juggler how many balls are needed for the pattern. $3/1 = 3$ meaning three balls are needed for this pattern.
2. A 0 represents an empty hand for a beat. Thus, 3 balls thrown at height 5 are written 55500.
3. A 1 represents a horizontal toss
4. A 2 represents a held ball for a beat.

With this in mind, jugglers can analyze string of numbers like a computer code. The sequence 53, for instance, means that the first ball is thrown at height 5 and the second and height three. This is a four ball juggling pattern since, $5+3+5+3 = 16/4$. Confusing and information scarce names can be reduced to a string of recognizable and useful numbers. The trick called Tennis can be described as 52233. The trick called Leak (apparently because it looks like a shower with a leak) can be written as 615150. This type of counting system

also allows almost limitless creation of patterns because the pattern can be created numerically before it is created physically.

This coding for juggling has also led to the creation of a new form of juggling called Prime Patterns. Prime Patterns contain no repeatable subsequences but differ from their number theory counterparts in that there are thought to be a finite number of prime patterns.

Several examples of prime patterns are:

42

441

4440

53502

Site Swap, while not universally known in the juggling community, has introduced exciting possibilities in the creation and transmission of juggling patterns, as well as a new counting system.³⁹

³⁹ All information for the counting system of Site Swap came from personal knowledge and conversations with Ricky Hawkins, Michael Mucciolo, Tim Ellis and Jeremy Warren, professional jugglers and circus artists



7

THE ART AND THE MATH: Adjacent Matrices and Advanced Juggling Patterns

Although Site Swap is a powerful tool when it comes to creating and transmitting juggling patterns, it has several drawbacks. First, it does not account for drops. Second, it can only transmit single pieces of information.

In 1995, Martin Probert, professional juggler and hobbyist mathematician, published an online book called *Four Ball Juggling*

40 Photo Credit: Karen Sokolow, 2009

which introduced the idea of mapping juggling patterns with adjacent matrices. Later, Charles Dancey refined Probert's ideas and introduced a useful tool for mapping juggling patterns and potential.⁴¹

Matrices have existed since ancient times, but the term "matrix" was not applied until 1850 by James Joseph Sylvester (1814-1897). A matrix, at its most general, is any place where something, like an answer to a set of data, can be formed or produced⁴².

A juggling matrix looks like this:⁴³

```

1111 . . . . . . . . . . . . . . + 4
1110 . . . . . . . . . . . . . . 4 X -
1101 . . . . . . . . . . . . . . 4 + X . .
1100 . . . . . . . . . . . . . . 4 . X - . .
1011 . . . . . . . . . . . . . . 4 . + X . . .
1010 . . . . . . . . . . . . . . 4 . . X - . . .
1001 . . . . 4 . . . . . . . . . . + X . . . . .
1000 . 4 . . . . . . . . . . . . . . X - . . . . .
0111 . . . . . . . . + 3 . . . . 2 . 1 0 .
0110 . . . . . . . . 3 X - . 2 . . 0 . . . .
0101 . . . . 3 + X . . . . . 1 0 . . . . .
0100 . 3 . . . X - . . . . . 0 . . . . .
0011 . . + 2 . 1 0 . . . . . . . . . .
0010 . 2 X - 0 . . . . . . . . . . .
0001 + 1 0 . . . . . . . . . . .
0000 0 - . . . . . . . . . . .

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0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1
1 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1
0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1
0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1

```

Above is a four ball juggling pattern. Although this seems overwhelming at first, the interpretation is rather simple. The 0 represents "ball not in hand" and the 1 represents "ball in hand". This is often called a landing schedule. The numbers within the matrix represent the height of the balls, as in Site Swap. The plus and minus symbols were added by Dancey to represent picked up and dropped balls. Imagine that you start with no balls (state 0000) and then pick up a ball. You are now in state 0001. By tracing routes through the matrix, you can account for any pattern with any dropped or picked up ball. This way of representing juggling patterns can be expanded to account for any number of balls, what happens with those balls and possible and impossible patterns.

41 The text to the online book can be found at juggling.org

42 From ualr.edu

43 From Dancey's interpretation of *Four Ball Juggling*

8

THE ART: Aerial Silks

Aerial Silks is a circus art where acrobats perform dances and contortions suspended from the ceiling on long strands of fabric, called tissue. Aerial Silks is, perhaps, the youngest of the circus arts. It was invented in the French Circus School in 1959 when a student chose to present her final rope climbing act on a blot of cloth. Although the art form started in 1959, it was not popularized until 1998 when two Cirque du Soleil performers performed it in the show *Quidam*. Since then, there has been an explosion of this grateful form of dance⁴⁴.

⁴⁴ From suspenseaerial.com



45

THE MATH: Knot Theory

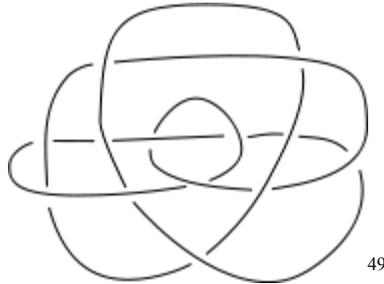
One of the major components in Silks is the creation of knots meant to support the artist as they are suspended from the air. Knot theory, unlike Aerial Silks, has a long history. The most striking examples of ornamental knots can be found in the *Book of Kells* produced by Celtic monks in about 800AD⁴⁶. The study of knots by mathematicians has been occurring for centuries. In 834, the problem of the Borromean Rings first came into existence. This consists of three rings interlocked such that if any one ring were cut, all three would fall apart. The study of knots was codified in the 1800s when Peter Guthrie Tait (1831-1901)⁴⁷ began tabulating all knots and their variations. He painstakingly drew hundreds of knots and then eliminated all invariants, those knots that were the identical. For

45 Photo Credit: Tim Ellis, 2010

46 Pickover, 24.

47 Pickover, 86.

years, the only way of codifying knots was by tabulation. In 1974, Vaughan Frederick Randal Jones invented a number to discover invariance called the Jones Polynomial⁴⁸. Now, Knot theory is integrated in the study of DNA, string theory, quantum mechanics and other important aspects of the fabric of reality.



With regards to silks, the concept of the Unknot becomes particularly important. An unknot is a knot that looks tangled, but is actually a closed loop. This allows it to be untangled. Most knots used in Silks are unknots because they need to be tied and untied quickly. The other mathematical concept used by Aerialists is the Sheepshank Conundrum. This is a knot studied because it is only a knot when tension is applied. In theoretical mathematics, this causes consternation, but in Aerial Silks it creates strength and safety with the weight of the performer and then unravels easily when the artist needs it to.

48 Pickover, 478.

49 Photo Credit: Wikipedia.com, knot theory



50 Photo Credit: Tim Ellis, 2010

THE ART: Poi Spinning

Poi is a dynamic circus art that originated with the Maori people in New Zealand. Poi is the Maori word for “ball”. Poi are ball hung from cords that are held in either hand and spun. It was originally used by women to keep their hands flexible and by men to build strength⁵¹. Traditionally poi were made of flax but today they are more commonly made from chain and Kevlar fabric so the ends can be soaked in camping fuel and burned. The burning of the ends makes patterns when the poi are spun.



52

THE MATH: Epicycloids

51 From homeofpoi.com

52 Ibid.

The patterns created by poi are forms of epicycloids. Epicycles were first described by Ptolomy (140 CE) to describe a path traced by a point on a circle rotating around another circle⁵³. Ole Roemer named the epicycloid in 1674 and the first curves were noted by Bernoulli in 1725. Epicycloids have interested astronomers and mathematicians like Kepler, Newton, Euler and others⁵⁴ who look to the heaven for similar patterns. The equation that describes epicycloids is written as:

$$\begin{aligned}x &= (a + b) \cos t + b \cos \left\{ \left[\frac{a + b}{b} \right] t \right\} \\y &= (a + b) \sin t + b \sin \left\{ \left[\frac{a + b}{b} \right] t \right\}\end{aligned}$$

Where a is radius 1, b is radius 2 and t is the angle⁵⁵.

In the case of poi, one circle is described by the radius of the arm and the other by the radius of the chain of poi. Poi spinning is a complicated form of epicycloids because there are actually two overlapping elements: Two arms, two chains and two angles. Both can move independently of one another, but most performers prefer for the two to move in slightly different rhythms so the epicycloid forms a complete picture.

53 Stillwell, 34.

54 Stillwell, 249-251.

55 Lindsley, Linda. *The Epicycloid*. Found on:
<http://online.redwoods.cc.ca.us/instruct/darnold/calproj/sp99/lindal/epicycloid.html>



56



57

56 Photo Credit: Karen Sokolow, 2010

57 Photo Credit: Karen Sokolow, 2009



10

THE ART: Acrobalancing

58 Photo Credit: Tim Ellis, 2010

Acrobalancing, also called acro-yoga, partner balancing and adagio, is the circus art of balancing to people together. There is no clear history of acrobalancing, but most think that it started with the advent of the circus. The earliest recording of circus performances date back almost 2500 years. In China, performances involving acrobatic have been recorded as early as 475 BC during the Warring States Period. Now it is a part of many dance and circus performances.



59



60

THE MATH: Vectors

Like juggling, Acrobalancing has a problem with transmission. Balances in different parts of the country do not have the same names and there are no standard international names. Unlike juggling's Site Swap, though, no one has yet attempted to codify acrobalancing. I have started a project to incorporate position graphs and vector graphs to help standardize acrobalance transmission.

From a mathematic perspective, vectors are weighted forces. Newton (1687) helped initiate the use of vectors with his laws of motion⁶¹. Since forces add vectorally, it became possible to speak

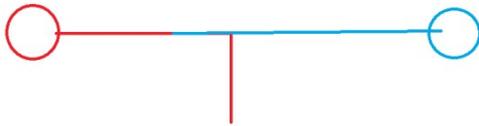
59 Photo Credit: From offthegroundcircus.com

60 Photo Credit: Ibid

61 Stillwell, 248

about forces acting together or in opposition. Vector analysis as we know it now was created by Oliver Heaviside by separating the scalar part of the vector from the imaginary part of the vector⁶².

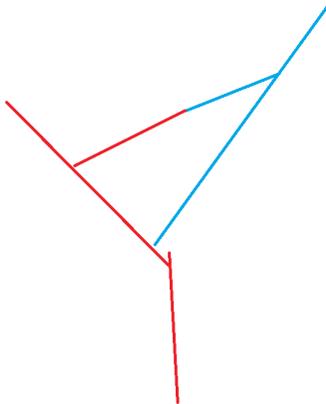
Although it is a work in process, below is a schematic of how the new graphing system I am working on would look:



⁶³

The red represent one person, the blue another. In this case, the vector ratio between the two are in balance.

In the picture below, the math is a little more complicated, The weight is generally distributed unevenly, but the angle of the two, when added, equals a vertical point meaning that this, too, is a stable balance. Though it is a work in progress, should this type of notation work, it would allow for greater sharing, creation and safety in acrobalance.



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62 Stillwell, 436

63 Photo Credit: Created by Karen Sokolow

64 Ibid.