

## Lecture 3: Geometry

### The butterfly theorem

Draw an arbitrary chord  $AB$  in a circle. Now draw two new arbitrary chords  $PQ, RS$  through the center  $M$  of  $AB$ . The line segments  $PR$  and  $QS$  now cut the chord  $AB$  in equal distance.

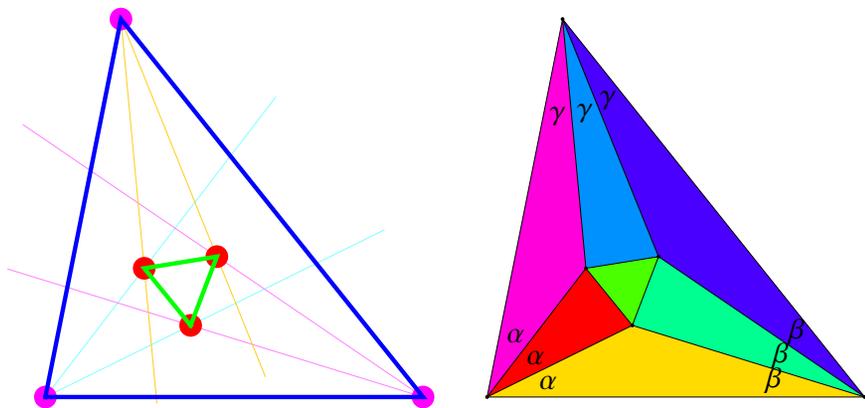
We will look at some aspects of this theorem in class.

### Morley's miracle

The following theorem was discovered in 1899 by **Frank Morley** at Haverford college near Philadelphia.

If one trisects the angles of a triangle, the corresponding trisector intersections form an equilateral triangle.

It is a beautiful result because it is not obvious, or even surprising. **John Conway** found an elegant proof: write  $a'$  for the angle  $a + \pi/3$  and  $a''$  for  $a + 2\pi/3$ . Build 7 triangles with angles  $(0', 0', 0'), (a, b', c'), (a', b, c'), (a', b', c), (a, b, c''), (a, b'', c), (a'', b, c)$  and cyclic. The triangles  $(a, b', c'), (a', b, c), (a', b', c)$  are determined by assuming their shortest side length is the one from the equilateral triangle  $(0', 0', 0')$ . The other three are required to have the same height than the triangle  $(0', 0', 0')$ . These 7 triangles can be put together to a large triangle with angles  $(a, b, c)$ .

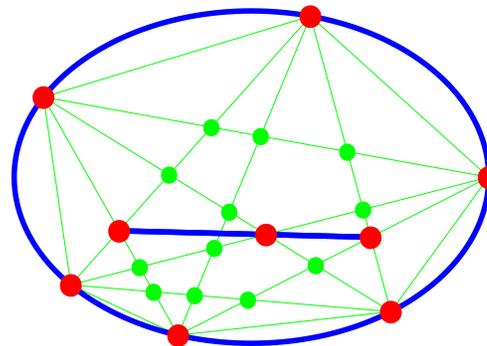


### Pascal's mystic hexagram

The following result has been found in 1640 by Pascal, when he was 17. He probably got the problem from his father who was a friend of Desargues. See Stillwell "mathematics by its history" page 95.

Pairs of opposite sides of a hexagon inscribed in a conic section meet in three collinear points.

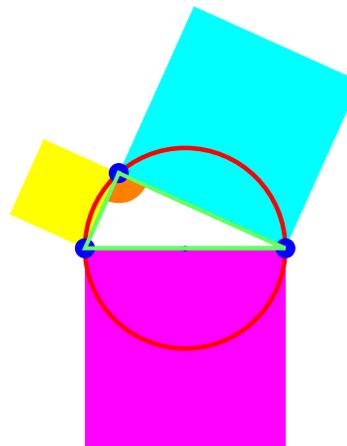
Also this result is not obvious. Pascal probably proved it first for circles. Applying a linear transformation on the picture preserves the linear incidence structure and gets it for all. We can also be seen as a consequence of the Pappus-Pascal theorem.

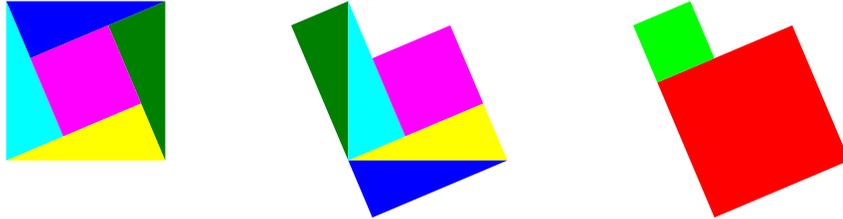


### Pythagoras theorem

For all right angle triangles of side length  $a, b, c$ , the quantity  $a^2 + b^2 - c^2$  is zero.

As shown in class, there are rearrangement proofs.





An other beautiful result is:

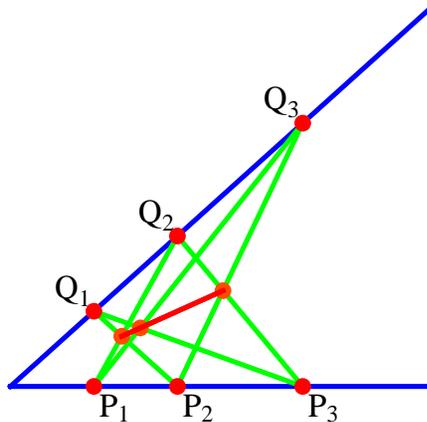
Given a circle of radius 1 and a point  $P$  inside the circle. For any line through  $P$  which intersects the circle at points  $A, B$  we have  $|PO|^2 - |PA||PB| = 1$ .

This is a consequence of Pythagoras. By scaling translation and rotation we can assume the circle is at the origin and that the line through the point  $P = (a, b)$  is horizontal. The intersection points are then  $(\pm\sqrt{1-a^2}, a)$ . Now  $(b - \sqrt{1-a^2})(b + \sqrt{1-a^2}) = b^2 - 1 + a^2$ .

## Pappus theorem

Pappus of Alexandria (290- 350) showed:

Take three points  $P_1, P_2, P_3$  on a first line and three points  $Q_1, Q_2, Q_3$  on a second line. Draw all possible connections  $P_i Q_j$  with  $i \neq j$ . The intersection points of the lines  $P_i Q_j$  and  $P_j Q_i$  are on a line.

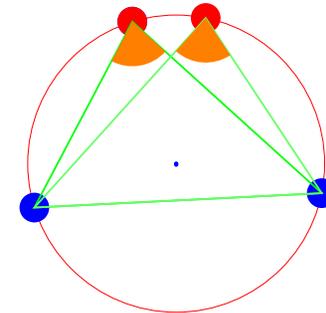


## Thales Theorem

Thales of Miletus (625 BC -546 BC) got the following beautiful result

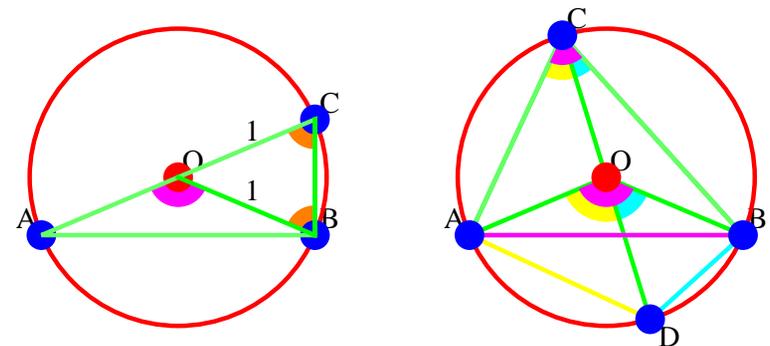
A triangle inscribed in a fixed circle is deformed by moving one of its points on the circle, then the angle at this point does not change.

The result is relevant also because Thales is considered the **first modern Mathematician**. Thales theorem is a prototype of a stability result. We look at a slightly more general case than usual treated which is called the "Fass kreis" theorem in Europe. In this worksheet we want to understand it and prove it.



Let's look first at the case when one side of the triangle goes through the center. a) The triangle  $BCO$  is an isosceles triangle.

b) The central angle  $AOB$  is twice the angle  $ACB$ .

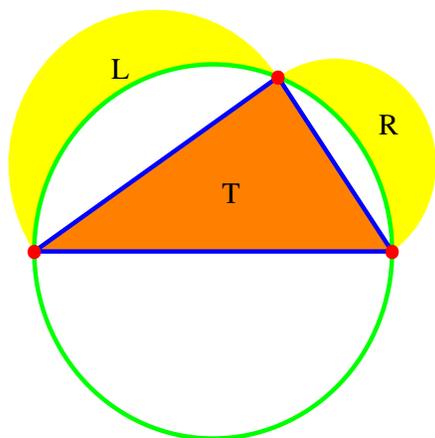


- Here are the steps to see the theorem: c) What is the relation between the angles  $AOD$  and  $ACD$ ?  
 d) What is the relation between the angles  $DOB$  and  $DCB$ ?  
 e) Find a relation between the central angle  $AOB$  and the angle  $ACB$ ?  
 f) Why does the angle  $ACB$  not change if  $C$  moves on the circle?

## Hippocrates Theorem

The quadrature of the Lune is a result of **Hippocrates of Chios** (470 BC - 400 BC) and also called Hippocrates theorem. It is the first rigorous quadrature of a curvilinear area. It states:

The sum  $L + R$  of the area  $L$  of the left moon and the area  $R$  of the right moon is equal to the area  $T$  of the triangle.

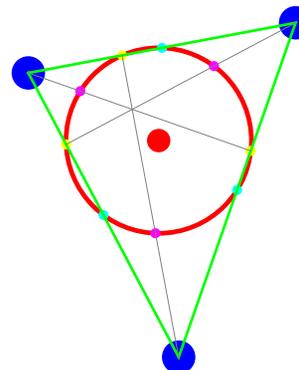


If  $A, B, C$  are the areas of the half circles build over the sides of the triangle, then  $A + B = C$ . If  $U$  is the area of the intersection of  $A$  with the upper half circle  $C$ . and let  $V$  be the area of the intersection of  $B$  with  $C$ . Let  $T$  be the area of the triangle. Then  $\overline{U + V + T = C}$ . Interpret  $L = A - U$  and  $R = B - V$  are the moon areas we can add them up and use the just shown relation to see  $L + R = T$ .

## Feuerbach's Theorem

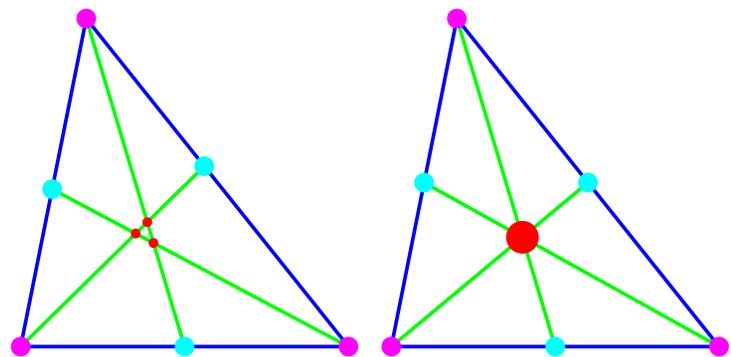
The 3 midpoints of each side, the 3 feet of each altitude and the three midpoints of the line segments from the vertices to the orthocenter lie on a common circle.

This result is attributed to **Karl Wilhelm Feuerbach** (1800-1834), who found a partial result of this in 1822. We will prove it with the computer in class. In the case of an equilateral triangle the midpoints and the height bases are the same and we have only 6 points. The Feuerbach circle is the circle inscribed into the triangle.



## The centroid

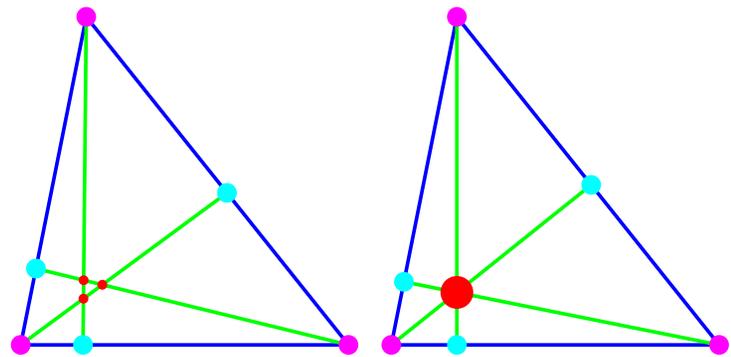
The **centroid** of a triangle is the intersection of the lines which connect the vertices of a triangle with the midpoints of the opposite side. It is not at all trivial that these three lines intersect in one point. It is a stability property of the triangle. Deforming a triangle does not change this property. If  $A, B, C$  are the coordinates of the vertices, then  $(A + B)/2, (A + C)/2$  and  $(B + C)/2$  are the midpoints of the sides. To verify the property just check that with  $P = (A + B + C)/3$ , the points  $A, P, (B + C)/2$  are on a line, the points  $(B, P, (A + C)/2)$  are on a line and the points  $(C, P, (A + B)/2)$  are on a line. There is an easier but more advanced way to see this: check it first for the equilateral triangle. Now, any triangle can be mapped into any other by a linear transformation. Because linear transformations preserve lines and ratios, the intersection property will stay true for all triangles.



To the left, we see the situation as we would expect it without "knowing" that the three intersection points agree. To the right, we see the actual situation.

### The orthocenter

The **orthocenter** is the intersection of the three altitudes of a triangle. Also here - a priori - we have three different points the intersection, for each pair of altitudes. Why do they meet in one point? It is not obvious and was not proven by the Greeks for example. One can take the intersection of two altitudes, get a point  $P$  and form the line from  $P$  to the third point in the triangle. The fact that this line is perpendicular to the third line can be seen by looking at the angles. The angles between two heights is the same as the angle between the two corresponding sides.

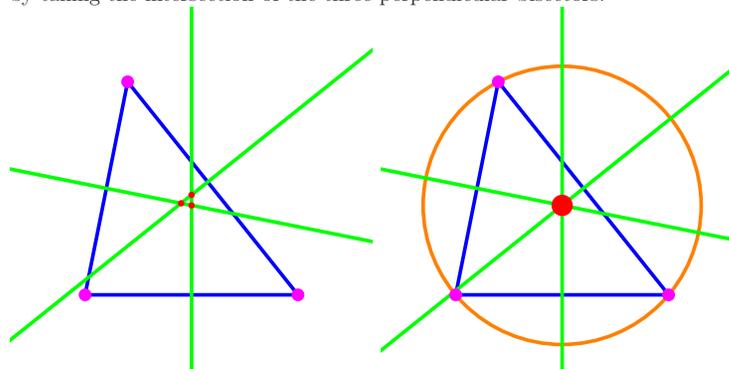


To the left, we see the situation as we would expect it without "knowing" that the three intersection points agree. To the right, we see the actual situation.

### The Center of the Circumscribed circle

Any circle which passes through two points  $A, B$  of a triangle lies on the perpendicular bisector of  $A$  and  $B$ . When moving a point  $M$  on that line and always drawing the circle centered at  $M$

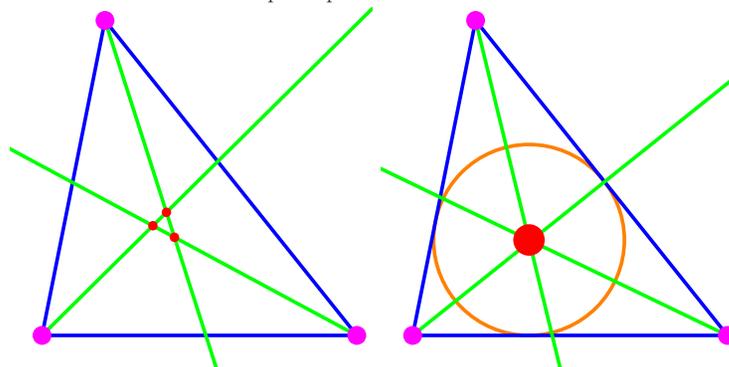
through  $A, B$ , then there will be a moment, where the distance to the third point  $C$  is equal to the distance to  $A$ . We have found the circumscribed circle of the triangle. The point can be obtained by taking the intersection of the three perpendicular bisectors.



To the left, we see the situation as we would expect it without "knowing" that the three intersection points agree. To the right, we see the actual situation.

### The Center of the inscribed circle

Any circle which is tangent to two sides of a triangle lies on the angle bisector at the intersection point of the sides. Take a circle on that line which is tangent to the two sides. If the center is close to the point then the circle is small and inside the triangle. Move the point along the line. There will be a moment, when the circle will touch the third side. This point is the intersection point of all angular bisectors. It is the center of the inscribed circle. The inscribed circle is the circumscribed circle of the pedal points.



To the left, we see the situation as we would expect it without "knowing" that the three intersection points agree. To the right, we see the actual situation.