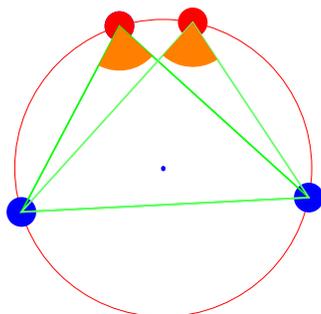


## Lecture 3: Worksheets

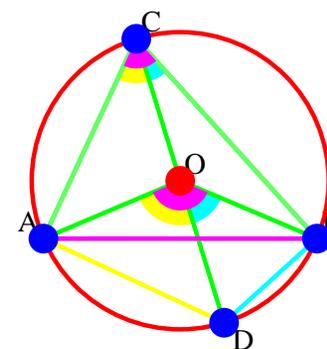
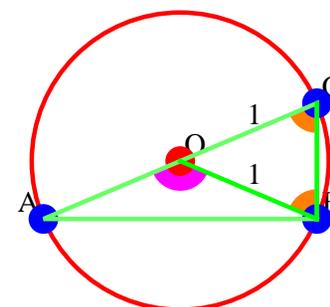
### Thales theorem

It is a result of **Thales of Miletus** (625 BC -546 BC) stating that if a triangle inscribed in a fixed circle is deformed by moving one of its points on the circle, then the angle at this point does not change. The result is relevant also because the discoverer Thales is considered the first modern Mathematician. Thales theorem is a prototype of a stability result. We look at a slightly more general case than usual treated which is called the "Fass kreis" theorem in Europe. In this worksheet we want to understand it and prove it.



Lets look first at the case when one side of the triangle goes through the center.

- The triangle  $BCO$  is an isosceles triangle.
- The central angle  $AOB$  is twice the angle  $ACB$ .

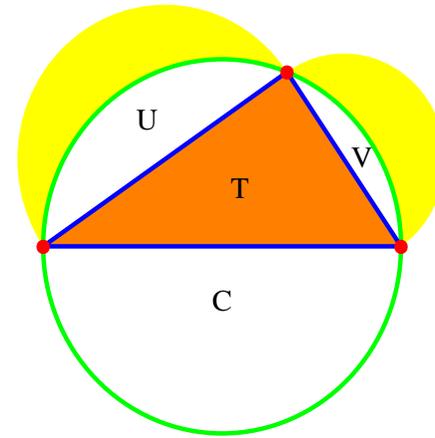
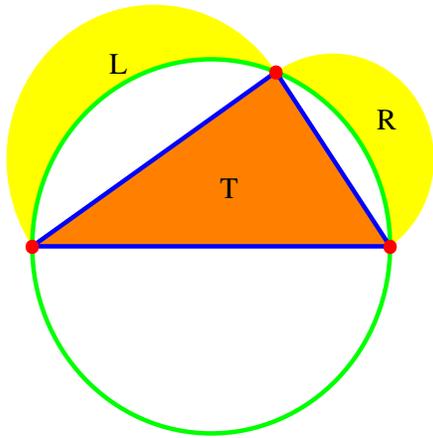


- What is the relation between the angles  $AOD$  and  $ACD$ ?
- What is the relation between the angles  $DOB$  and  $DCB$ ?
- Find a relation between the central angle  $AOB$  and the angle  $ACB$ ?
- Why does the angle  $ACB$  not change if  $C$  moves on the circle?

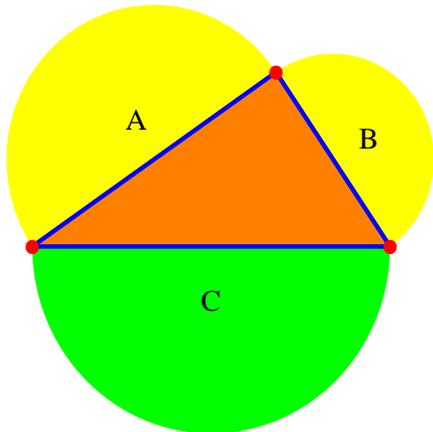
In this worksheet, we prove the quadrature of the Lune, a result of **Hippocrates of Chios** (470 BC - 400 BC). It is the first rigorous quadrature of a curvilinear area.

## 2. Hippocrates theorem

The sum  $L + R$  of the area  $L$  of the left moon and the area  $R$  of the right moon is equal to the area  $T$  of the triangle.



a) Relate first the area of each half circle with the corresponding triangle side length. Let now  $A, B, C$  the areas of the half circles build over the sides of the triangle. Show that  $A + B = C$ . The picture shows the half disc below the hypotenuse.

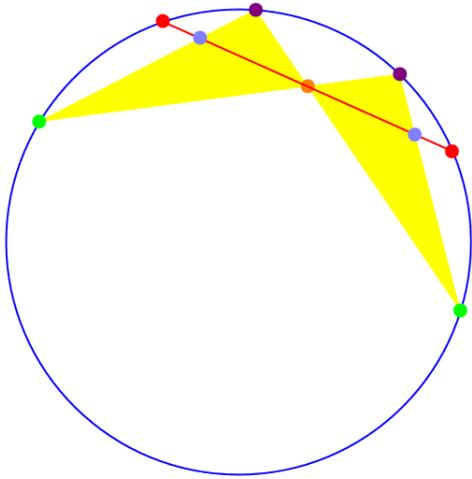


b) Let  $U$  be the area of the intersection of  $A$  with the upper half circle  $C$ . and let  $V$  be the area of the intersection of  $B$  with  $C$ . Let  $T$  be the area of the triangle. Why is  $U + V + T = C$ ?

c) The number  $A - U$  is the area of a region. Which one. Similarly, what is  $B - V$ ? Can you finish the proof of the theorem  $L + R = T$ ?

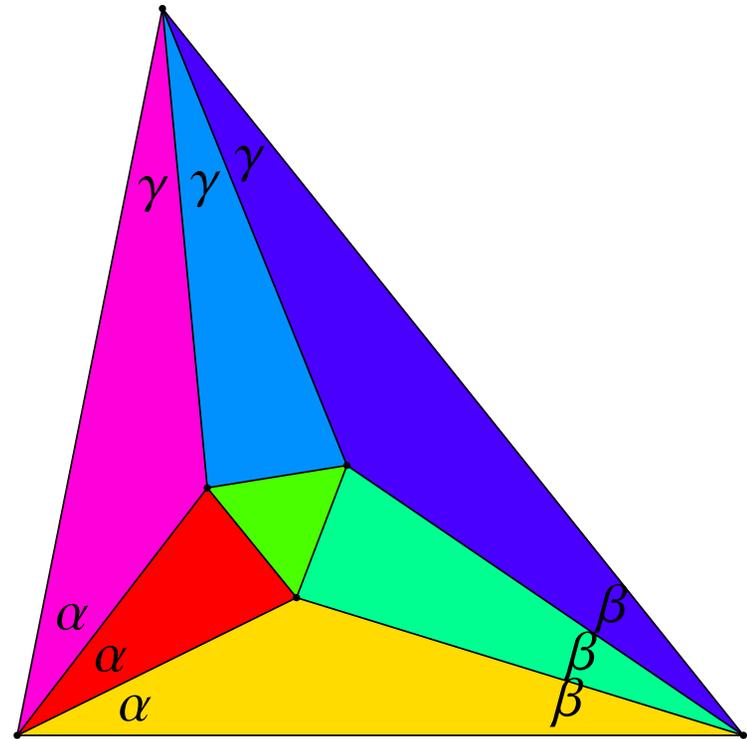
## The butterfly theorem

We want to see why the butterfly wings in the butterfly theorem are similar triangles:



## Morley's miracle

We want to understand the proof of Morley's miracle and build the triangle up with 7 triangles.



## Fasskreis theorem

Given a circle of radius 1 and a point  $P$  inside the circle. For any line through  $P$  which intersects the circle at points  $A, B$  we have  $|PO|^2 - |PA||PB| = 1$ .

Proof with Pythagoras. By scaling translation and rotation we can assume the circle is at the origin and that the line through the point  $P = (a, b)$  is horizontal. The intersection points are then  $(\pm\sqrt{1-a^2}, a)$ . Now  $(b - \sqrt{1-a^2})(b + \sqrt{1-a^2}) = b^2 - 1 + a^2$ .