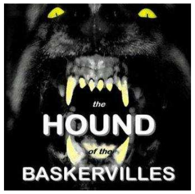
Lecture 8: Probability

1. Combinatorics

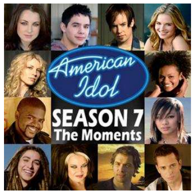
Here are the most important combinatorics problems:

How many ways are there to:	The answer is:
permute n elements	$n! = n(n-1)...2 \cdot 1$
choose k from n with repetitions	n^k
pick k different from n if order matters	$\frac{n!}{(n-k)!}$
pick k different from n where order does not matter	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Problem 1: You play "scrabble". You are stuck with the letters *STORY*. How many single words of length 5 can you write?



Problem 2: the "Hound of Baskervilles" has 338'787 words. You can assume an alphabet of 30 including space and punctuations.



Problem 3: How many ways are there to chose 3 people from a contestant group of 12 if the order does not matter?

Problem 4: A combination lock has 40 numbers $0-39$. A lock combination consists of 3 different numbers, where the order matters. How many different lock combinations are there?

The Monty Hall Problem

We want to understand the famous **Monty Hall problem**



You have to choose from three doors. Behind one door is a car and behind the others are goats. You pick a door. The host, who knows what's behind the doors, opens an another door, one which has a goat. You have the choice to choose the door or to switch. What is better?

The "three door problem" is also called **"Monty Hall problem"** and became a sensation and controversy in 1991. Intuitive argumentation can lead to the conclusion that it does not matter whether to change the door or not. When asked, a large majority of test persons tell that it does not matter.

1) We first assume that we decide not to switch.
You choose a door. Note that the revelation of the host does not affect your choice.

What is the probability that you win in this case?

2) Now we switch. We look at three possibilities now.
What happens if you initially chose the door with the car? Do you win or lose in this case?

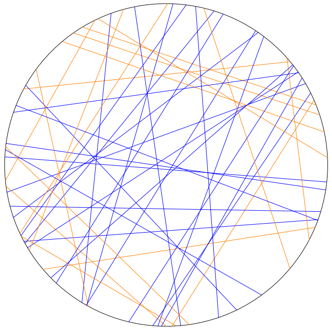
3) What happens if you initially chose the door with the goat? Do you win or lose in this case?

We have computed the winning probability with no switching and then in 2-3) the winning probability with switching. What do you conclude?
4) For the following question, most people would say 1/2.

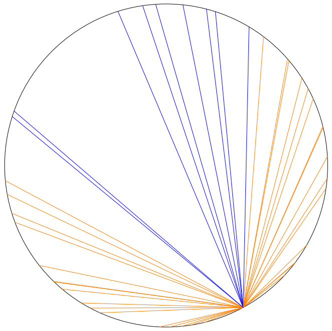
Dave has 2 kids. One of them is a boy. What is the probability that the other kid is a girl?

The Bertrand paradox

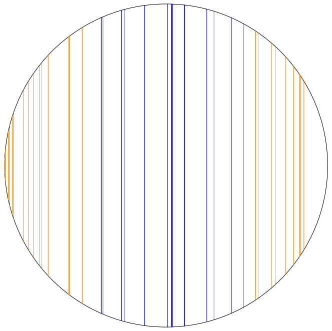
The **Bertrand paradox** illustrates that one has to be clear on how to setup a probabilistic model in a concrete situation.



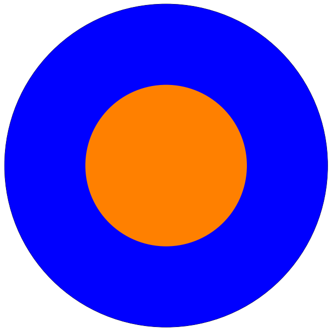
We throw random lines onto the unit disc. What is the probability that the line intersects the disc with a length $\geq \sqrt{3}$, the length of the inscribed equilateral triangle?



Problem 1) Take an arbitrary point on the boundary of the disc. The set of all lines through that point are parameterized by an angle ϕ. For which midpoints is the length of the chord longer than the equilateral triangle length? By comparing angles, what is the probability?



Problem 2) Take now all lines perpendicular to a fixed diameter. The entire diameter has length 2. Where do the chords hit the diameter so that it is longer than $\sqrt{3}$? By comparing lengths, what is the probability?



Problem 3) Look at the midpoints of the chords. Where does such a midpoint have to be so that the chord is longer than $\sqrt{3}$? By comparing areas, what is the probability now?

The Petersburg Paradox

The origins of probability was in gambling. We look here closely at the Petersburg paradox, which had been devised by Daniel Bernoulli in 1738. You pay a fixed entrance fee C and you get the prize 2^T, where T is the number of times, the casino flips a coin until "head" appears.
For example, you enter 10 dollars. If the sequence of coin experiments would give "tail, tail, tail, head", you win $2^3 - 10 = 8 - 10 = -2$ dollars. This means you have lost 2 dollars in this game.

1) Build groups of 2-4. One is the casino, the others play the casino. Choose an entrance fee which you think is fair and play as many times as time allows. In the end, record your winning.

2) What is the probability that you lose your entire winning? That is, what is the chance that we have "head" the first time? Note that $T = 0$ in this case.

3) What is the probability that we have "head" the second time? Note that $T = 2$ in this case. How much do we win or lose in this case?

4) What is the probability that "head" appears the third time? Note that $T = 3$ in this case. How much did you win or lose in this case?

5) What is the probability that "head" appears at time $T = n$ the first time? How much did you win or lose in this case?

Fair would be an entrance fee which is equal to the expectation of the win, which is $1 \cdot P[T = 0] + 2 \cdot P[T = 1] + 5 \cdot P[T = 2] + ...$ What does "fair" mean? For example, the situation $T = 20$ is so improbable that it never occurs in the life-time of a person. Therefore, for any practical reason, one has not to worry about large values of T. This, as well as the finiteness of money resources is the reason, why casinos do not have to worry about the following bullet proof **martingale strategy** in roulette: bet c dollars on red. If you win, stop, if you lose, bet $2c$ dollars on red. If you win, stop. If you lose, bet $4c$ dollars on red. Keep doubling the bet. Eventually after n steps, red will occur and you will win $2^n c - (c + 2c + \cdots + 2^{n-1}c) = c$ dollars.

How does one solve the Petersburg paradox? What would be a reasonable entrance fee in "real life"? Bernoulli proposed to replace the expectation $\mathrm{E}[G]$ of the profit $G = 2^T$ with the expectation $(\mathrm{E}[\sqrt{G}])^2$, where $u(x) = \sqrt{x}$ is called a **utility function**. This would lead to a fair entrance

$$(\mathrm{E}[\sqrt{G}])^2 = (\sum_{k=1}^{\infty} 2^{k/2} 2^{-k})^2 = \frac{1}{(\sqrt{2}-1)^2} \sim 5.828... .$$

Similar effects appear in political situations as in **voting systems**, where different voting systems can produce different winners. The following example is by Donald Saari:

"Consider 15 people deciding what beverage to serve at a party. Six prefer milk first, wine second, and beer third; five prefer beer first, wine second, and milk third; and four prefer wine first, beer second, and milk third. In a plurality vote, milk is the clear winner. But if the group decides instead to hold a runoff election between the two top contenders milk and beer, then beer wins, since nine people prefer it over milk. And if the group awards two points to a drink each time a voter ranks it first and one point each time a voter ranks it second, suddenly wine is the winner."