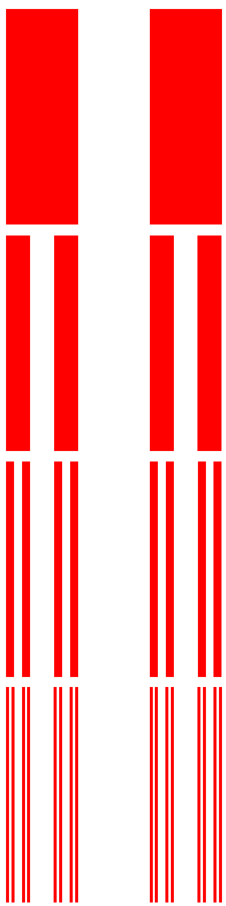
Lecture 10: Analysis

The Cantor Set

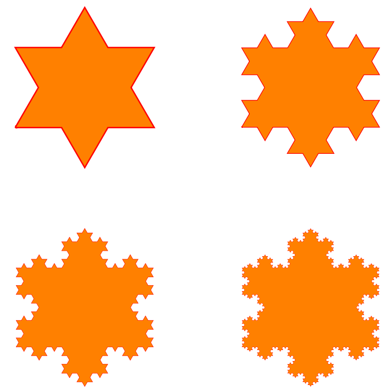


Analysis is a huge field. Our goal is to understand **fractals**, objects with fractional dimension. Fractals enter many parts of analysis: spectral theory, harmonic analysis, calculus of variations, functional analysis. But because these fields need some time to learn and explain, the subject of fractals looks like a nice entry point. The story becomes so mostly pictorial. In essence, we want to understand the formula

$$\dim(X) = \frac{-\log(n)}{\log(r)} \, .$$

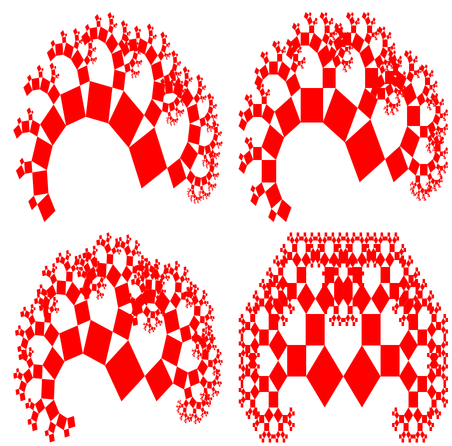
It tells that if we want to find the dimension of an object we cover it with boxes of size r and count how many we need: n. The dimension is what happens if r goes to zero. The prototype of a fractal is the **Cantor set** which was discovered in 1875 by **Henry Smith** Start with the unit interval. Cut the middle third, then cut the middle third from both parts then the middle parts of the four parts etc. What is left in the end is the Cantor set for which the dimension is $\log(3)\ \log(2)$.

The Koch Curve



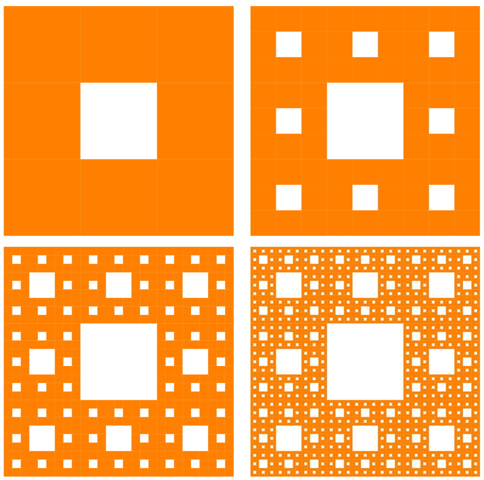
The **Koch snowflake** is an example of a fractal, where the dimension is between 1 and 2. It was first described by the Swedish mathematician Helge von Koch (1870-1924). The Koch curve was described by him in 1904. It is a simple model for a **snowflake**.

The Tree of Pythagoras



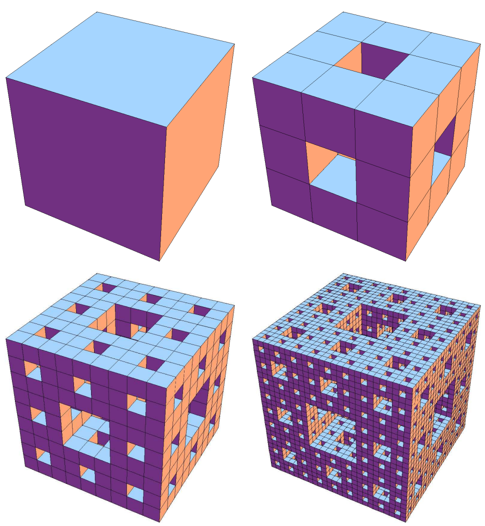
The **tree of Pythagoras** is an example of a fractal, where the dimension is between 1 and 2. It was first described by the Swedish mathematician **Helge von Koch** (1870-1924). The Koch curve was described by him in 1904. It comes close to actual **trees**. It inspired antenna designs.

The Sierpinski Carpet



The **Sierpinski carpet** is a fractal in the plane. Its dimension is $\log(8)/\log(2)$. It was described by **Waclav Sierpinski** in 1916.

The Menger Sponge



The **Menger sponge** is a fractal in space. Its dimension is between 2 and 3. It was first described by Karl Menger (1902-1985). Its dimension is $\log(20)/\log(3)$ which is about 2.7.

The Mandelbrot set

We introduce complex numbers $z = a + ib$ and define complex multiplication

$$(a + ib)(u + iv) = au - bv + (av + bu)i$$

Now look at the map $T(z) = z^2 + c$ where c is a fixed complex number. Start with $z = i$ for example, we get $T(z) = i + c$ and $T^2(z) = T(T(z)) = (i + c)^2 + c$ etc. The **Mandelbrot set** is the set of complex numbers $c = a + ib$ for which $T^n(0)$ stays bounded. The **filled in Julia set** J_c of c is the set of z such that $T^n(z)$ stays bounded. The **Julia set** is the boundary of that set.

For example, for $c = 0$, the map is $T_0(z) = z^2$. Since $|z^n| = |z|^n$ we see that the disc $\{|z| \leq 1\}$ is the filled in Julia set for $c = 0$ and the unit circle $\{|z| = 1\}$ is the Julia set.

The following picture (Peitgen-Richter-Saupe) shows the Mandelbrot set in the c plane and a few Julia sets. The circle is shown at the bottom.