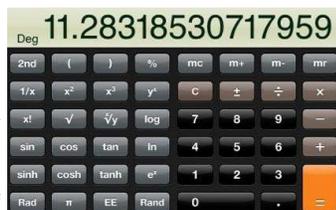


## Lecture 12: Dynamical systems and Chaos

### Chaos

Simple transformations can produce chaotic outcome. Make sure your calculator is in the "Rad" mode. Remember that  $2\pi$  radians is equal to 360 degrees. You can check whether your calculator is in Radian mode, by computing  $\cos(\pi)$  and get  $-1$ . Make sure your calculator is in rad mode. Use a scientific calculator. In the iPhone calculator for example, turn the device to get to the scientific mode.



The Scientific Calculator built in by default in the Iphone/Ipod appears when you turn the device.

### Order and Chaos with the calculator

1) Take a calculator, and pushing repetitively the button cos. What do you observe?

2) Now repeat pushing the sin x button. What do you observe?

3) Now push  $x^2$  repetitively.

4) What do you see if you push the buttons sin, then type  $1/x$  and repeat this process again and again?

5) Experiment with the button tan. Also here, change tan and cot.

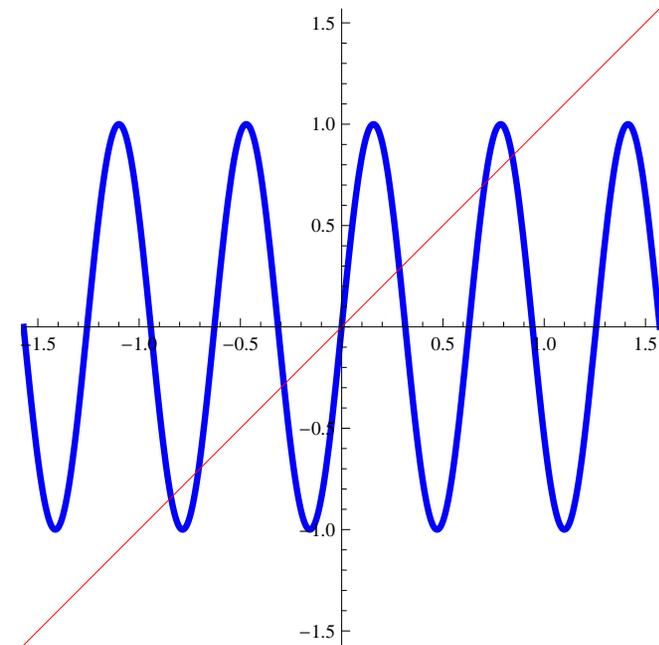
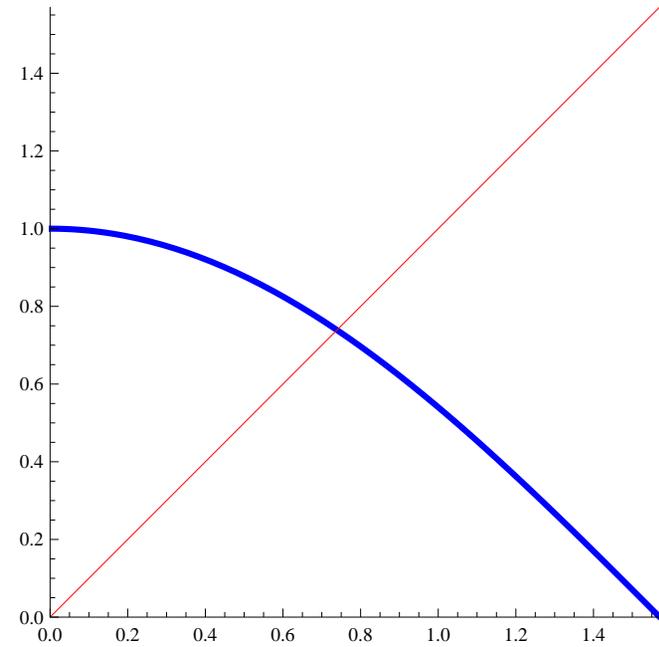
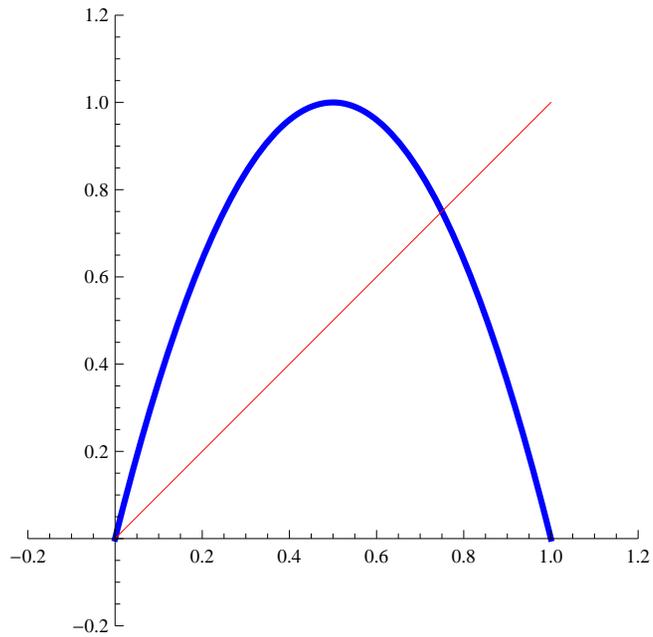
6) Look for other "chaotic" key combinations? Experiment also with Deg and Rad changes.

## Part II: The Cobweb construction

### 1. Objective

We graphically compute a few iterates of one dimensional maps.

### 2. Stability



## Lecture 12: The Ulam-Collatz system

### 1. Objective

We look at a dynamical system of number theoretical nature.

### 2. The Collatz system

In the **Collatz system**, we start with an integer and map it with the following rule:

$$T(x) = \begin{cases} x/2 & x \text{ even} \\ 3x+1 & x \text{ odd} \end{cases}$$

The question is whether the orbit always ends up with 1.

For example:  $x = 7$  produces 7, 22, 11, 34, 17.

### 3. Experiment

1) Start with the initial condition 26:

2) Start with the initial condition 9:

3) Start with the initial condition 2048:

4) What is wrong with the following proof of the Collatz conjecture?

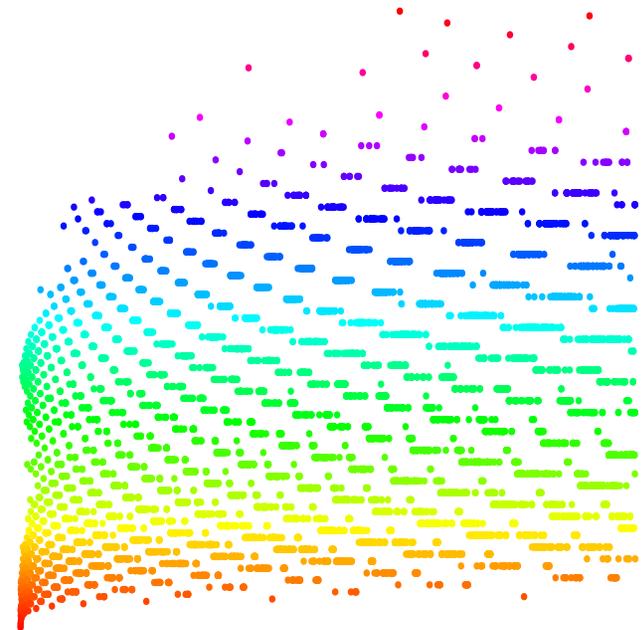
Proof. Consider only the odd numbers in the Collatz sequence. We show that each odd number is in average  $3/4$  times smaller than the previous one:

With probability  $1/2$  the number  $3x+1$  is divisible by 2 and not 4: this increases  $x$  by  $3/2$

With probability  $1/4$  the number  $3x+1$  is divisible by 4 and not 8: this decreases  $x$  by  $3/4$

With probability  $1/8$  the number  $3x+1$  is divisible by 8 and not 16: this decreases  $x$  by  $3/8$

To compute the probability, we take logarithms and compute  $a = \sum_{n=1}^{\infty} \frac{1}{2^n} \log(3/2^n)$ . The average decay rate of the size of a number is the factor  $e^a = 3/4$ .



3) The Collatz system certainly can be modified. Can you find one, for which there is a nontrivial loop?

# Lecture 12: Part IV: Cellular automata

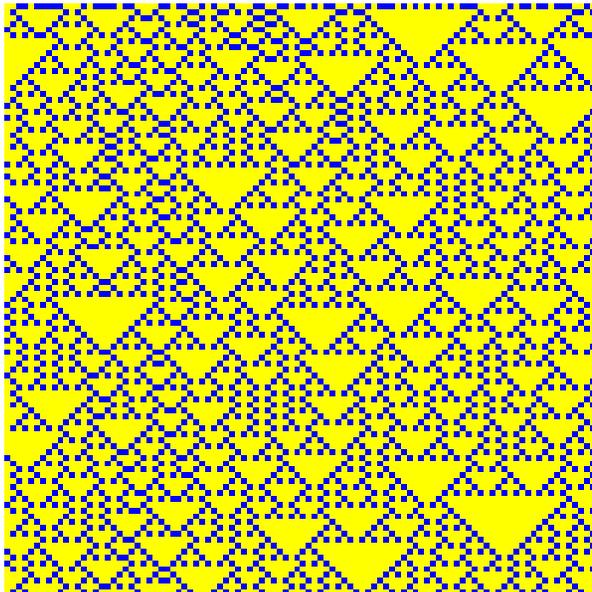
## 1. Objective

We look at a dynamical systems called Cellular automata. These are continuous maps on sequence spaces in which the evolution rule is translational invariant.

## 2. The Rule 18 CA

neighborhood	new middle cell
111	0
110	0
101	0
100	1
011	0
010	0
001	1
000	0

Because only cell neighborhoods of the form 100 and 001 lead to an offspring 1, we and  $100 = 4, 001 = 1$  in binary, we have  $2^4 + 2^1 = 18$ .



## 3. Run it

1	0	1	0	0	0	1	0	0	1	0	1	1	0	1	0

### Cellular Automata Offer New Outlook on Life, the Universe, and Everything

What kind of world do we live in? The question has been bugging about for thousands of years by philosophers, theologians, and politicians. More recently a collection of folk shows hints have weighed in on the subject. So far, no one's come up with an answer that everyone can agree on. Mathematicians have considered the same question. But where others worry over the blurred boundaries of Good and Evil, mathematicians ponder a sharper dichotomy: the Continuous versus the Discrete.

Continuous mathematics, exemplified by calculus and differential equations, has long dominated mathematical descriptions of the world. But discrete mathematics is making a bid for primacy. With modern computers, researchers have discovered astonishingly complex behavior in seemingly simple, finite systems. The results have led some theorists to speculate that discrete models, which lend themselves to digital computation, are the "right" way to study nature.

Erica Jen, a mathematician at Los Alamos National Laboratory in Los Alamos, New Mexico, is one of a growing number of researchers who believe that discrete mathematics can mirror many aspects of physical reality fully as well as the more customary continuous theories. Jen has been studying mathematical properties of discrete systems called cellular automata. These systems, she says, are useful models for many types of complex physical, chemical, or biological systems. They also have an amazing life of their own.

Cellular automata "exhibit an extremely rich and diverse range of pattern formation," Jen says. Among the most interesting are "self-organizing" patterns. Highly structured patterns that seem to emerge spontaneously from a "random soup" of random binary



Erica Jen. (Photo courtesy of Erica Jen.)

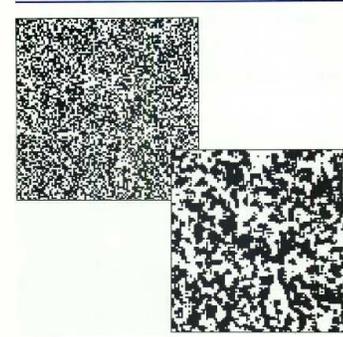


Figure 2. A  $100 \times 100$  "snapshot" of cellular automata proceeds from a random initial state (top) to a stable state (left) to a noisy state (right). The cells are black, white, or gray. The value of the cell at time  $t$  and position  $(x, y)$  is the value of the cell at time  $t-1$  and position  $(x, y)$  plus the value of the cell at time  $t-1$  and position  $(x+1, y)$  plus the value of the cell at time  $t-1$  and position  $(x-1, y)$ .

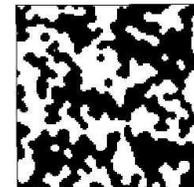
digits. Jen and her colleagues hope to understand exactly how "self-organizing" patterns arise and precisely what properties they possess. By studying cellular automata with mathematical tools from areas such as abstract algebra and number theory, Jen hopes to bring theoretical rigor to a subject that is often as much art as science.

Loosely speaking, a cellular automaton is a "by-product" of space and time. Instead of seeing continuously from point to point and moment to moment, cellular automata consist of discrete "cells" with discrete values that change instantaneously at discrete intervals, much like frames in a movie. The spatial frame, moreover, is a grid that permits the study of each cell's value change depending on the values of nearby cells.

The possible state for complete is a "neighboring state." Each cell in a system of black and white squares could be programmed to switch color if the majority of its immediate neighbors are in the opposite color (see Figure 2). Another rule might specify that the value of each cell change in the sum of the values of the cells surrounding it, or, reflecting things in black and white, again, to the parity of the sum (black could be odd and white even).

"The essential feature of cellular automata is that they are discrete in space, time, and state values; they evolve according to local interaction rules; and their rules apply

Jen hopes to bring theoretical rigor to a subject that is often as much art as science.



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