

## Lecture 4: Number Theory

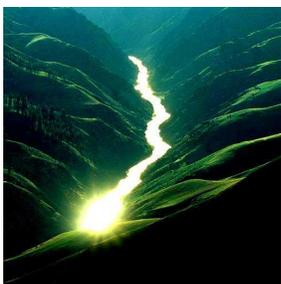
### Twin prime conjecture



There are infinitely many prime twins  $p, p + 2$ .

The first twin prime is  $(3, 5)$ . The largest known prime twins  $(p, p + 2)$  have been found in 2011. It is  $3756801695685 \cdot 2^{666669} \pm 1$ . There are analogue problems for **cousin primes**  $p, p + 4$ , **sexy primes**  $p, p + 6$  or **Germaine primes**, where  $p, 2p + 1$  are prime.

### Goldbach conjecture



Every even integer  $n > 2$  is a sum of two primes.

The Goldbach conjecture has been verified numerically until  $4 \cdot 10^{18}$ . It is known that every sufficiently large odd number is the sum of 3 primes. One believes this "weak Goldbach conjecture" for 3 primes is true for every odd integer larger than 7.

### Andrica conjecture



The prime gap estimate  $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$  holds.

For example  $\sqrt{p_{1000}} - \sqrt{p_{999}} = \sqrt{7919} - \sqrt{7907} = 0.067\dots$ . An other prime gap estimate conjectures is **Polignac's conjecture** claiming that there are infinitely many prime gaps for every even number  $n$ . It is stronger than the twin prime conjecture. It includes for example the claim that there are infinitely many cousin primes or sexy primes. **Legendre's conjecture** claims that there exists a prime between any two perfect squares. Between  $16 = 4^2$  and  $25 = 5^2$ , there is the prime 23 for example.

### Odd perfect numbers

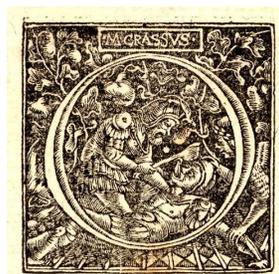


Probably the oldest problems in mathematics is the question

There is an odd perfect number.

A perfect number is equal to the sum of all its proper positive divisors. Like  $6 = 1 + 2 + 3$ . The search for perfect numbers is related to the search of large prime numbers. The largest prime number known today is  $p = 2^{43112609} - 1$ . It is called a Mersenne prime. Every even perfect number is of the form  $2^{n-1}(2^n - 1)$  where  $2^n - 1$  is prime.

### Diophantine equations



Many problems about Diophantine equations, equations with integer solutions are unsettled. Here is an example:

Solve  $x^5 + y^5 + z^5 = w^5$  for  $x, y, z, w \in \mathbb{N}$ .

Also  $x^5 + y^5 = u^5 + v^5$  has no nontrivial solutions yet. Probabilistic considerations suggest that there are no solutions. The analogue equation  $x^4 + y^4 + z^4 = w^4$  had been settled by Noam Elkies in 1988 who found  $2682440^4 + 15365639^4 + 18796760^4 = 20615673^4$ .

### ABC Conjecture



The abc conjecture is:

If  $a + b = c$ , then  $c \leq (\prod_{p|abc} p)^2$ .

For example, for  $10 + 22 = 32$ , the prime factors of  $abc = 7040$  are 2, 5, 11 and indeed  $32 \leq (2 \cdot 5 \cdot 11)^2 = 12100$ . The abc-conjecture is open but implies Fermat's theorem for  $n \geq 6$ : assume  $x^n + y^n = z^n$  with coprime  $x, y, z$ . Take  $a = x^n, b = y^n, c = z^n$ . The abc-conjecture gives  $z^n \leq (\prod_{p|abc} p) \leq (abc)^2 < z^6$  establishing Fermat for  $n \geq 6$ . The cases  $n = 3, 4, 5$  to Fermat have been known for a long time. In August 2012, there were rumors of an attack by Shinichi Mochizuki.