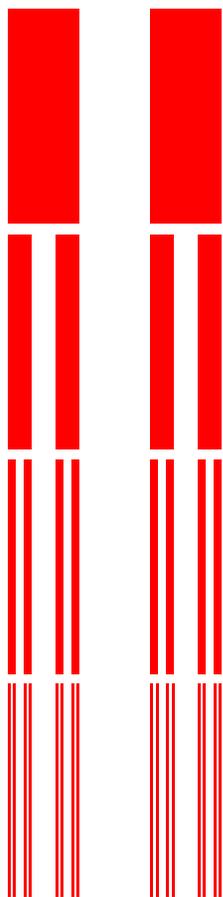


## Lecture 10: Analysis

### The Cantor Set

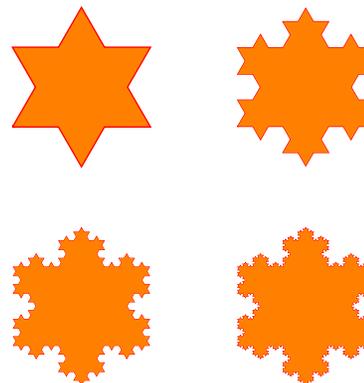


Analysis makes up a large part of mathematics. To get a glimpse, our goal is to understand **fractals**, objects with fractional dimension. Fractals enter many parts of analysis: spectral theory, complex analysis, harmonic analysis, calculus of variations, functional analysis. But because these fields need some time to learn and explain, the subject of fractals looks like a nice entry point. Our story will become pictorial but there is a formula we want to understand:

$$\dim(X) = \frac{-\log(n)}{\log(r)}.$$

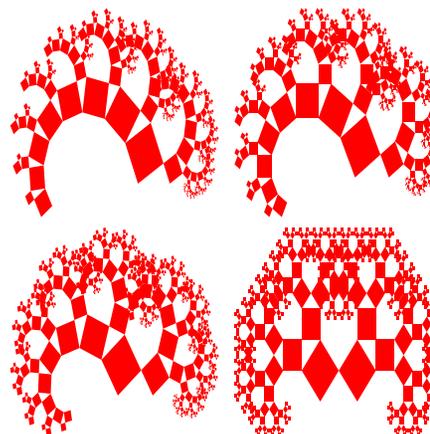
It tells that if we want to find the dimension of an object we cover it with boxes of size  $r$  and count how many we need:  $n$ . The dimension is what happens if  $r$  goes to zero. The prototype of a fractal is the **Cantor set** which was discovered in 1875 by **Henry Smith**. Start with the unit interval. Cut the middle third, then cut the middle third from both parts then the middle parts of the four parts etc. What is left in the end is the Cantor set for which the dimension is  $\log(2) / \log(3)$ .

### The Koch Snowflake



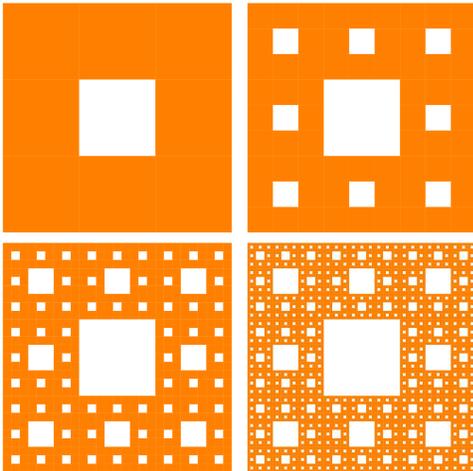
The **Koch snowflake** is an example of a fractal, where the dimension is between 1 and 2. It was first described by the Swedish mathematician **Helge von Koch** (1870-1924) who described it in 1904. It is a simple model for a **snowflake**. There is a simplified version which just is defined over an interval. It is called the **Koch curve**.

### The Tree of Pythagoras



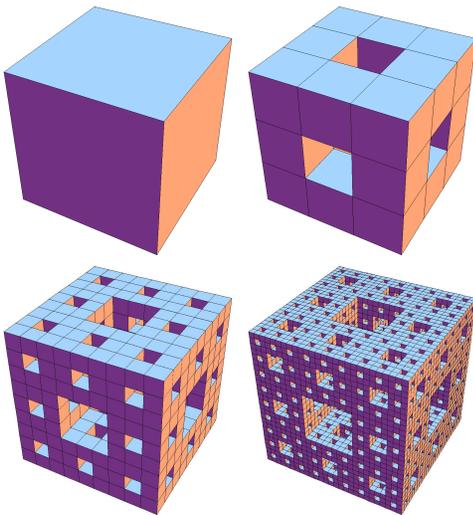
The **tree of Pythagoras** is an example of a fractal, where the dimension is between 1 and 2. It was first described by the Swedish mathematician **Helge von Koch** (1870-1924). The Koch curve was described by him in 1904. It comes close to actual **trees**. It inspired antenna designs.

## The Sierpinski Carpet



The **Sierpinski carpet** is a fractal in the plane. Its dimension is  $\log(8)/\log(2)$ . It was described by **Waclav Sierpinski** in 1916.

## The Menger Sponge



The **Menger sponge** is a fractal in space. Its dimension is between 2 and 3. It was first described by Karl Menger (1902-1985). Its dimension is  $\log(20)/\log(3)$  which is about 2.7.

## The Mandelbrot set

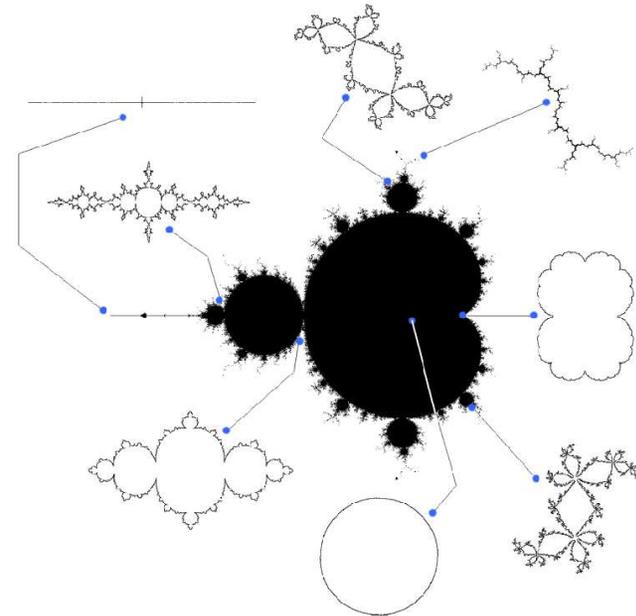
We introduce complex numbers  $z = a + ib$  and define complex multiplication

$$(a + ib)(u + iv) = au - bv + (av + bu)i$$

Now look at the map  $T(z) = z^2 + c$  where  $c$  is a fixed complex number. Start with  $z = i$  for example, we get  $T(z) = i + c$  and  $T^2(z) = T(T(z)) = (i + c)^2 + c$  etc. The **Mandelbrot set** is the set of complex numbers  $c = a + ib$  for which  $T^n(0)$  stays bounded. The **filled in Julia set**  $J_c$  of  $c$  is the set of  $z$  such that  $T^n(z)$  stays bounded. The **Julia set** is the boundary of that set.

For example, for  $c = 0$ , the map is  $T_0(z) = z^2$ . Since  $|z^n| = |z|^n$  we see that the disc  $\{|z| \leq 1\}$  is the filled in Julia set for  $c = 0$  and the unit circle  $\{|z| = 1\}$  is the Julia set.

The following picture (Peitgen-Richter-Saupe) shows the Mandelbrot set in the  $c$  plane and a few Julia sets. The circle is shown at the bottom.



A three dimensional version of the Mandelbrot set is called the **Mandelbulb**. It uses spherical coordinates which have been introduced by Euler.