Lecture 3: Geometry

Geometry is the science of **shape, size and symmetry**. While arithmetic dealt with numerical structures, geometry deals with metric structures. Geometry is one of the oldest mathematical disciplines and early geometry has relations with arithmetics: we have seen that that the implementation of a commutative multiplication on the natural numbers is rooted from an interpretation of $n \times m$ as an area of a **shape** that is invariant under rotational **symmetry**. Number systems built upon the natural numbers inherit this. Identities like the **Pythagorean triples** $3^2+4^2 = 5^2$ were interpreted geometrically. The **right angle** is the most "symmetric" angle apart from 0. Symmetry manifests itself in quantities which are **invariant**. Invariants are one the most central aspects of geometry. Felix Klein's **Erlanger program** uses symmetry to classify geometries depending on how large the symmetries of the shapes are. In this lecture, we look at a few results which can all be stated in terms of invariants. In the presentation as well as the worksheet part of this lecture, we will work us through smaller miracles like **special points in triangles** as well as a couple of gems: **Pythagoras**, **Thales**,**Hippocrates**, **Feuerbach**, **Pappus**, **Morley**, **Butterfly** which illustrate the importance of symmetry.

Much of geometry is based on our ability to measure **length**, the **distance** between two points. A modern way to measure distance is to determine how long light needs to get from one point to the other. This **geodesic distance** generalizes to curved spaces like the sphere and is also a practical way to measure distances, for example with lasers. It bypasses the problem to determine first the underlying nature of the space in which we do geometry. Having a distance $d(A, B)$ between any two points A, B, we can look at the next more complicated object, which is a set A, B, C of 3 points, a **triangle**. Given an arbitrary triangle ABC, are there relations between the 3 possible distances $a = d(B, C), b = d(A, C), c = d(A, B)$? If we fix the scale by $c = 1$, then $a + b \geq 1, a + 1 \geq b, b + 1 \geq a$. For any pair of (a, b) in this region, there is a triangle. After an identification, we get an abstract space, which represent all triangles uniquely up to similarity. Mathematicians call this an example of a **moduli space**.

A **sphere** $S_r(x)$ is the set of points which have distance r from a given point x. In the plane, the sphere is called a **circle**. A natural problem is to find the circumference $L = 2\pi$ of a unit circle, or the area $A = \pi$ of a unit disc, the area $F = 4\pi$ of a unit sphere and the volume $V = 4 = \pi/3$ of a unit sphere. Measuring the length of segments on the circle leads to new concepts like **angle** or **curvature**. Because the circumference of the unit circle in the plane is $L = 2\pi$, angle questions are tied to the number π, which Archimedes already approximated by fractions.

Also **volumes** were among the first quantities, Mathematicians wanted to measure and compute. A problem on **Moscow papyrus** dating back to 1850 BC explains the general formula $h(a^2+ab+b^2)/3$ for a truncated pyramid with base length a, roof length b and height h. Archimedes achieved to compute the **volume of the sphere**: place a cone inside a cylinder. The complement of the cone inside the cylinder has on each height h the area $\pi - \pi h^2$. The half sphere cut at height h is a disc of radius $(1 - h^2)$ which has area $\pi(1 - h^2)$ too. Since the slices at each height have the same area, the volume must be the same. The complement of the cone inside the cylinder has volume $\pi - \pi/3 = 2\pi/3$, half the volume of the sphere.

The first geometric playground was **planimetry**, the geometry in the flat two dimensional space. Highlights are **Pythagoras theorem**, **Thales theorem**, **Hippocrates theorem**, and **Pappus theorem**. Discoveries in planimetry have been made later on: an example is the Feuerbach theorem from the 19th century or the Sadov theorem for quadrilaterals. Greek Mathematics is closely related to history. It starts with **Thales** goes over Euclid's era at 300 BC, and ends with the threefold destruction of Alexandria 47 BC by the Romans, 392 by the Christians and 640 by the Muslims. Geometry was also a place, where the **axiomatic method** was brought to mathematics: theorems are proved from a few statements which are called axioms like the 5 axioms of Euclid:

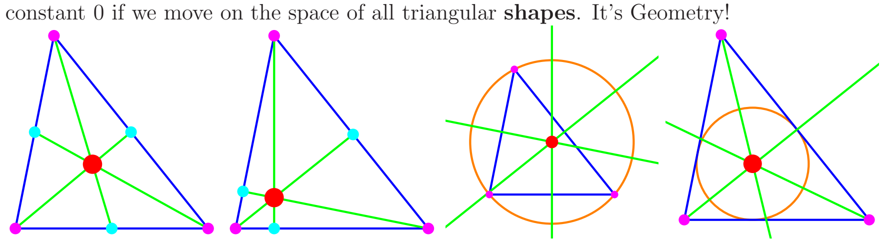
1. Any two distinct points A, B determines a line through A and B.
2. A line segment $[A, B]$ can be extended to a straight line containing the segment.
3. A line segment $[A, B]$ determines a circle containing B and center A.
4. All right angles are congruent.
5. If lines L, M intersect with a third so that inner angles add up to $< \pi$, then L, M intersect.

Euclid wondered whether the fifth postulate can be derived from the first four and called theorems derived from the first four the "absolute geometry". Only much later, with **Karl-Friedrich Gauss** and **Janos Bolyai** and **Nicolai Lobachevsky** in the 19'th century in **hyperbolic space** the 5'th axiom does not hold. Indeed, geometry can be generalized to non-flat, or even much more abstract situations. Basic examples are geometry on a sphere leading to **spherical geometry** or geometry on the Poincare disc, a **hyperbolic space**. Both of these geometries are non-Euclidean. **Riemannian geometry**, which is essential for **general relativity theory** generalizes both concepts to a great extent. An example is the geometry on an arbitrary surface. Curvatures of such spaces can be computed by measuring length alone, which is how long light needs to go from one point to the next.

An important moment in mathematics was the **merge of geometry with algebra**: this giant step is often attributed to **René Descartes**. Together with algebra, the subject leads to algebraic geometry which can be tackled with computers: here are some examples of geometries which are determined from the amount of symmetry which is allowed:

Euclidean geometry	Properties invariant under a group of rotations and translations
Affine geometry	Properties invariant under a group of affine transformations
Projective geometry	Properties invariant under a group of projective transformations
Spherical geometry	Properties invariant under a group of rotations
Conformal geometry	Properties invariant under angle preserving transformations
Hyperbolic geometry	Properties invariant under a group of Möbius transformations

Here are four pictures about the 4 special points in a triangle and with which we will begin. We will see why in each of these cases, the 3 lines intersect in a common point. It is a manifestation of a **symmetry** present on the space of all triangles. **size** of the distance of intersection points is constant 0 if we move on the space of all triangular **shapes**. It's Geometry!



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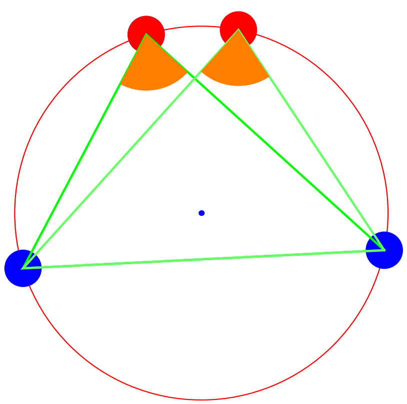
Thales Theorem

Thales of Miletus (625 BC -546 BC) got the following beautiful result

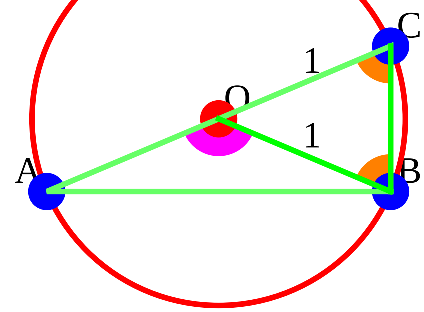
> A triangle inscribed in a fixed circle is deformed by moving one of its points on the circle, then the angle at this point does not change.

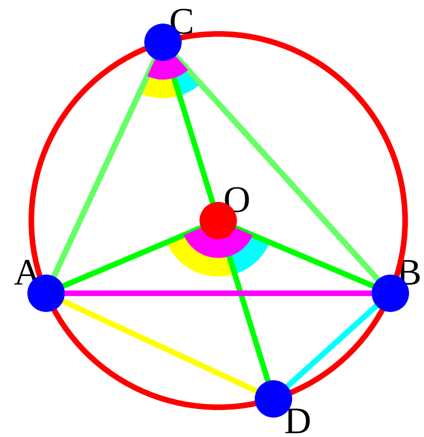
The result is relevant also because Thales is considered the **first modern Mathematician**. Thales theorem is a prototype of a stability result. In this worksheet we want to understand it and prove it.



Lets look first at the case when one side of the triangle goes through the center.

a) The triangle BCO is an isosceles triangle.
b) The central angle AOB is twice the angle ACB.





Here are the steps to see the theorem:

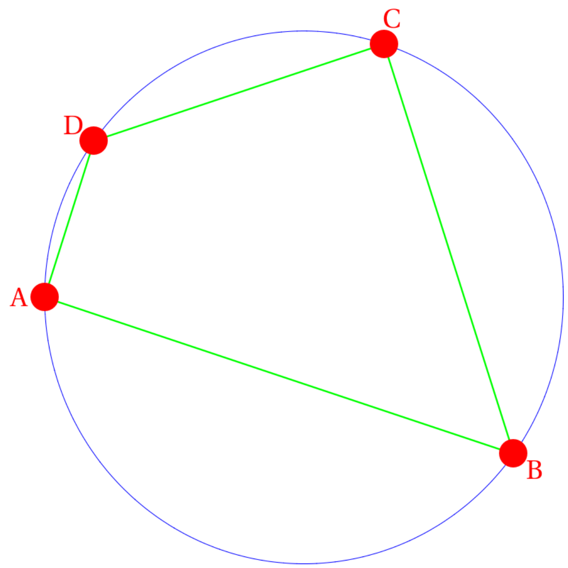
c) What is the relation between the angles AOD and ACD?
d) What is the relation between the angles DOB and DCB?
e) Find a relation between the central angle AOB and the angle ACB?
f) Why does the angle ACB not change if C moves on the circle?

Sadov's theorem

Here is an example of a geometric theorem which has been found only 10 years ago:

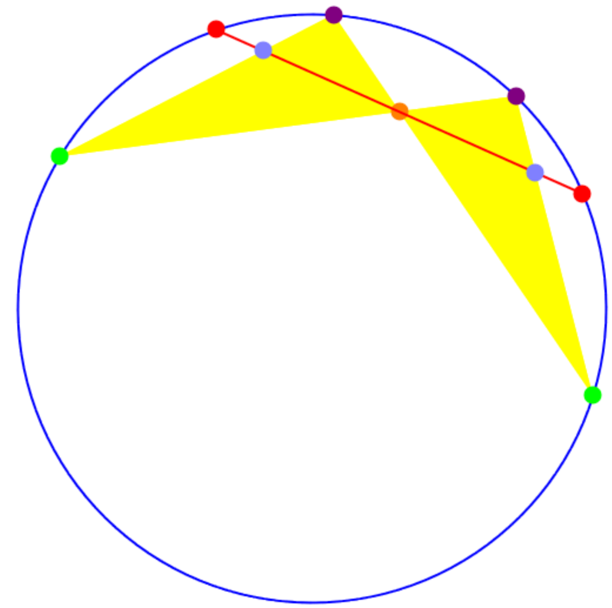
> The quadrilateral $ABCD$ is on a circle if and only if
>
> $$|AB||BC||CA| + |AC||CD||DA| = |BC||CD||DB| + |AB||BD||DA|$$

We will demonstrate a verification given by Shalosh B. Ekhadof, a computer of Doron Zeilberg at Rutgers University. The theorem has appeared in a text of Hadamard, an examination of Ramanujan and in a text by J. Vojtech. The full theorem was first proven computer assisted (2003-2005) and independently by M.A. Rashid, A.O. Ajibade in 2003. Source: Sergey Sadov, Memorial University of Newfoundland, Candada.



The butterfly theorem

Draw an arbitrary chord AB in a circle. Now draw two new arbitrary chords PQ, RS through the center M of AB. The line segments PR and QS now cut the chord AB in equal distance.

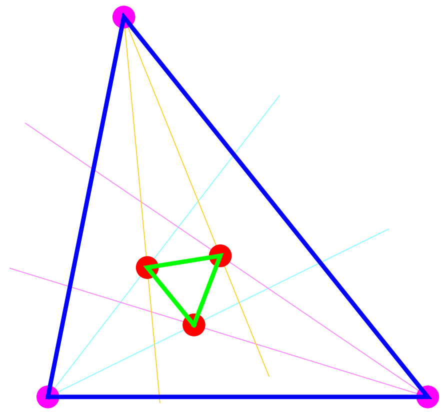


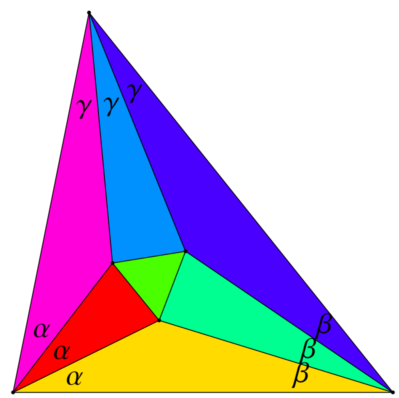
Morley's miracle

The following theorem was discovered in 1899 by **Frank Morley** at Haverford College near Philadelphia.

If one trisects the angles of a triangle, the corresponding trisector intersections form an equilateral triangle.

It is a beautiful result because it is not obvious, or even surprising. **John Conway** found an elegant proof: write a' for the angle $a + \pi/3$ and a'' for $a + 2\pi/3$. Build 7 triangles with angles $(0', 0', 0'), (a, b', c'), (a', b, c'), (a', b', c), (a, b, c''), (a, b'', c), (a'', b, c)$ and cyclic. The triangles $(a, b', c'), (a', b, c'), (a', b', c)$ are determined by assuming their shortest side length is the one from the equilateral triangle $(0', 0', 0')$. The other three are required to have the same height than the triangle $(0', 0', 0')$. These 7 triangles can be put together to a large triangle with angles (a, b, c).

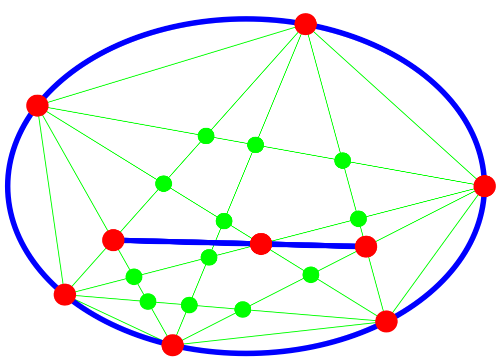




Pascal's mystic hexagram
The following result has been found in 1640 by Pascal, when he was 17. He probably got the problem from his father who was a friend of Desargues. See Stillwell "mathematics by its history" page 95.

> Pairs of opposite sides of a hexagon inscribed in a conic section meet in three collinear points.

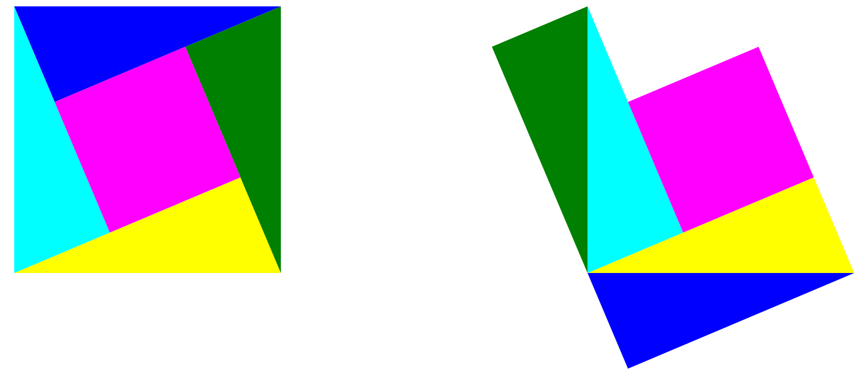
Also this result is not obvious. Pascal probably proved it first for circles. Applying a linear transformation on the picture preserves the linear incidence structure and gets it for all. We can also been seen as a consequence of the Pappus-Pascal theorem.



Pythagoras theorem

> For all right angle triangles of side length a, b, c, the quantity $a^2 + b^2 - c^2$ is zero.

As shown in class, there is simple rearrangement proof:



An other beautiful result is:

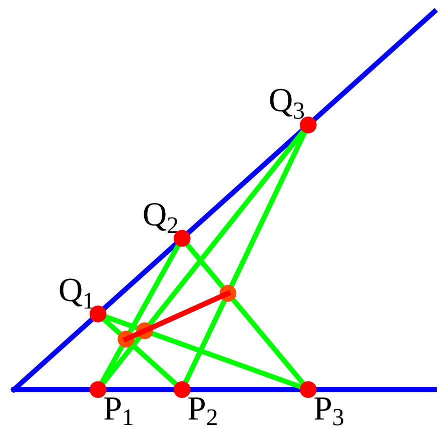
> Given a circle of radius 1 and a point P inside the circle. For any line through P which intersects the circle at points A, B we have $|PO|^2 - |PA||PB| = 1$.

This is a consequence of Pythagoras. By scaling translation and rotation we can assume the circle is at the origin and that the line through the point $P = (a, b)$ is horizontal. The intersection points are then $(\pm\sqrt{1-a^2}, a)$. Now $(b - \sqrt{1-a^2})(b + \sqrt{1-a^2}) = b^2 - 1 + a^2$.

Pappus theorem

Pappus of Alexandria (290- 350) showed:

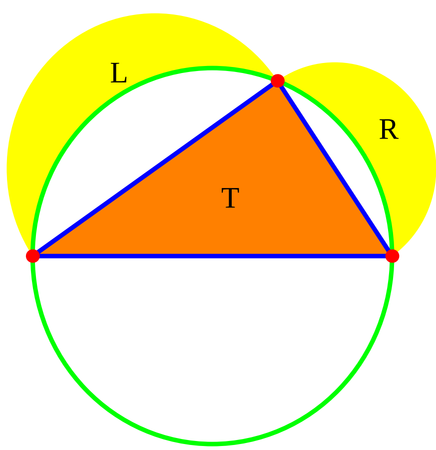
> Take three points P_1, P_2, P_3 on a first line and three points Q_1, Q_2, Q_3 on a second line. Draw all possible connections P_iQ_j with $i \neq j$. The intersection points of the lines P_iQ_j and P_jQ_i are on a line.



Hyppocrates Theorem

The quadrature of the Lune is a result of **Hippocrates of Chios** (470 BC - 400 BC) and also called Hyppocrates theorem. It is the first rigorous quadrature of a curvilinear area. It states:

> The sum $L+R$ of the area L of the left moon and the area R of the right moon is equal to the area T of the triangle.

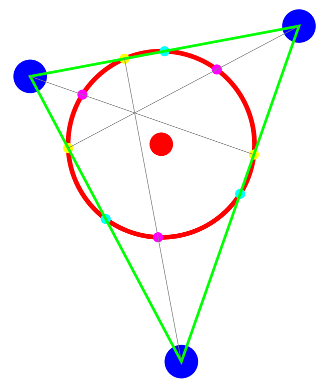


If A, B, C are the areas of the half circles build over the sides of the triangle, then $A+B=C$. If U is the area of the intersection of A with the upper half circle C. and let V be the area of the intersection of B with C. Let T be the area of the triangle. Then $\boxed{U+V+T=C}$. Interpret $L=A-U$ and $R=B-V$ are the moon areas we can add them up and use the just shown relation to see $L+R=T$.

Feuerbach's Theorem

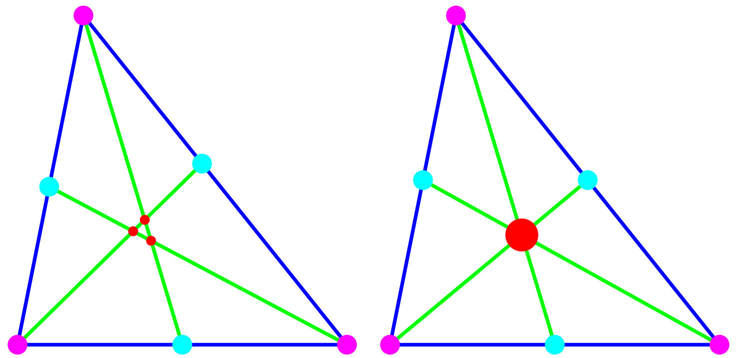
> The 3 midpoints of each side, the 3 foots of each altitude and the three midpoints of the line segments from the vertices to the orthocenter lie on a common circle.

This result is attributed to **Karl Wilhelm Feuerbach** (1800-1834), who found a partial result of this in 1822. We will prove it with the computer in class. In the case of an equilateral triangle the midpoints and the height bases are the same and we have only 6 points. The Feuerbach circle is the circle inscribed into the triangle.



The centroid

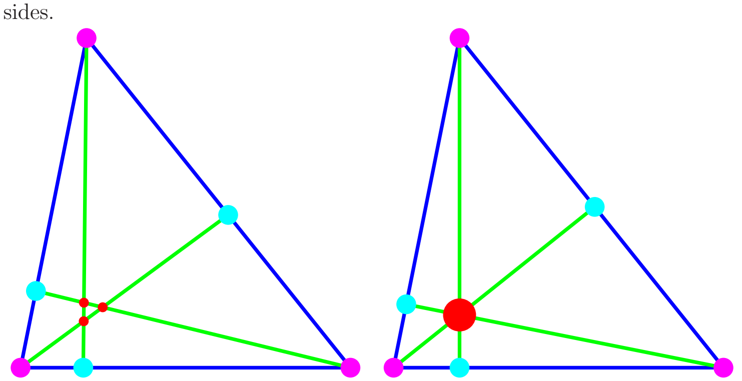
The **centroid** of a triangle is the intersection of the lines which connect the vertices of a triangle with the midpoints of the opposite side. It is not at all trivial that these three lines intersect in one point. It is a stability property of the triangle. Deforming a triangle does not change this property. If A, B, C are the coordinates of the vertices, then $(A+B)/2, (A+C)/2$ and $(B+C)/2$ are the midpoints of the sides. To verify the property just check that with $P=(A+B+C)/3$, the points $A, P, (B+C)/2$ are on a line, the points $(B, P, (A+C)/2$ are on a line and the points $(C, P, (A+B)/2$ are on a line. There is an easier but more advanced way to see this: check it first for the equilateral triangle. Now, any triangle can be mapped into any other by a linear transformation. Because linear transformations preserve lines and ratios, the intersection property will stay true for all triangles.



To the left, we see the situation as we would expect it without "knowing" that the three intersection points agree. To the right, we wee the actual situation.

The orthocenter

The **orthocenter** is the intersection of the three altitudes of a triangle. Also here - a priory - we have three different points the intersection, for each pair of altitudes. Why do they meet in one point? It is not obvious and was not proven by the Greeks for example. One can take the intersection of two altitudes, get a point P and form the line from P to the third point in the triangle. The fact that this line is perpendicular to the third line can be seen by looking at the angles. The angles between to heights is the same than the angle between the two corresponding sides.



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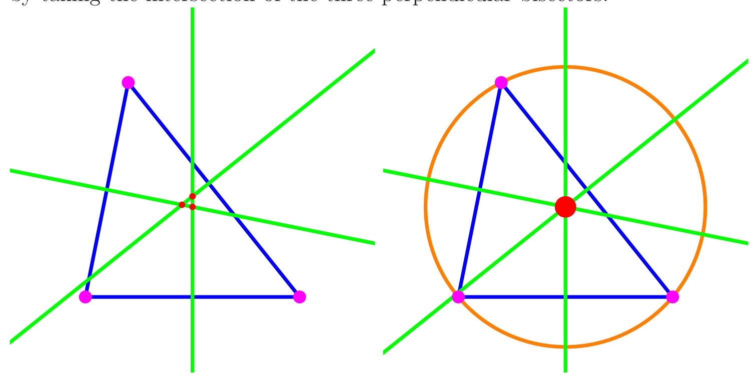
The Center of the Circumscribed circle

Any circle which passes through two points A,B of a triangle lies on the perpendicular bisector of A and B. When moving a point M on that line and always drawing the circle centered at M through A,B, then there will be a moment, where the distance to the third point C is equal to the distance to A. We have found the circumscribed circle of the triangle. The point can be obtained by taking the intersection of the three perpendicular bisectors.



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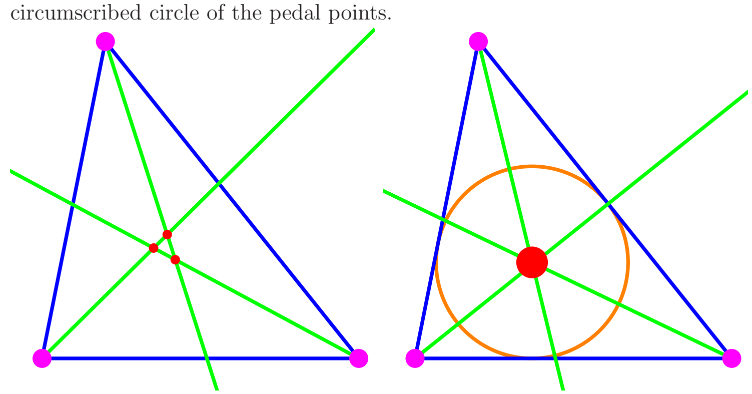
The Center of the inscribed circle

Any circle which is tangent to two sides of a triangle lies on the angle bisector at the intersection point of the sides. Take a circle on that line which is tangent to the two sides. If the center is close to the point then the circle is small and inside the triangle. Move the point along the line. There will be a moment, when the circle will touch the third side. This point is the intersection point of all angular bisectors. It is the center of the inscribed circle. The inscribed circle is the circumscribed circle of the pedal points.



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