Lecture 6: Worksheets

We stack disks onto each other building n layers and count the number of discs. The number sequence we get are called **triangular numbers**.

1	3	6	10	15	21	36	45	...

This sequence defines a **function** on the natural numbers. For example, $f(4) = 10$.

1 Can you find $f(200)$? The task to find this number was given to Carl Friedrich Gauss in elementary school. The 7 year old came up quickly with an answer. How?

Carl-Friedrich Gauss, 1777-1855

Tetrahedral numbers

We stack spheres onto each other building n layers and count the number of spheres. The number sequence we get are called **tetrahedral numbers**.

1	4	10	20	35	56	84	120	...

Also this sequence defines a **function**. For example, $g(3) = 10$. But what is $g(100)$? Can we find a formula for $g(n)$?

2 Verify that $g(n) = n(n+1)(n+2)/6$, satisfies $Dg(n) = g(n) - g(n-1) = n(n+1)/2$.

3 **Problem:** Given the sequence $1, 1, 2, 3, 5, 8, 13, 21, \dots$ which satisfies the rule $f(x) = f(x-1) + f(x-2)$. It defines a function on the positive integers. For example, $f(6) = 8$. What is the function $g = Df$, if we assume $f(0) = 0$?

4 **Problem:** Take the same function f given by the sequence $1, 1, 2, 3, 5, 8, 13, 21, ...$ but now compute the function $h(n) = Sf(n)$ obtained by summing the first n numbers up. It gives the sequence $1, 2, 4, 7, 12, 20, 33, ...$ What sequence is that?

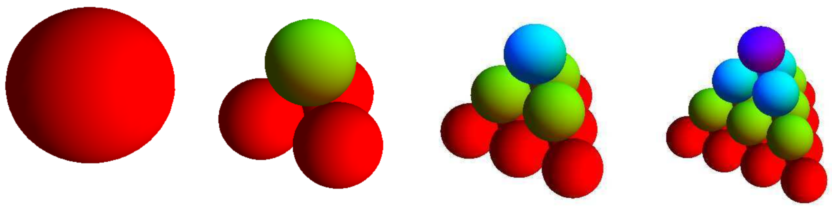
5 **Problem:** Find the next term in the sequence

2 6 12 20 30 42 56 72 90 110 132 .

6 **Problem:** Find the next term in the sequence

$$3, 12, 33, 72, 135, 228, 357, 528, 747, 1020, 1353... .$$

To do so, compute successive derivatives $g = Df$ of f, then $h = Dg$ until you see a pattern.



n=1 n=2 n=3 n=4

Lecture 6: convergence of series

1. Objective

Sums can be counterintuitive. This worksheet can illustrate why calculus needed some time to be developed.

2. Poincare's example

Henri Poincaré mentions in his "New Methods of Celestial Mechanics" the two series

$$S_n = \sum_{n=0}^{\infty} \frac{1000^n}{n!}$$

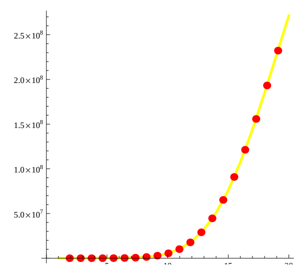
and

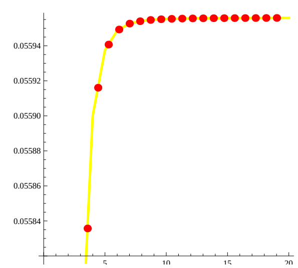
$$S_n = \sum_{n=0}^{\infty} \frac{n!}{1000^n} \, .$$

1) Why does the first series have a limit? Can you give its value? You might have to look up the series for the exponential function.

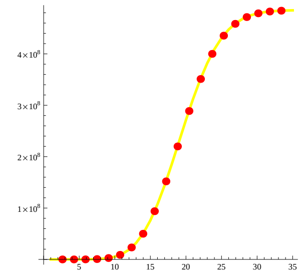
2) Why does the second series not have a limit? Just take some large n and see what the terms are you are summing up.

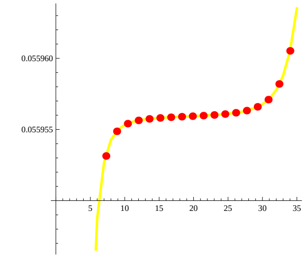
Experimental evidence would rule the first to be divergent and the second to be convergent. The experiments would be too extreme in Poincaré's example. Therefore, we replace 1000 with 20. Lets look at the first 20 values of S_n





Only if we sum up higher up, we see what is going on.





3. The harmonic series

The harmonic series

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

diverges. Lets see why.

1) Why is the third and forth therm together larger than 1/2?

$$\frac{1}{3} + \frac{1}{4} \, .$$

2) Why is the sum of the fifth up to eighth term larger than 1/2?

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \, .$$

3) How do we continue the argument?

Experimentally, the series seems to stay bounded. To get to 100, we would need $e^{100} = 10^{43}$ steps. But the universe is only 10^{17} seconds old.

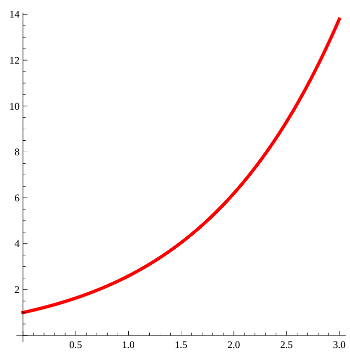
Lecture 6: the exponential function

1. Objective

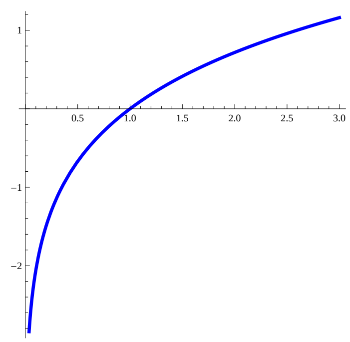
Having defined exp as the limit of $\exp_n(x) = (1+x/n)^n$ and log as the limit of $\log_n(y) = n(y^{1/n}-1)$ allows to define a^b for any positive a any real b. We look at a few examples of the exponential and logarithm function.

The exponential and logarithm

1) Verify that $\exp_n(0) = 1$ and $\log_n(1) = 0$ for all n.



The polynomial exponential $\exp_{10}(x)$.



Its inverse $\log_{10}(x)$.

2. The exponential to an arbitrary base

We have seen that the functions exp and log are inverses of each other and that

$$\exp(a+b) = \exp(a) * \exp(b), \log(a*b) = \log(a) + \log(b) .$$

We can define $a^b = e^{b\log(a)}$ for positive a. Similarly, we can define the logarithm $\log(b)$ to the base a as the inverse of the function $f(x) = a^x$.

1) What is 100^x for $x = 2$ and for $x = 1/2$?

2) The definition implies $a^b \cdot a^c = a^{b+c}$ How can one use this to show verify $5^{1/2} = \sqrt{5}$?

3) For rational p/q we have $a^{p/q} = (a^p)^{1/q}$. What is $27^{5/3}$?

2. The logarithm to an arbitrary base

4) What is $\log(1000)$ to the base 10?

5) What is $\log(256)$ to the base 2?

6) The logarithm $\log(a)$ to base 10 is defined as the number x such that $10^x = a$. Verify that $x = \log(a)/\log(10)$ by showing $10^x = a$.

2. The Zeno paradoxes of motion

History shows that one had great difficulties to understand the concept of limit.

> ”In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead.”

> ”That which is in locomotion must arrive at the half-way stage before it arrives at the goal.”

> ”If everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless.”

2. Calculus books

My efforts to reduce things to the core is a reaction to an ”focus free” attitude towards calculus which manifests in the **insane inflation** of text book sizes. **Eli Maor** expresses it well in his book ”The facts on file: Calculus Handbook, 2003”:

”Over the past 25 years or so, the typical college calculus textbook has grown from a modest 350-page book to a huge volume of some 1,200 pages, with thousands of exercises, special topics, interviews with career mathematicians, 10 or more appendixes, and much, much more. But as the old adage goes, more is not always better. The enormous size and sheer volume of these monsters (not to mention their weight!) have made their use a daunting task. Both student and instructor are lost in a sea of information, not knowing which material is important and which can be skipped. As if the study of calculus is not a challenge already, these huge texts make the task even more difficult.”

I personally find that there truth in that. Textbooks like **Snyder's Elementary Textbook on the Calculus** was 200 pages long. Also Maor's book is a short dictionary or glossary and does not substitute a textbook. But it can help to focus.

3. History

Here are some pioneers of single variable calculus and calculus teaching.

Zeno of Elea 490-430 Notion of derivative
Democritus 460-370 Cone and Pyramid. Atomic structure of matter
Eudoxus 408-355 BC method of exhaustion
Archimedes 287-212 BC area of disc, volume of sphere
Johannes Kepler 1571-1630, velocity and acceleration
Rene Descartes 1596-1650, tangents, rule of signs
Bonaventura Cavalieri 1598-1647 Cavalieri principle
Pierre de Fermat 1601-1665 Maxima, Integral of power function
John Wallis 1616-1703 integral calculus with x^a, infinite series
Christiaan Huygens 1629-1695 Waves, gravity,
Blaise Pascal 1623-1662, expectation, Pascal triangle
Isaac Barrow 1630-1677 Calculating tangents
James Gregory 1638-1675 Fundamental theorem of calculus
Robert Hooke 1635-1703 Inverse square law
Isaac Newton 1643-1727 Fluxions = Derivatives
Gottfried Leibniz 1646-1716 Modern version
Michel Rolle 1652-1719 Critic of calculus, Roles theorem
Guillaume de L'Hopital 1661-1704 Textbook, Hopitals law
Johann Bernoulli 1667-1748 First textbook (written with L'Hopital)
Brook Taylor 1685-1731 Taylor series, Difference calculus
Leonard Euler 1707-1783 Basel problem, analytic geometry
Maria Agnesi 1718-1799 Textbook in calculus
Bernard Bolzano 1781-1848 Rigor, $\epsilon - \delta$, Extremal value theorem
Augustin Cauchy 1789-1857, continuity, complex calculus
Karl Weierstrass 1815-1897 Rigorous foundation of calculus
Bernhard Riemann 1826-1866 Riemann integral, Zeta functions
Henri Lebesgue 1875-1941 Modern integration

For multivariable calculus, one would have to add mathematicians like:

Joseph-Louis Lagrange 1736-1813 Lagrange Multipliers
Pierre-Simon Laplace 1749-1827, Potential theory
Carl Friedrich Gauss 1777-1855 Greens theorem
George Green 1793-1841 Gauss theorem
Michael Ostrogradsky 1801-1862, Stokes theorem
George Gabriel Stokes 1819-1903 Stokes theorem

Lecture 6: the fundamental theorem of calculus

1. Objective

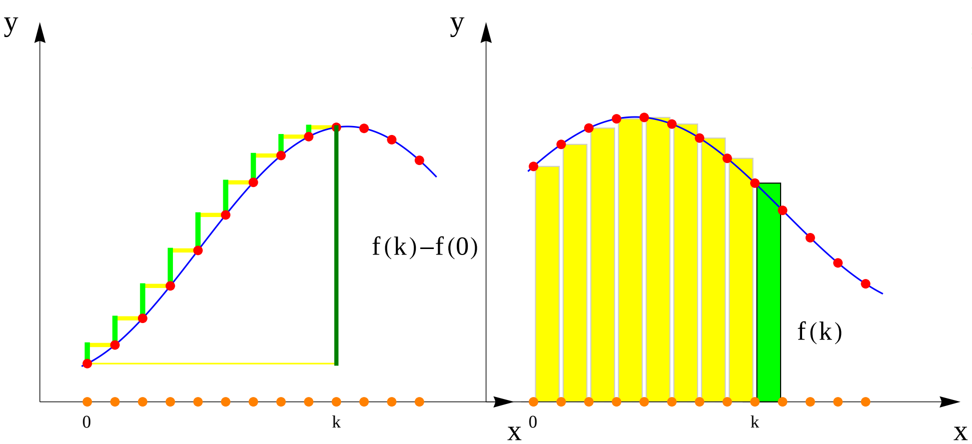
We want to have a closer look at the quantum fundamental theorem of calculus.

1. Reminders

The difference	$D_n f(x) = n(f(x+\frac{1}{n}) - f(x))$	becomes the	**derivative**	$f'(x)$.
The sum	$S_n f(x) = \frac{1}{n}\sum_{0\leq\frac{k}{n}<x} f(\frac{k}{n})$	becomes the	**integral**	$\int_0^x f(t)\,dt$.

The **fundamental theorem of calculus**:

$S_n D_n f(\frac{k}{n}) = f(\frac{k}{n}) - f(0)$	becomes in the limit	$\int_0^x f'(t)\,dt = f(x) - f(0)$
$D_n S_n f(\frac{k}{n}) = f(\frac{k}{n})$	becomes in the limit	$(\int_0^x f(t)\,dt)' = f(x)$



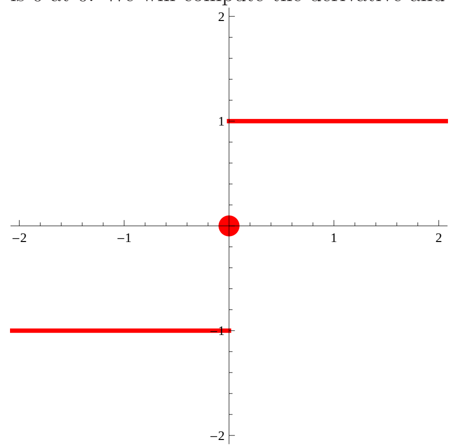
Sum the differences and get $S_n D_n f(\frac{k}{n}) = f(\frac{k}{n}) - f(0)$

Difference the sum and get $D_n S_n f(\frac{k}{n}) = f(\frac{k}{n})$

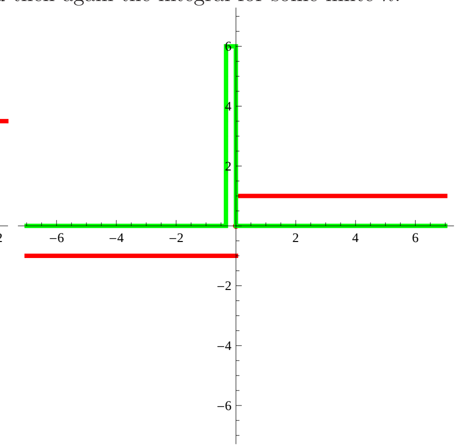
The result shows that S_n is the left and right inverse of D_n up to a constant. Integration and differentiation are inverse operations to each other.

1. An example

If we do not take the limit, the fundamental theorem theorem holds for all functions. We illustrate this here with the **signum function**, which is -1 for negative x and 1 for positive x and which is 0 at 0. We will compute the derivative and then again the integral for some finite n.



The Signum function



The first derivative of the Signum function

2. Problems

Draw the first derivate in the following situation: