

Lecture 7: Set Theory and Logic

We will mostly focus on the work of two mathematicians: **Georg Cantor** and **Kurt Gödel**. Their mathematics changed our way we think about mathematics. In both cases, the mathematics community needed time to absorb the implications of the revolutions. Hilbert said about Cantor "Nobody will drive us from the paradise that Cantor has created for us". Cantor clarified the term "cardinality" is, showed that certain infinities like that the cardinality of points in the plane or points in space are the same and most importantly showed that different infinities exist. Gödel's theorems show that mathematics and knowledge in general can not be exhausted by listing a sequence of basic truths from which everything follows. Whenever we make such a list, there are statements which are independent of the system. It would be a mistake to take this as a limitation of mathematics, in contrary it shows that mathematics is inexhaustible: there is always something more to explore.

Counting: Set theory

We first demonstrate that one can compute with sets like with numbers. There is an addition, the symmetric difference and a multiplication, the intersection. With these two operations, we prove the familiar rules of arithmetic

$$A + B = B + A, A \cdot B = B \cdot A, A \cdot (B + C) = A \cdot B + A \cdot C$$

hold. This is a Boolean algebra. There is a set which plays the role of 0. Which one is it? There is also a set which plays the role of 1. Which one is it?

Counting: Hilbert's Hotel

Hilbert's hotel is located on route 8. It has countably many rooms numbered $1, 2, 3, \dots$. The hotel is fully booked. As a newcomer arrives. David, the hotel manager is mortified. David has an idea and moves guest in room i to room $i + 1$ and gives the newcomer the first room 1.

Another day, the hotel is empty but a large group arrives. They are the "fractions" on their way to a cardinal match with the "squares". Can David accommodate them? He thinks hard and finally manages.

In the summer, the "reals" appear. David is not there but has George, the apprentice is in the office. The group consists of all real numbers between 0 and 1. Can George accommodate them? As much as he tries to shift and renumber, he can not do it.

Counting: the interval

The interval $(-1, 1)$ has the same cardinality than the real line. The function $f(x) = \tan(\pi x/2)$ maps the interval $(-1, 1)$ onto the real line, one to one.

The square $(0, 1) \times (0, 1)$ has the same cardinality than the real line. A bijection can be constructed by $f(0.a_1a_2a_3a_4\dots) = (0.a_1a_3a_5, 0.a_2a_4a_6\dots)$.

Arithmetic with sets

One can calculate with sets as with numbers. They form a "Boolean ring".

Addition: $A + B = A \Delta B$ with the zero element \emptyset

Multiplication: $A \cdot B = A \cap B$ with the one element Ω .

All the rules of the real numbers apply but there are additional consequences which appear a bit strange $A + A = 0$ and $A^2 = A \cdot A = A$. This means that A is its own additive inverse.

Paradoxa

We have seen a few paradoxa like the **Liars paradox** "I lie", the **barbers paradox** "the barber is the person who shaves everybody who does not shave him or herself", the **surprise exam problem** "it is impossible to make a surprise exam problem", the **heap problem** "take a grain away from a heap keeps it a heap", the **biographer's problem** "who needs one year to write one day of his biography", Here is another, the **Berry paradox** which comes somehow close to the Goedel numbering:

The smallest integer not definable in less than 11 words.

The problem is that this number is defined with 10 words. This looks like a stupid example but it illustrates that there are properties of numbers like "the shortest way to describe the number" which is not computable.

Axiom of choice

The **axiom of choice (C)** has a nonconstructive nature which can lead to seemingly paradoxical results like the **Banach Tarski paradox**: one can cut the unit ball into 5 pieces, rotate and translate the pieces to assemble two identical balls of the same size than the original ball. Gödel and Cohen showed that the axiom of choice is logically independent of the other axioms ZF.