

Lecture 13: Experimental Mathematics

We look at problems where computers could play a role for the solution. Usually, in the case a counter example exists.

2. Goldbach's conjecture

The statement is that every even number larger than 2 is the sum of two primes. While it is unlikely that a computer search will find a counter example, it might be that computer search finds the solution to the problem. How?

Here is a complex analytic approach which is a caricature of much more sophisticated methods developed since 90 years. Search for function $f(x) = \sum_p a_p x^p$ with a_p positive so that f can be written in terms of functions for which integration theory works well (trig functions, exponentials, polynomials, hypergeometric functions etc). Then check (theoretically) that the Taylor coefficients of $f(x)^2$ are positive. One can do that by integration in the complex plane. This approach is a long shot since it is unlikely that a closed form for $f(x)$ can be found which works. More likely is that one can approximate things well enough to push the threshold higher above which counter examples must appear.

3. Euler Cuboid

It is not known whether there exists a cuboid with integer side length such that all face diagonals are integers and additionally, also the large diagonal is an integer. Computer searches have found nothing. One can search on parametrized families of Euler cubes but the **perfect Euler cuboid** - if it exists - could be so large that no computer could find it by brute force search.

4. Riemann hypothesis

This is a prototype, where experiments led to more and more support that the Riemann hypothesis is true. One approach looks numerically for roots of the Riemann zeta function on the critical line. An other approach is the Merten's approach. Define $\mu(n) = 1$ if n is the product of an even number of different primes and -1 if n is the product of an odd number of different primes. In all other cases, that is if n has a square factor larger than 1, we have $\mu(n) = 0$. Is μ sufficiently random so that $S_n = \sum_{k=1}^n \mu(k)$ grows like the iterated law of logarithm? While it looks as if the $\mu(n)$ behave like a random sequence, there are some correlations.

5. Billiards

One does not know whether there are triangular billiards without periodic points. One also does not know whether there are smooth convex billiards besides the ellipse for which one has **integrability** in the sense that all points are either periodic, asymptotic to a periodic point or almost periodic in the sense that the dynamics is equivalent to a translation on a finite or infinite dimensional torus. Integrability implies that one can compute the future orbit arbitrarily well without running into the sensitive dependence on initial condition problems.

6. Chaos

Verify that for $c > 2$ and all $n \frac{1}{n} \int_0^1 \int_0^1 \log |\partial_x f_n(x, y)| dx dy \geq \log(\frac{c}{2})$. I myself have tried over a decade to prove this using methods from quantum mechanics, calculus of variations and complex analytic methods. The problem is open. The left hand side converges to the average of the Lyapunov exponents which is in this case also the **entropy** of the map.

5. Prime twins?

This is a problem, which can first of all be investigated statistically. Similarly as Gauss looked numerically for a law describing the frequency of the prime numbers, one can first see, how many prime twins one has to expect in a certain interval and then see whether this expectation is confirmed. Furthermore one can look whether there are any patterns on arithmetic subsequences. Maybe there are some unexpected sequences along which there are more prime twins. Related to the twin primes problem is the problem to estimate the minimal distance between two Gaussian primes in the complex plane.

5. Is π normal?

The decimal digits of π appear random enough so that every digit appears with the same frequency.

6. Are there odd perfect numbers?

While a brute force search is unlikely, there are other approaches which are more likely to find an odd perfect number. Any perfect number satisfies $\sigma(n)/n = 2$, where $\sigma(n)$ is the sum of all the divisors of n including n . Take a large set $B = \{p_1, \dots, p_s\}$ of primes. For every $k = (k_1, \dots, k_s)$, form the number $n = p_1^{k_1} p_2^{k_2} \dots p_s^{k_s}$. We have $\sigma(p^k) = (1 + p + p^2 + \dots + p^k)/p^k = (p - p^{-k})/(p - 1)$. Let $a(p, k) = \log(\sigma(p^k))$. The goal is to find (k_1, \dots, k_s) such that

$$\log(a(p_1, k_1)) + \dots + \log(a(p_l, k_l)) - k_1 \log(p_1) - \dots - k_s \log(p_s) = \log(2) .$$

The idea is to keep first the primes and change only the "dials" k_j in a controlled way. Certain dial changes will produce a very small net change of the left hand side. For large primes and large n the first order change will dominate and methods from Diophantine geometry could be used and linear algebra to get close to the right hand side. If lucky (provided of course there is an odd perfect number), one could hit it like this. If we would take 1000 primes each 1000 digits long and deal with exponents of the order of 1000, we would investigate numbers with billions of digits.

Lecture 13: integer partition problem

1. Objective

The following problem is known to be **NP-complete**. This means that if it could be solved in polynomial time, then all NP problems would be polynomial and $P=NP$.

In other words, if you can design a method, which solves the problem in a manner which is polynomial in n , you win a Million dollars and you would have solved the most important problem in computer science.

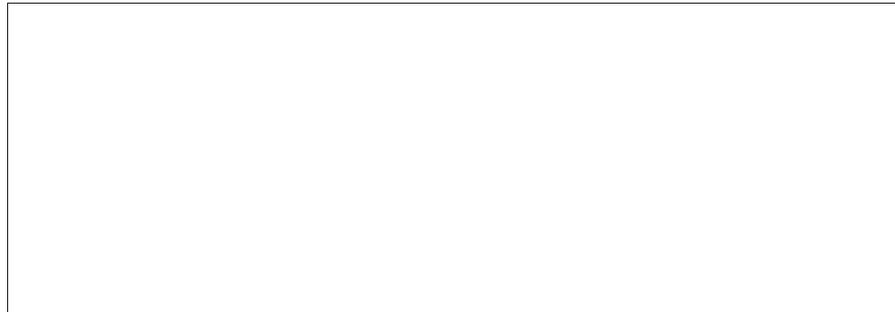
2. The problem

Given n positive integers a_1, \dots, a_n , divide them up into two subsets, so that the sum of these numbers in one set and the sum of numbers in the other set are as close together as possible.

For example, let

$$A = \{2, 3, 5, 7, 11, 13, 17\}.$$

We can apply the "cruel school selection" algorithm: the largest two elements are the team leaders of a team, who alternatively select the best remaining candidate. What do you get?



2. Can you do better than that? How well can you divide up the numbers better?

