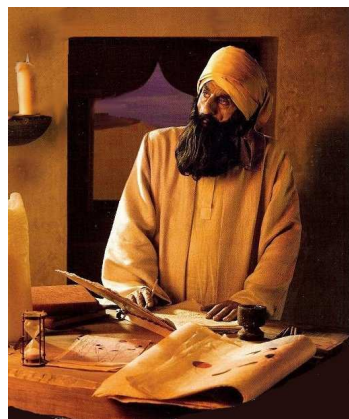


## Lecture 5: Worksheets

We look at the quadratic and the cubic equation and then at puzzles like the 15 puzzle or Rubik type puzzles.

### The quadratic equation

The solution of the quadratic equation  $x^2 + bx + c = 0$  is one of the major achievements of early algebra. It relies on the method of **completion of the square** and is due to the Persian mathematician **Al Khwarizmi**.



Artist rendering of Al Khwarizmi for an advertisement.

The completion of the square is the idea to add  $b^2/4$  on both sides of the equation and move the constant to the right. Like this  $x^2 + bx + b^2/4$  becomes a square  $(x + b/2)^2$ . Geometrically, one has added a square to a region to get a square. From  $(x + b/2)^2 = -c + b^2/4$  we can solve  $x$  and get the famous formula for the solution of the quadratic equation

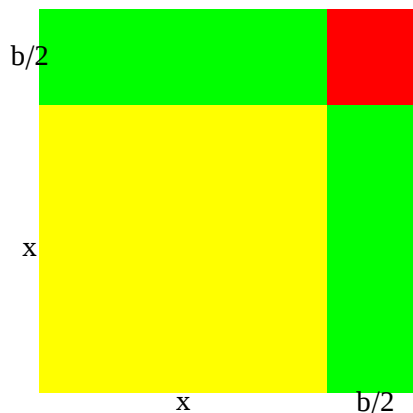
$$x = \sqrt{\frac{b^2}{4} - c} - \frac{b}{2}.$$

Since one can take both the positive and the negative square root, there are two solutions.

1) Write down the solution formula for the equation  $ax^2 + bx + c = 0$ .

2) If  $x_1, x_2$  are the two solutions to  $x^2 + bx + c = 0$ , then the sum of the two solutions is  $x_1 + x_2 = -b$ .

3) If  $x_1, x_2$  are the two solutions of  $x^2 + bx + c$ , then the product of the solutions is  $x_1 x_2 = c$ .



4) What are the solutions to is  $x^4 - 4x^2 + 3 = 0$ ?

5) Find the solutions to  $x^6 - 4x^4 + 3x^2 = 0$ .

## The cubic equation

1) Lets look at the cubic equation  $x^3 - 7x + 6$ . Can you figure out the roots?

2) Verify that if  $a, b, c$  are solutions to a cubic equation satisfy  $a + b + c = 0$  if and only if it is depressed:  $x^3 + px + q = 0$ . Hint: Write  $(x - a)(x - b)(x - c)$ .

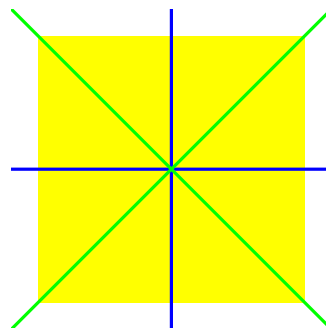
## Lecture 5: Symmetry groups

We look at all the rotational symmetries of a square and realize it as a group. Then, we do the same for all rotational and reflection symmetries of a rectangle.

### The rotation symmetries of a square

Given a square in the plane centered at the origin. We can rotate the square by 90, 180 or 270 degrees and get the same shape. Given two such rotations, we can perform one after the other and get another rotation. All the rotations leaving the square invariant form a **group**: one can "add" these operations and get a new operation.

| +        | turn 0   | turn 90  | turn 180 | turn 270 |
|----------|----------|----------|----------|----------|
| turn 0   | turn 0   | turn 90  | turn 180 | turn 270 |
| turn 90  | turn 90  | turn 180 | turn 270 | turn 0   |
| turn 180 | turn 180 | turn 270 | turn 0   | turn 90  |
| turn 270 | turn 270 | turn 0   | turn 90  | turn 180 |



We can write the **multiplication table** in a more compact way by writing 1 for the turn 90 degrees, 2 for the turn 180 degrees and 3 for the turn 270 degrees:

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 0 |
| 2 | 2 | 3 | 4 | 1 |
| 3 | 3 | 0 | 0 | 2 |

If we look at all rotations and reflections which leave the square invariant, there are 8 group elements. Beside the four rotations, we have 2 reflections at the diagonals and 2 reflections at the main axes.

- 1) What do you obtain if you reflect at the  $x$  axes, then reflect at the diagonal  $x=y$ ?

- 2) What do you obtain if you reflect first at the diagonal  $x=y$  and then reflect at the  $x$  axes?

- 3) We have seen above that the rotations which leave the square invariant, form a group which satisfies  $xy = yx$ . It is commutative. Does the group of rotations and reflections form a symmetry group which is commutative?

## The symmetries of a rectangle

Assume now we have a rectangle which is not a square. We look at all possible symmetries of this object. They include the identity, the reflection at the axes as well as the reflection at the center (which is a turn 180 degrees).

- 4) What do you obtain if you compose a reflection at the  $x$ -axes with the reflection at the  $y$ -axes?

- 5) Is the symmetry group of a rectangle commutative?

The rotational symmetry group as well as the full rotation-reflection symmetry group can be introduced for any geometrical object. Like triangles, cubes, octahedrons or polyhedra for tilings in the plane. Understanding the symmetries of an object produces a link between algebra and geometry.

## The 15 puzzle

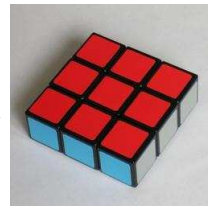
The **15 puzzle** was invented by **Noyes Palmer Chapman** in 1874. Chapman was a postmaster from Canastota in New York. From there the puzzle moved over to Syracuse, Watch-hill, Hartford and was first seriously sold in Boston. **Sam Loyd** offered a 1000 dollar prize for the solution of the case, when two pieces are switched. Since the number of transpositions plus the distance of the empty space to the 15'th position is always an even number, it can not be solved if it is odd initially.



1) Why does the puzzle have less or equal than  $16!$  group elements if the hole can be anywhere? It is a bit harder to see that there are exactly  $16!/2$  group elements with the whole at the end. It is better to not assume that the hole has to be at the end since otherwise, one has no group. The god number is still not known. It is between 152 and 208 for single tile moves.

## The floppy

The **floppy cube** was designed by Katsuhiko Okamoto. With 192 possible positions it is much less complex than the Rubik cube. We will learn how to solve it in class.



## The Rubik's cube

The **Rubik's cube** is quite a large puzzle. 3) Argue that the rubic cube has less than  $8! \cdot 12! \cdot 3^8 \cdot 2^{12}$  group elements.

