Lecture 9: Topology

Topology studies properties of geometric objects which do not change under continuous reversible deformations. For a topologist, a coffee cup with a single handle is the same as a doughnut. One can deform one into the other without punching any holes in it or ripping things apart. Similarly, a plate and a croissant are the same. But a croissant is not equivalent to a doughnut. On a doughnut, there are closed curves which can not be deformed to a point. For a topologist the letters O and P are the same but different from the letter B. The mathematical setup is beautiful: a **topological space** is a set X with a set O of subsets of X containing both $\emptyset$ and X such that finite intersections and arbitrary unions in O are in O. Sets in O are called **open sets** and O is called a **topology**. The complement of an open set is called **closed**. Examples of topologies are the **trivial topology** $O = \{\emptyset, X\}$, where no open sets besides the empty set and X exist or the discrete topology $O = \{A \subset X\}$, where every subset is open. But these are in general not interesting. An important example on the plane X is the collection O of sets U in the plane X for which every point is the center of a small disc still contained in U. A special class of topological spaces are **metric spaces**, where a set X is equipped with a **distance function** $d(x, y) = d(y, x) \geq 0$ which satisfies the **triangle inequality** $d(x, y) + d(y, z) \geq d(x, z)$ and for which $d(x, y) = 0$ if and only if $x = y$. A set U in a metric space is open if to every x in U, there is a **ball** $B_r(x) = \{y | d(x, y) < r\}$ of positive radius r contained in U. Metric spaces are topological spaces but not all topological spaces are metric: the trivial topology for example is not in general. For doing **calculus** on a topological space X, each point has a neighborhood called **chart** which is topologically equivalent to a disc in Euclidean space. Finitely many such neighborhoods covering X form an **atlas** of X. If the charts are glued together with identification maps on the intersection one obtains a **manifold**. Two dimensional examples are the **sphere**, the **torus**, the projective plane or the **Klein bottle**. Topological spaces X, Y are called **homeomorphic** meaning "topologically equivalent" if there is an invertible map from X to Y which is also induces an invertible map on the corresponding topologies. A basic task is to decide whether two spaces are equivalent in this sense or not. The surface of the coffee cup for example is equivalent in this sense to the surface of a doughnut but it is not equivalent to the surface of a sphere.

Many properties of geometric spaces can be understood by replacing them with **graphs** forming a skeleton of the space. A graph is a finite collection of vertices V together with a finite set of edges E, where each edge connects two points in V. For example, the set V of cities in the US where the edges are pairs of cities connected by a street is a graph. The **Königsberg bridge problem** was a trigger puzzle for the study of graph theory. **Polyhedra** were an other start in graph theory. It study is losely related to the analysis of surfaces. The reason is that one can see polyhedra as discrete versions of surfaces. In computer graphics for example, surfaces are rendered as finite graphs, using triangularizations.

The **Euler characteristic** of a convex polyhedron is a remarkable topological invariant. It is

$$V - E + F = 2 ,$$

where V is the number of vertices, E the number of edges and F the number of **faces**. This number is equal to 2 for connected polyhedra in which every closed loop can be pulled together to a point. This formula for the Euler characteristic is also called **Euler's gem**, a fact which comes with a rich history. **René Descartes** seems have stumbled upon it and written it down in a secret notebook. It was Leonard Euler in 1752 was the first to proved the formula for convex polyhedra. A convex polyhedron is called a **platonic solid**, if all vertices are on the unit sphere, all edges have the same length and all faces are congruent polygons. A theorem of Theaetetus states that there are only 5 platonic solids: [Proof: Assume the faces are regular n-gons and m of them meet at each vertex. Beside the Euler relation $V + E + F = 2$, a polyhedron also satisfies the relations $nF = 2E$ and $mV = 2E$ which are obvious from counting vertices or edges in different ways. This gives $2E/m - E + 2E/n = 2$ or $1/n + 1/m = 1/E + 1/2$. From $n \geq 3$ and $m \geq 3$ we see that it is impossible that both m and n are larger than 3. There are now nly two possibilities: either $n = 3$ or $m = 3$. In the case $n = 3$ we have $m = 3, 4, 5$ in the case $m = 3$ we have $n = 3, 4, 5$. The five possibilities $(3, 3), (3, 4), (3, 5), (4, 3), (5, 3)$ represent the 5 platonic solids.] The pairs (n, m) are called the **Schläfly symbol** of the polyhedron:

Name	V	E	F	V-E+F	Schläfli	Name	V	E	F	V-E+F	Schläfli
tetrahedron	4	6	4	2	$\{3,3\}$						
hexahedron	8	12	6	2	$\{4,3\}$	dodecahedron	20	30	12	2	$\{5,3\}$
octahedron	6	12	8	2	$\{3,4\}$	icosahedron	12	30	20	2	$\{3,5\}$

The Greeks proved this more geometrically: Euclid showed in his "Elements" that at each vertex, we can attach 3,4 or 5 equilateral triangles, 3 squares or 3 regular pentagons. (6 triangles, 4 squares or 4 pentagons would lead to a total angle which is too large because each corner must have at least 3 different edges). **Simon Antoine-Jean L'Huilier** refined in 1813 Euler's formula to situations with holes: $V - E + F = 2 - 2g$, where g is the number of holes. For a doughnut with one hole we have $V - E + F = 0$. . Cauchy first proved that there are exactly 4 non-convex regular **Kepler-Poinsot** polyhedra. Their Euler characteristic can be different.

Name	V	E	F	V-E+F	Schläfli
small stellated dodecahedron	12	30	12	-6	$\{5/2,5\}$
great dodecahedron	12	30	12	-6	$\{5,5/2\}$
great stellated dodecahedron	20	30	12	2	$\{5/2,3\}$
great icosahedron	12	30	20	2	$\{3,5/2\}$

If two different face types are allowed but each vertex still look the same, one obtains 13 **semi-regular polyhedra.** They were first studied by **Archimedes** in 287 BC. Since his work is lost, **Johannes Kepler** is considered the first person since antiquity to describe the whole set of thirteen in his "Harmonices Mundi". The Euler characteristic $\chi = 2 - 2g$ is also useful for surfaces. One can reduce the question to graphs, triangularizations of the surface. The Euler characteristic completely characterizes smooth compact surfaces if they are orientable. A non-orientable surface, the **Klein bottle** can be obtained by gluing ends of the Möbius strip. Classifying higher dimensional manifolds is more difficult and finding more invariants is part of modern research. Higher analogues of polyhedra are called **polytopes** (Alicia Boole Stott). **Regular polytopes** are the analogue of the platonic solids in higher dimensions. Here they are for the first few dimensions:

dimension	name	Schläfli symbols
2:	Regular polygons	$\{3\}, \{4\}, \{5\}, ...$
3:	Platonic solids	$\{3,3\}, \{3,4\}, \{3,5\}, \{4,3\}, \{5,3\}$
4:	Regular 4D polytopes	$\{3,3,3\}, \{4,3,3\}, \{3,3,4\}, \{3,4,3\}, \{5,3,3\}, \{3,3,5\}$
≥ 5:	Regular polytopes	$\{3,3,3,\dots,3\}, \{4,3,3,\dots,3\}, \{3,3,3,\dots,3,4\}$

Ludwig Schläfly found in 1852 that there are exactly six convex regular convex 4-polytopes or **polychora**. The expression "choros" is Greek for "space". Schlaefli's polyhedral formula tells that for any **convex polytope** in four dimensions, the relation $V - E + F - C = 0$ holds, where C is the number of 3-dimensional **chambers**. In dimensions 5 and higher, there are only 3 types of polytopes: the higher dimensional analogues of the tetrahedron, octahedron and the cube. A general formula $\sum_{i=0}^{d-1} (-1)^i V_i = 1 - (-1)^d$ gives the Euler characteristic of a convex polytop in d dimensions with i-dimensional parts V_i.

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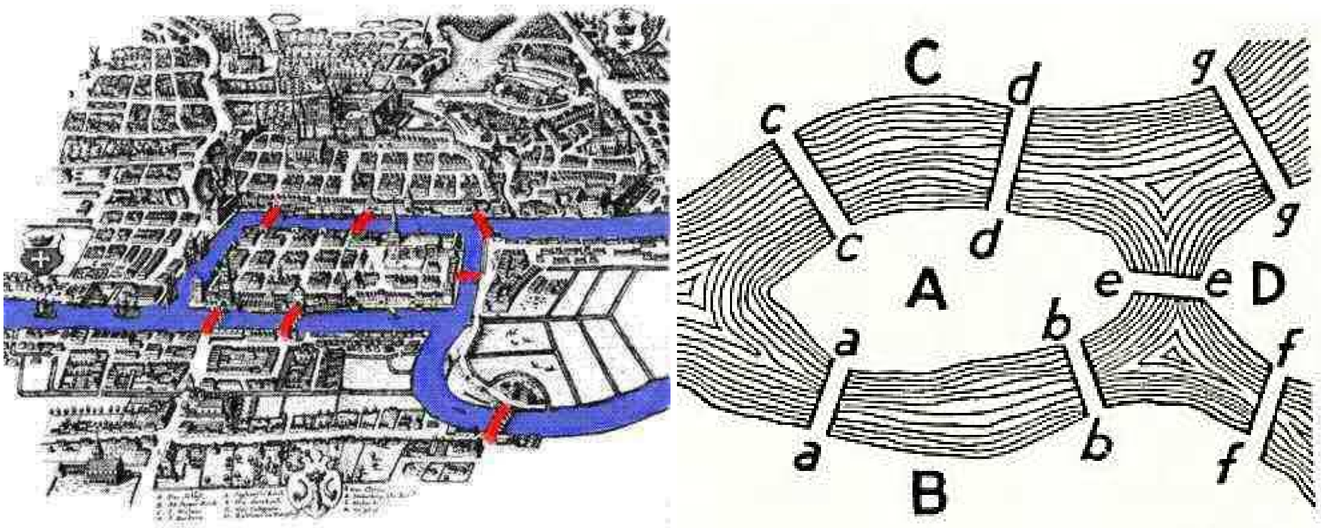
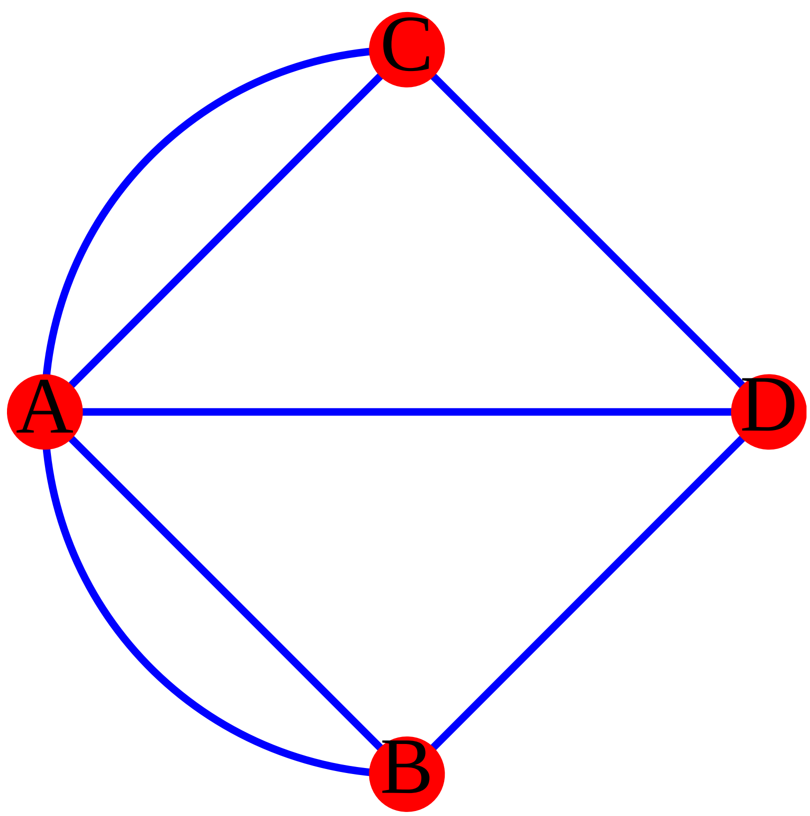
For a topologist, a triangle and a circle are equivalent as they can be deformed into each other. The Boston subway map is a topologically equivalent representation of the actual subway paths. Lengths need not be to scale. Topology is "rubber geometry".

One of the starting points of topology is the Königsberg bridge problem. Is it possible to find a path which which crosses every bridge once and only once?



Euler realized that one can see this as an abstract problem about graphs. It it possible find a path through the graph which covers the entire graph but no edge twice? Such a path is called an **Eulerian path**. If the start and end point is the same it is called an **Eulerian circuit**.



Euler presented his work in 1735. It was published as "Solutio problematis ad geometriam situs pertinentis" in 1741.

It is historically significant, because it is one of the first results in graph theory an area of mathematics closely related to topology because a lot of topology have analogue results on graphs.

There is more to that problem than just a new field of mathematics:

The problem shows how mathematical abstraction can simplify a problem.