Lecture 12: Dynamical systems

Dynamical systems theory is the science of time evolution. If time is **continuous** the evolution is defined by a **differential equation** $\dot{x} = f(x)$. If time is **discrete** then we look at the **iteration of a map** $x \to T(x)$.

The goal is to **predict the future** of the system when the present state is known. A **differential equation** is an equation of the form $d/dtx(t) = f(x(t))$, where the unknown quantity is a path $x(t)$ in some "phase space". We know the **velocity** $d/dtx(t) = \dot{x}(t)$ at all times and the initial configuration $x(0)$), we can to compute the **trajectory** $x(t)$. What happens at a future time? Does $x(t)$ stay in a bounded region or escape to infinity? Which areas of the phase space are visited and how often? Can we reach a certain part of the space when starting at a given point and if yes, when. An example of such a question is to predict, whether an asteroid located at a specific location will hit the earth or not. An other example is to predict the weather of the next week.

An examples of a dynamical systems in one dimension is the differential equation

$$x'(t) = x(t)(2 - x(t)), x(0) = 1$$

It is called the **logistic system** and describes population growth. This system has the solution $x(t) = 2e^t/(1 + e^{2t})$ as you can see by computing the left and right hand side.

A **map** is a rule which assigns to a quantity $x(t)$ a new quantity $x(t + 1) = T(x(t))$. The state $x(t)$ of the system determines the situation $x(t + 1)$ at time $t + 1$. An example is is the **Ulam map** $T(x) = 4x(1 - x)$ on the interval $[0, 1]$. This is an example, where we have no idea what happens after a few hundred iterates even if we would know the initial position with the accuracy of the Planck scale.

Dynamical system theory has applications all fields of mathematics. It can be used to find roots of equations like for

$$T(x) = x - f(x)/f'(x) .$$

A system of number theoretical nature is the **Collatz map**

$$T(x) = \frac{x}{2}\ (\text{evenx}), 3x + 1 \text{ else} .$$

A system of geometric nature is the **Pedal map** which assigns to a triangle the pedal triangle.

About 100 years ago, **Henry Poincaré** was able to deal with **chaos** of low dimensional systems. While **statistical mechanics** had formalized the evolution of large systems with probabilistic methods already, the new insight was that simple systems like a **three body problem** or a **billiard map** can produce very complicated motion. It was Poincaré who saw that even for such low dimensional and completely deterministic systems, random motion can emerge. While physisists have dealt with chaos earlier by assuming it or artificially feeding it into equations like the **Boltzmann equation**, the occurrence of stochastic motion in geodesic flows or billiards or restricted three body problems was a surprise. These findings needed half a century to sink in and only with the emergence of computers in the 1960ies, the awakening happened. Icons like

wing of a butterfly can produce a tornado in Texas in a few weeks. The reason for this statement is that the complicated equations to simulate the weather reduce under extreme simplifications and truncations to a simple differential equation $\dot{x} = \sigma(y-x), \dot{y} = rx - y - xz, \dot{z} = xy - bz$, the **Lorenz system**. For $\sigma = 10, r = 28, b = 8/3$, Ed Lorenz discovered in 1963 an interesting long time behavior and an aperiodic "attractor". Ruelle-Takens called it a **strange attractor**. It is a **great moment** in mathematics to realize that attractors of simple systems can become fractals on which the motion is chaotic. It suggests that such behavior is abundant. What is chaos? If a dynamical system shows **sensitive dependence on initial conditions**, we talk about **chaos**. We will experiment with the two maps $T(x) = 4x(1-x)$ and $S(x) = 4x - 4x^2$ which starting with the same initial conditions will produce different outcomes after a couple of iterations.
The sensitive dependence on initial conditions is measured by how fast the derivative dT^n of the n'th iterate grows. The exponential growth rate γ is called the **Lyapunov exponent**. A small error of the size h will be amplified to $he^{\gamma n}$ after n iterates. In the case of the Logistic map with $c = 4$, the Lyapunov exponent is $\log(2)$ and an error of 10^{-16} is amplified to $2^n \cdot 10^{-16}$. For time $n = 53$ already the error is of the order 1. This explains the above experiment with the different maps. The maps $T(x)$ and $S(x)$ round differently on the level 10^{-16}. After 53 iterations, these initial fluctuation errors have grown to a macroscopic size.
Here is a famous open problem which has resisted many attempts to solve it: Show that the map $T(x,y) = (c\sin(2\pi x) + 2x - y, x)$ with $T^n(x,y) = (f_n(x,y), g_n(x,y))$ has sensitive dependence on initial conditions on a set of positive area. More precisely, verify that for $c > 2$ and all n $\frac{1}{n}\int_0^1\int_0^1 \log|\partial_x f_n(x,y)|\,dxdy \geq \log(\frac{c}{2})$. The left hand side converges to the average of the Lyapunov exponents which is in this case also the **entropy** of the map. For some systems, one can compute the entropy. The logistic map with $c = 4$ for example, which is also called the **Ulam map**, has entropy $\log(2)$. The **cat map**

$$T(x,y) = (2x + y, x + y) \bmod 1$$

has entropy $\log|(\sqrt{5}+3)/2|$. This is the logarithm of the larger eigenvalue of the matrix.

While questions about simple maps look artificial at first, the mechanisms prevail in other systems: in astronomy, when studying planetary motion or electrons in the van Allen belt, in mechanics when studying coupled penduli or nonlinear oscillators, in fluid dynamics when studying vortex motion or turbulence, in geometry, when studying the evolution of light on a surface, the change of weather or tsunamis in the ocean. Dynamical systems theory started historically with the problem to understand the **motion of planets**. Newton realized that this is governed by a differential equation, the **n-body problem**

$$x_j''(t) = \sum_{i=1}^n \frac{c_{ij}(x_i - x_j)}{|x_i - x_j|^3} ,$$

where c_{ij} depends on the masses and the gravitational constant. If one body is the sun and no interaction of the planets is assumed and using the common center of gravity as the origin, this reduces to the **Kepler problem** $x''(t) = -Cx/|x|^3$, where planets move on **ellipses**, the radius vector sweeps equal area in each time and the period squared is proportional to the semi-major axes cubed. A great moment in astronomy was when Kepler derived these laws empirically. An other great moment in mathematics is Newton's theoretically derivation from the differential equations.

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Dynamics

Dynamical systems theory studies the time evolution of systems.

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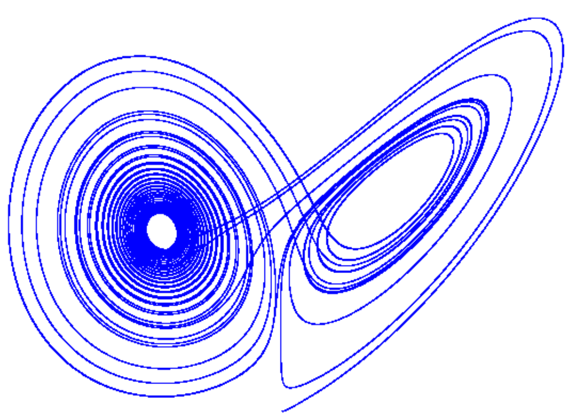
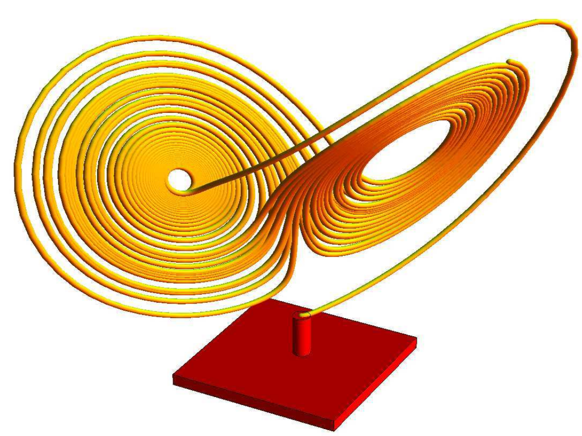
Here is the prototype of a differential equation

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \ . \end{aligned}$$

the **Lorenz system**. There are three parameters. For $\sigma = 10, r = 28, b = 8/3$, one observes a **strange attractor**.

The second picture below is a printable Lorentz attractor.

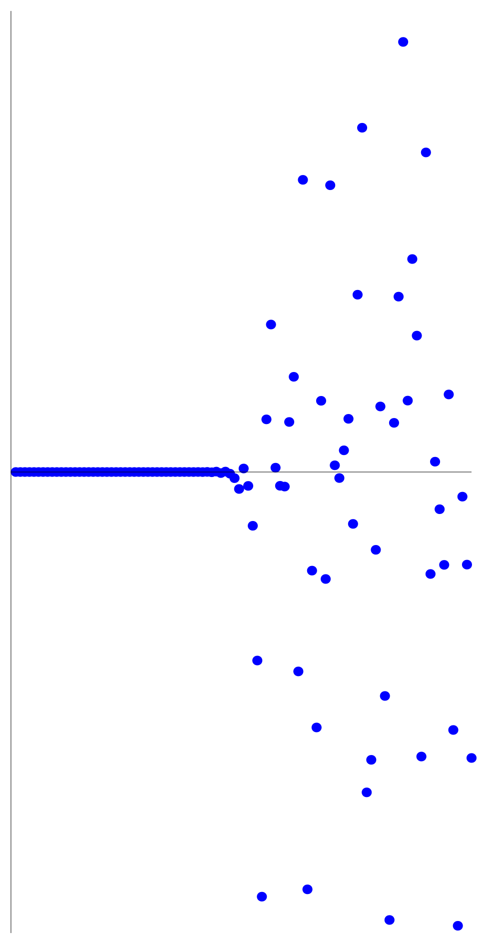




Chaos

There are various definitions of chaos. One of them is "sensitive dependence on initial conditions". The smallest change of the initial state produce large changes in the future. We illustrate this with the example of the logistic, where already for $n = 60$, there is a big difference between the orbit of $T(x) = 4x(1-x)$ and $S(x) = 4x - 4x^2$.

The following picture shows the difference of $T^n(x) - S^n(x)$ for $n = 0$ to 100 starting with 0.3. With 16 digits of accuracy of computation it does not make sense to talk about the value of $T^{100}(0.3)$. It is undefined evenso the map is very concrete and deterministic. Increasing the accuracy does not help much. The error doubles in each step so that we expect to see an error of the order 1 after $\log_2(10^{16}) = 53$ steps. If we computed with 100 digits accuracy, then we would have lost all information after $\log_2(10^{100}) = 332$ steps. Because $\log_2(10) = 3.32193$ we have only have to iterate 3.4 times the given accuracy to have no idea where the $n'th$ iterate is. As we can see, the value depends then for example on how we have written down the equations.



Already simple maps produce, when iterated unpredictable results.