Magical networks

Objective

In this first lecture, we look at a mysterious mathematical structure. It will almost certainty be unknown to you. The goal is to see how mathematics can produce complex structure from relatively simple rules.

The square function

Lets look at all positive integers smaller than some number n and represent them as points on a sheet of paper. Now look at each number k and compute the **remainder** of k^2 when dividing by n. This is again one of the number m. For example, if $n = 11$ and $k = 5$ then $k^2 = 25$ leaves the remainder 3 when dividing by 11. We write $3 = 5^2 \mod 11$. Now connect 5 with 3. Do that with every number and look at the **graph**. It is a collection of points and connections between them. Lets call this the **orbital graph** defined by the function $f(x) = x^2$ and the number n.

Lets construct the orbital graph for $n = 11$.

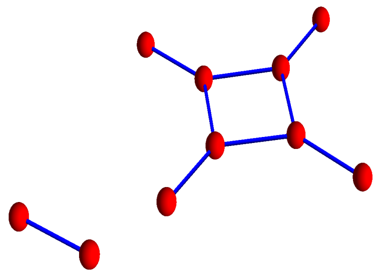
k	k^2
1	1
2	4
3	9
4	5
5	3
6	3
7	5
8	9
9	4
10	1

Now it is up to you. Construct the orbital graph for $n = 17$.

k	k^2
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	
16	

We see that on the right hand side the numbers $1, 3, 4, 5, 9$ appear and that $2, 6, 8, 10, 11$ do not appear on the right. The numbers which appear have a fancy name and are called **quadratic residues**. The others are quadratic non-residues. Can you see an other pattern in the table?

Lets look at the picture. Beside you draw the picture for $n = 17$. There is a fundamental difference between these two cases. What is it?

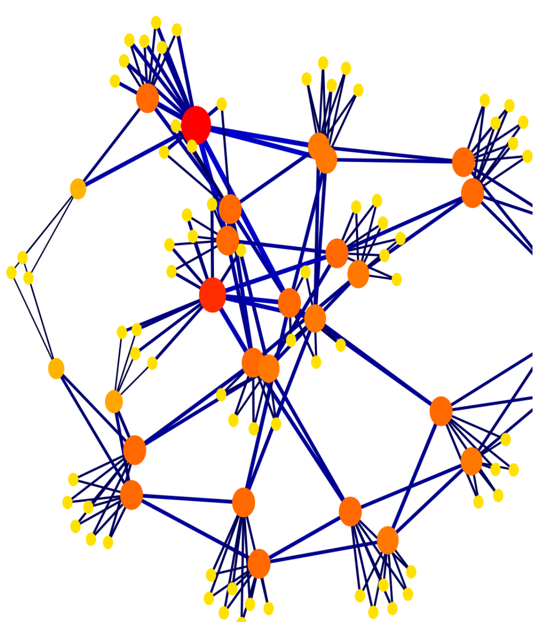


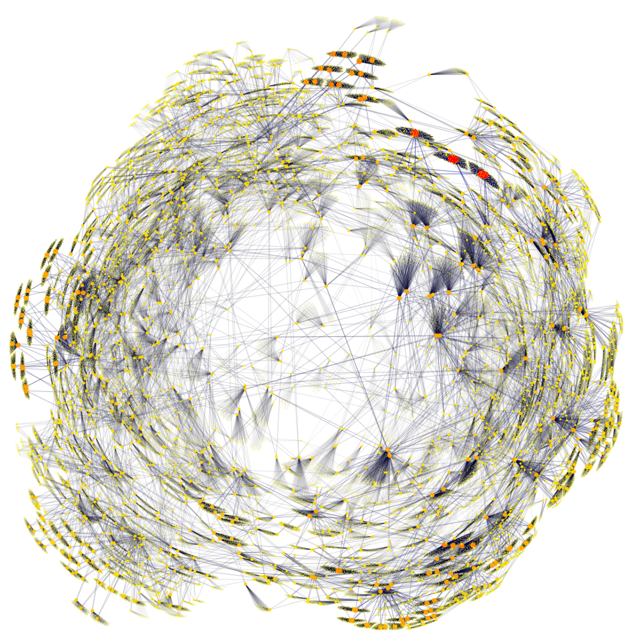
n=11 Is already drawn. n=17 It is your turn!

Networks

What we have seen so far are graphs. Graphs are objects which were introduced at the dawn of a mathematical field called topology. There are a lot of geometric properties one can explore with graphs like whether one can draw them in the plane without intersections. We have defined these graphs using number theoretical construction, using basic arithmetic. Lets look at larger graphs defined in the same way. Investigating large graphs is part of network theory, a branch of graph theory and also computer science, because many data structures are graphs.

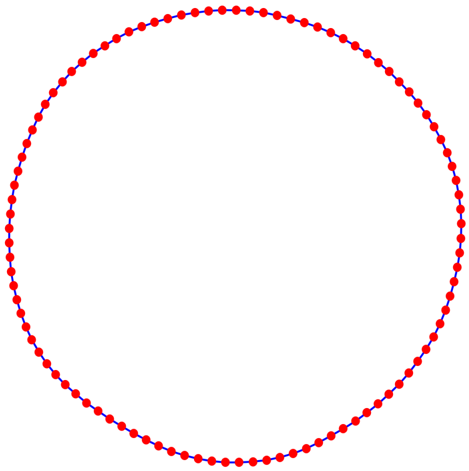
Now lets look at two functions $f(x) = x^2 + a$ and $g(x) = x^2 + b$. Connect n with $f(n)$ and n with $g(n)$. This network is now more complicated.



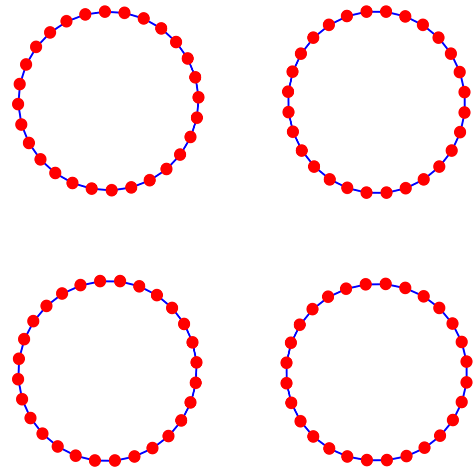


A simpler case

The game can be played also other functions. One can replace $x^2 + c$ with any other type of function. A simpler case is $f(x) = ax$. Lets look at some pictures

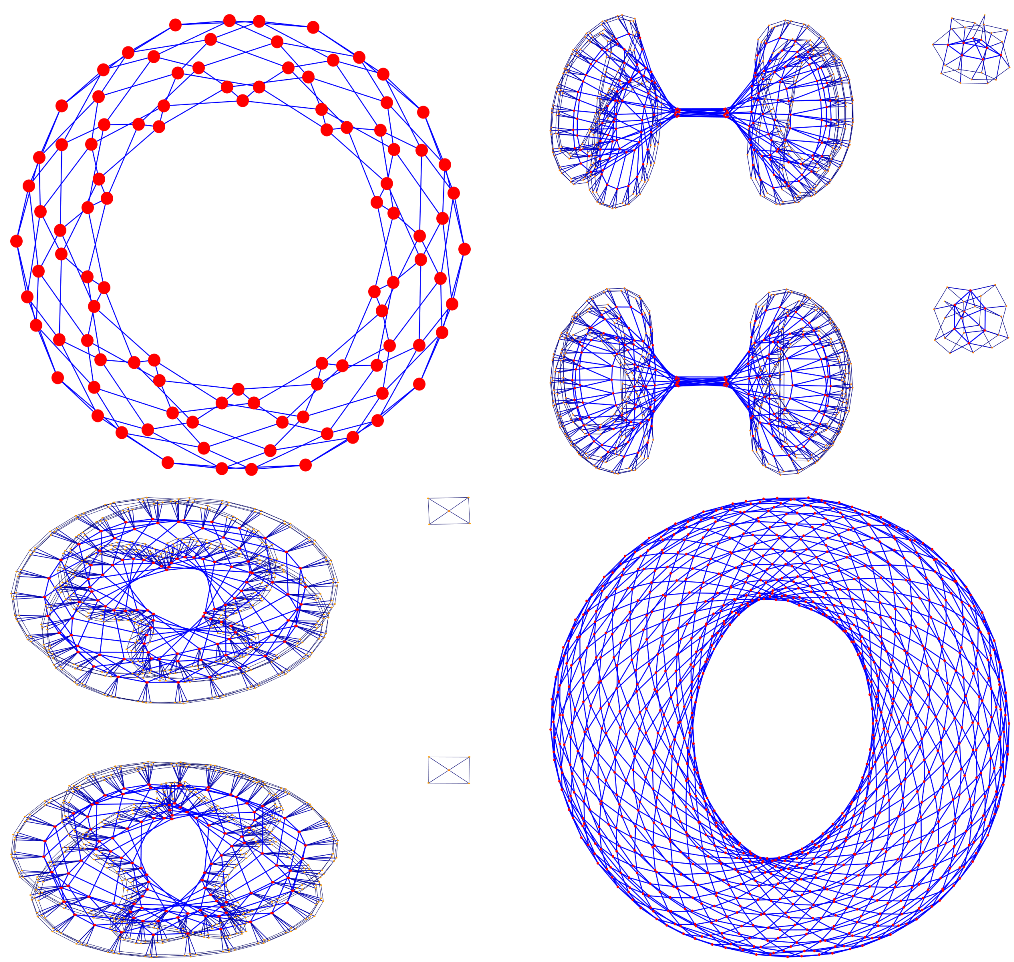


$f(x) = 2x$, $n = 101$



$f(x) = 2x$, $n = 113$

Here are some graphs with two generators:
The first for $n = 101$ and maps $T(x) = 3x, S(x) = 2x$
The first for $n = 1026$ and maps $T(x) = 3x, S(x) = 5x$
The first for $n = 1030$ and maps $T(x) = 3x, S(x) = 5x$
The first for $n = 1013$ and maps $T(x) = 101x, S(x) = 11x$



More square cases

Here we see the square graphs for the first few primes. If you want to produce the graph yourself, you can check here whether you get the same thing.