# Coloring problems

Coloring maps

In this first lecture, we look at one of the most famous problems in mathematics: the **4 color theorem**. We want to understand the problem and place it into the context of mathematics. Lets look at the following map of Switzerland from a book of Tietze from 1946.

You see that the map can be colored by 4 colors in such a way that no adjacent countries have the same color.
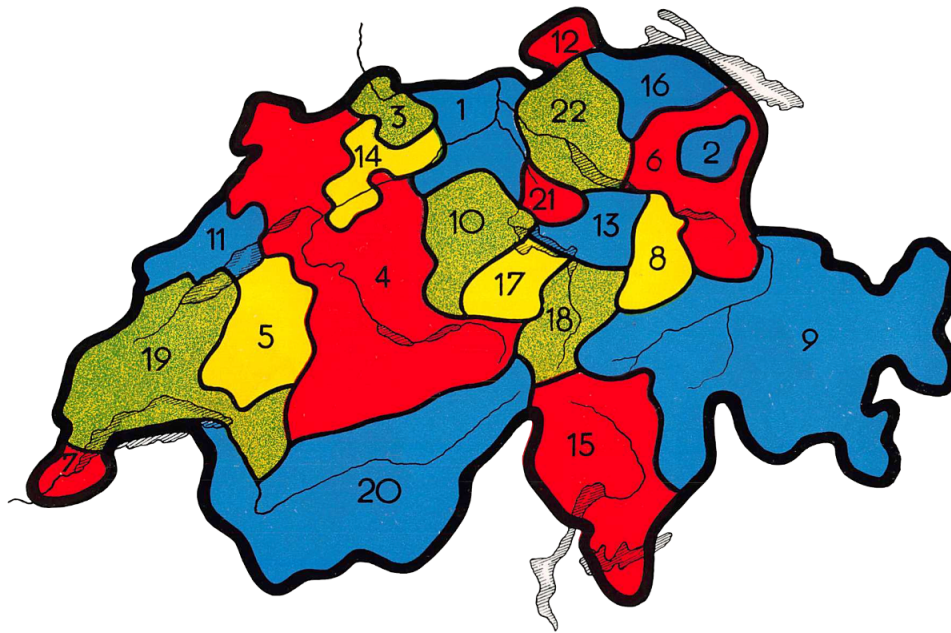

If each of the countries is represented by a node and two node are connected if the two countries have a common boundary, we get a graph.
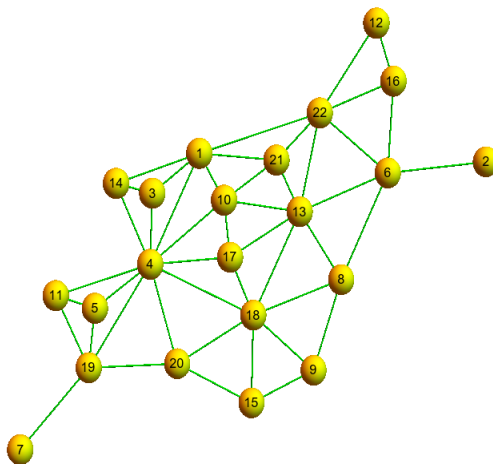
1) Lets look at the four nodes $1, 3, 4, 14$ or $4, 5, 11, 19$ in the Swiss county graph. They are called "tetrahedra". How many colors do we need to color each of them?

2) Lets look at the nodes neighboring the county 10. These are the counties $1, 4, 17, 13, 21$. They form a circle = circular graph. How many colors are need to color the circle, how many are needed to color the disc?

3) Lets look at the nodes neighboring 21. These are the counties $1, 10, 13, 22$, also forming a circle = circular graph. How many colors are needed to disc of radius 1 centered at 21. A disc of radius 1 is a wheel graph.
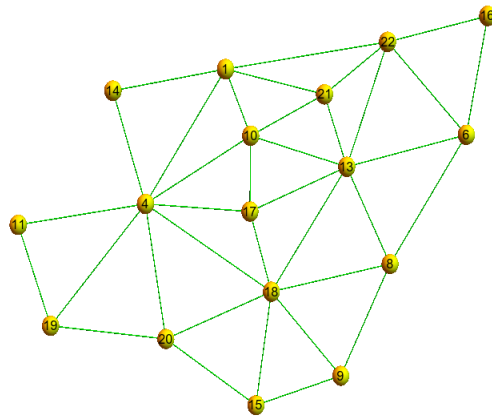
4) Can you find a general rule telling how many colors are needed to color a wheel graph. This is the situation where a country has n neighboring countries which are arranged as a circle around it. Under which conditions can we color this patch with 3 colors, when do we need 4 colors. Or do we need more colors?

5) Lets look at the countries $2, 7, 12, 3, 5$ in the Swiss county graph. Can you see that if we can color the situation without them, then we color the entire graph?

6) After taking away the countries $2, 7, 12, 3, 5$ we have a disc with boundary. (We mean with this that we have a graph for which there are interior vertices with circular unit sphere and vertices for which we have a noncircular interval graph. Can you glue two such discs together at the boundary to get a sphere? Argue why if we can color this sphere, we have solved the original problem and colored the disc.

7) A wheel graph has a central vertex connected to $n \geq 3$ boundary points, where the later are all connected in a circular manner, connected to the center. Lets call this wheel graph W_n. How many colors do we need to color such a graph? What happens if we take two wheel graphs W_4 and glue them together along the circular boundary (as in 6)). What geometric object do we get? (It is one of the platonic solids). How many colors do we need in this case?

8) How many colors are needed to color the faces of a cube such that two adjacent faces have different colors? If we produce a vertex for each face and connect two such vertices if the faces are adjacent, what graph do we get?
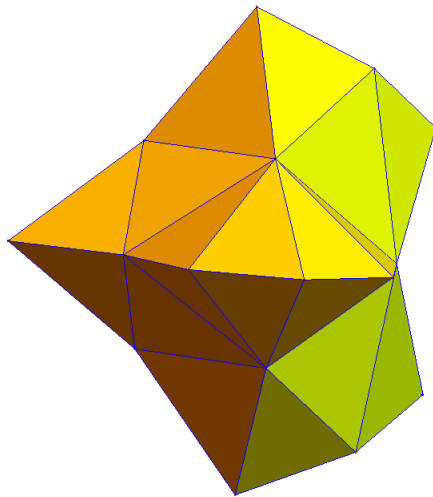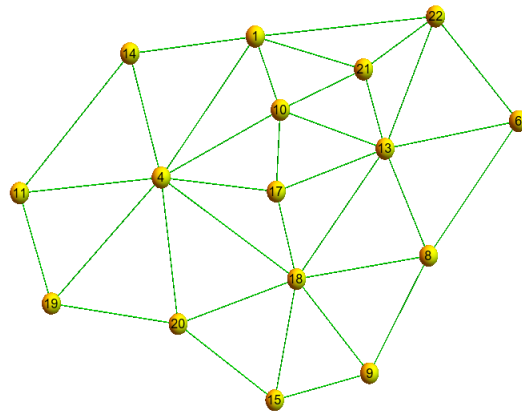
Some history

In the book "Math through the ages" we read "History often helps by adding context. Mathematics, after all, is a cultural product. It is created by people in a particular time and place and is often affected by that context. Knowing more about this helps us understand how mathematics fits in with other human activities."

Lets look at the history of the 4 color problem as it is fascinating and touches on some basic areas of mathematics. The problem has connections with topology, geometry, algebra combinatorics and probability theory.

The mathematician Möbius raised in 1840 a precursor problem called **Möbius-Weiske puzzle**.

"Once upon a time in the Far East there lived a Prince with five sons. These sons were to inherit the kingdom after his death. But in his will, the Prince made the condition that each of the five parts into which the kingdom was to be divided must border on every other. After the death of the father, the five sons worked hard to find a division of the land which would conform to his wishes; but all their efforts were in vain."
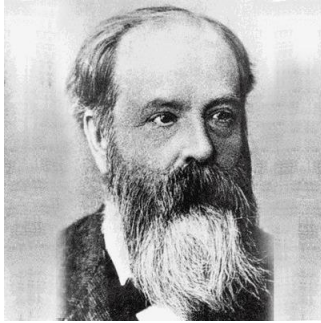
Francis Guthrie

The actual problem was posed first by **Francis Guthrie**. His brother Frederick communicated it to his teacher Augustus de Morgan, a former teacher of Francis, in 1852. The later communicated it to William Hamilton. Arthur Cayley in 1878 helped to spread the problem by putting in first in print and giving it the status of an open problem. A **"victorian comedy of errors"** followed: Alfred Kempe published a proof in 1879 which turned out to be incomplete. The gap was noticed by Heawood 11 years later in 1890. There was also an unsuccessful attempt by Peter Tait in 1880.

Ken Appel and Wolfgang Haken

After considerable work including Charles Pierce, George Birkhoff, Oswald Veblen, Philip Franklin, Hassler Whitney, Hugo Hadwiger, Leonard Brooks, William Tutte, Yoshio Shimamoto, Heinrich Heesch and Karl Dürre and computer analysis by Frank Allaire and Edwart Swart, a computer assisted proof of the 4-color theorem was obtained by Ken Appel and Wolfgang Haken in 1976. A new proof appeared in 1997 by Neil Robertson, Daniel Sanders, Paul Seymour, and Robin Thomas. and the computer programs by these four authors are still publicly available today. George Gonthier produced in 2004 a fully machine-checked proof of the four-color theorem.