Lecture 10: Fractals

1. Objective

We want to compute the dimension of various objects in the plane.

2. Definition of Dimension

If we need n squares of side length r to cover an object X. the dimension is defined as

$$d = \frac{-\log(n)}{\log(r)}$$

when r gets zero.

3. Dimension of a curve

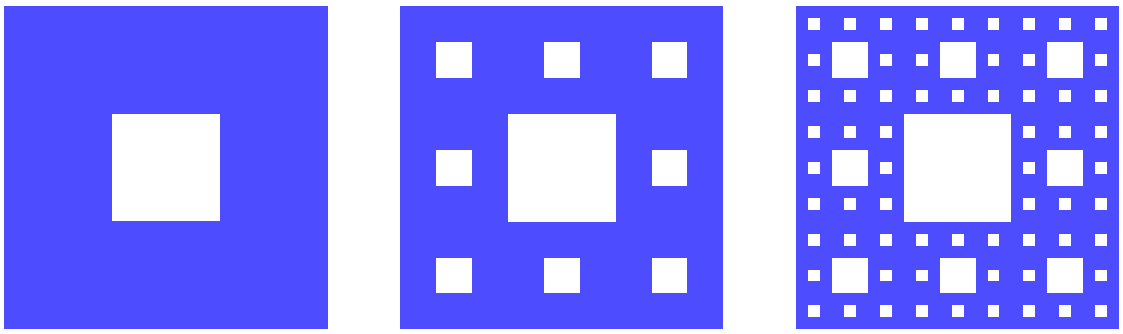
1) Assume a curve is given as the boundary of the unit square. How many squares of length $r = 1/10$ do we need to cover the curve? If we call n this number, what is $-\log(n)/\log(r)$?

4. Dimension of a region

2) Assume a region is the unit square. How many squares of length $r = 1/10$ do we need to cover the square? If we call n this number, what is $-\log(n)/\log(r)$?

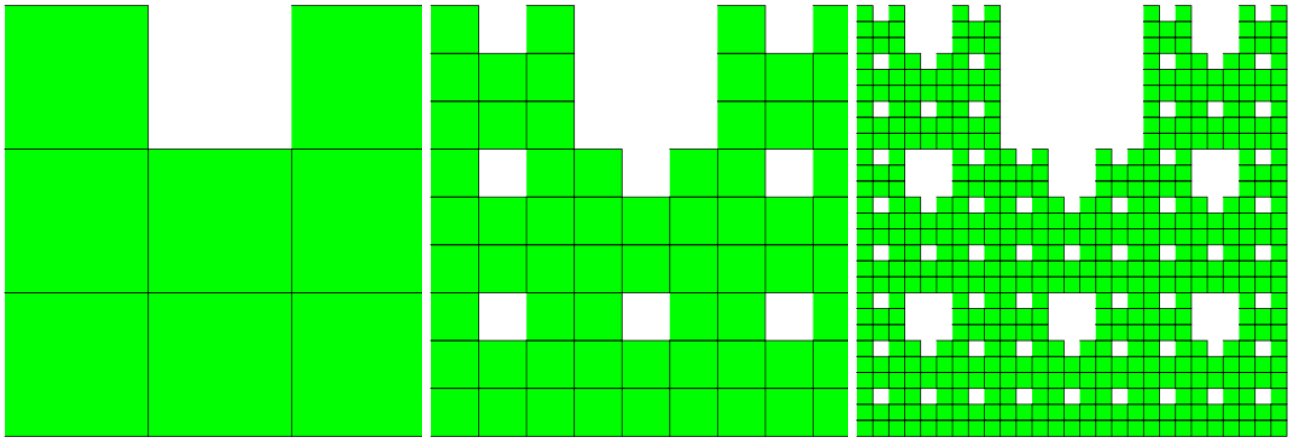
5. The Sirpinski carpet

The **Sirpinski carpet** is constructed recursively by dividing a square in 9 equal squares and cutting away the middle one, repeating this procedure with each of the squares etc. At the k'th step, we need $n = 8^k$ squares of length $r = 1/3^k$ to cover the carpet. What is the dimension?



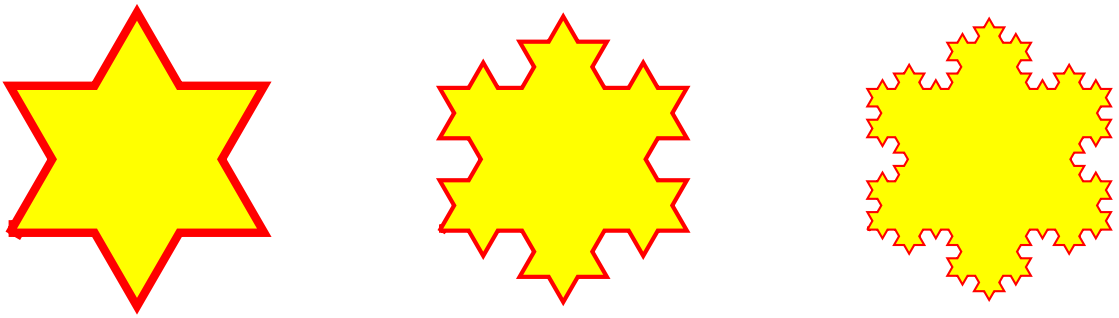
6. A castle

What is the dimension of the following fractal from which we see the first levels of construction?



7. The Koch Snowflake

What is the dimension of the Koch snowflake? How large is n, the number of squares we need to covert the flake if the square has size $1/3^k$ assuming that the first triangle has side length 1.



Lecture 10: Julia and Mandelbrot set

1. Objective

We want to understand the definition of the Julia sets and the Mandelbrot set. Define

$$T(z) = z^2 + c \, ,$$

where c is a fixed parameter. The **filled in Julia set** J_c is the set of points z for which the orbit $z, T(z), T(T(z))...$ stays bounded. The **Mandelbrot set** M is the set of c for which the point 0 is in the filled in Julia set J_c. It is the set of c such that $0, T(0) = c, T(T(0)) = c^2 + c, T(T(T(0))) = (c^2 + c)^2 + c...$ stays bounded. The **Julia set** finally is the boundary of the filled in Julia set.

2. Complex Arithmetic

1) What is the square root of -9?

2) Add $2 + 6i$ with $6 + 8i$.

3) Multiply $2 + 6i$ with $6 + 8i$.

4) What is the length of the complex number $3 + i4$?

3. Drawing an orbit

5) Assume $c = 2$. Compute the first 3 steps of the orbit of $z = 1$ of the quadratic map.

4. Drawing the Mandelbrot set

6) Verify that 0 is inside the Mandelbrot set. Verify that -1 is inside the Mandelbrot set. Verify that i is inside the Mandelbrot set.

7) Verify that 2 is outside the Mandelbrot set. Verify that 1 is outside the Mandelbrot set.

8) Can you verify that $1/4$ is the largest real number in the Mandelbrot set and -2 the smallest real number in the Mandelbrot set?

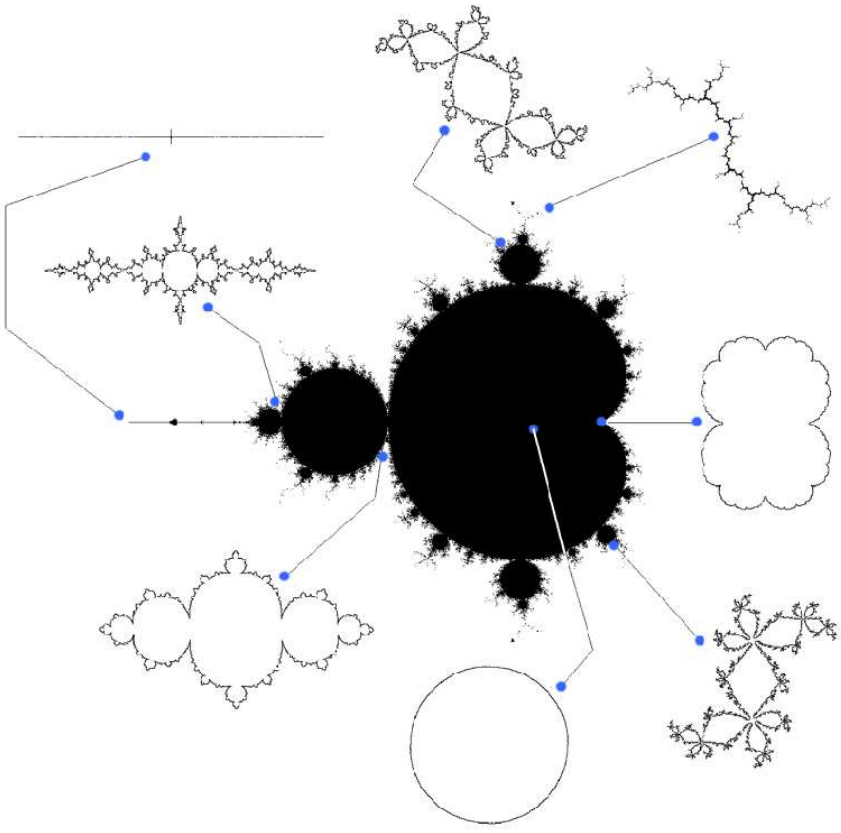


Image credit: MIQL.com