Lecture 1: Mathematical roots

1.1. Similarly, as one has distinguished the **canons of rhetorics**: memory, invention, delivery, style, and arrangement, or combined the **trivium**: grammar, logic and rhetorics, with the **quadrivium**: arithmetic, geometry, music, and astronomy, to obtain the seven **liberal arts and sciences**, one has tried to **organize all mathematical activities**. The organization of the possible branches of mathematics is a **taxonomy problem**. It is rather arbitrary and also has historically grown.
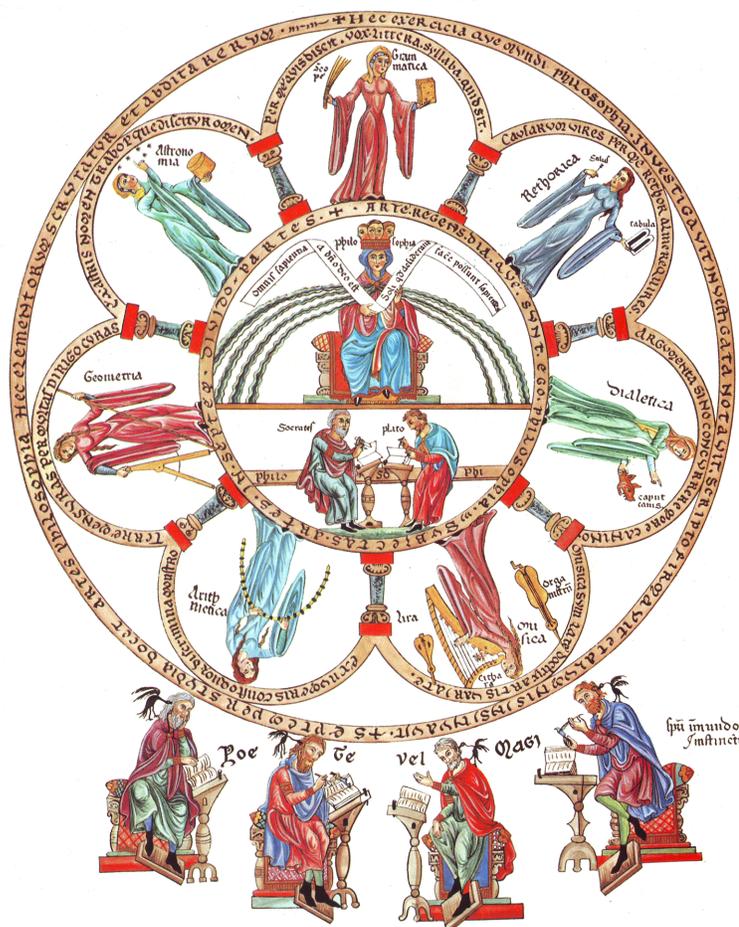

FIGURE 1.   Philosophia at septem artes liberales (Philosophy within the seven liberal arts and sciences) by Herrad of Landsberg (1125-1195), an 12'th century abbess at the castle of Landsberg. The picture is in the public domain and part of her work "Hortus deliciarum" (Garden of delights).

1.2. A natural approach which appears in the **eight ancient roots of mathematics** is based on practical tasks which humans usually do. Each activity in turn suggests also a key area in mathematics:

counting and sorting	**arithmetic**
spacing and distancing	**geometry**
positioning and locating	**topology**
surveying and angulating	**trigonometry**
balancing and weighing	**statics**
moving and hitting	**dynamics**
guessing and judging	**probability**
collecting and ordering	**algorithms**

1.3. Modern mathematics has grown into a sophisticated building. Modern classification systems split math into about 100 subfields. To morph the above 8 roots to the 12 mathematical areas, we complemented the ancient roots with calculus, numerics and computer science, merge trigonometry with geometry, separate arithmetic into number theory, algebra and arithmetic and turn statics into analysis. Let us call this more modern but still rather arbitrary adaptation the **12 modern roots of Mathematics**:

counting and sorting	**arithmetic**
spacing and distancing	**geometry**
positioning and locating	**topology**
dividing and comparing	**number theory**
balancing and weighing	**analysis**
moving and hitting	**dynamics**
guessing and judging	**probability**
collecting and ordering	**algorithms**
slicing and stacking	**calculus**
operating and memorizing	**computer science**
optimizing and planning	**numerics**
manipulating and solving	**algebra**

1.4. While relating **mathematical areas** with **human activities** is useful, it makes sense to also select specific topics in each of this area. Indeed, any of the subjects has grown to a huge tree itself. For our course, we will select a part in each of the 12 topics to build a lecture.

Arithmetic	numbers and number systems
Geometry	invariance, symmetries, measurement, maps
Number theory	Diophantine equations, factorizations
Algebra	algebraic and discrete structures
Calculus	limits, derivatives, integrals
Set Theory	set theory, foundations and formalisms
Probability	combinatorics, measure theory, statistics
Topology	polyhedra, topological spaces, manifolds
Analysis	extrema, estimates, variation, measure
Numerics	numerical schemes, codes, cryptology
Dynamics	differential equations, maps
Algorithms	computer science, artificial intelligence

1.5. Like any classification, this chosen division is rather arbitrary and a matter of personal preferences. The **MCC2020 AMS classification** distinguishes 63 main areas of mathematics. Many of them are broken off into even finer pieces. Additionally, there are fields which relate with other

areas of science, like economics, biology or physics:

00 General
01 History and biography
03 Mathematical logic and foundations
05 Combinatorics
06 Lattices, ordered algebraic structures
08 General algebraic systems
11 Number theory
12 Field theory and polynomials
13 Commutative rings and algebras
14 Algebraic geometry
15 Linear algebra; matrix theory
16 Associative rings and algebras
17 Non-associative rings and algebras
18 Category theory, homological algebra
19 K-theory
20 Group theory and generalizations

22 Topological groups, Lie groups
26 Real functions
28 Measure and integration
30 Functions of a complex variable
31 Potential theory
32 Several complex variables, analytic spaces
33 Special functions
34 Ordinary differential equations
35 Partial differential equations
37 Dynamical systems and ergodic theory
39 Difference and functional equations
40 Sequences, series, summability
41 Approximations and expansions
42 Fourier analysis
43 Abstract harmonic analysis
44 Integral transforms, operational calculus

45 Integral equations
46 Functional analysis
47 Operator theory
49 Calculus of variations, optimization
51 Geometry
52 Convex and discrete geometry
53 Differential geometry
54 General topology
55 Algebraic topology
57 Manifolds and cell complexes
58 Global analysis, analysis on manifolds
60 Probability theory, stochastic processes
62 Statistics
65 Numerical analysis
68 Computer science
70 Mechanics of particles and systems

74 Mechanics of deformable solids
76 Fluid mechanics
78 Optics, electromagnetic theory
80 Classical thermodynamics, heat transfer
81 Quantum theory
82 Statistical mechanics, structure of matter
83 Relativity and gravitational theory
85 Astronomy and astrophysics
86 Geophysics
90 Operations research, Programming
91 Game theory, Economics
92 Biology and other natural sciences
93 Systems theory and control
94 Information, communication, circuits
97 Mathematics education

1.6. What are **fancy developments** in mathematics today? Michael Atiyah identified in the year 2000 the following **six hot spots**:

$$
\begin{array}{lcr}
\text{local} & \text{and} & \text{global} \\
\text{low} & \text{and} & \text{high dimension} \\
\text{commutative} & \text{and} & \text{non-commutative} \\
\text{linear} & \text{and} & \text{nonlinear} \\
\text{geometry} & \text{and} & \text{algebra} \\
\text{physics} & \text{and} & \text{mathematics}
\end{array}
$$

1.7. Also this choice is of course highly personal. One can easily add 12 other **polarizing** quantities which help to distinguish or parametrize different parts of mathematical areas. The use of ambivalent pairs can be used to slice through the different areas:

regularity	and	randomness
integrable	and	chaotic
invariants	and	perturbative
experimental	and	deductive
polynomial	and	exponential
applied	and	abstract

discrete	and	continuous
existence	and	construction
finite dim	and	infinite dim
topological	and	differential geometric
practical	and	theoretical
axiomatic	and	example based

1.8. An other possibility to refine the fields of mathematics is to **combine** different areas. Examples are **probabilistic number theory**, **algebraic geometry**, **numerical analysis**, **geometric number theory**, **numerical algebra**, **algebraic topology**, **geometric probability**, **algebraic number theory**, **dynamical probability = stochastic processes**. Almost every pair is an actual field.

1.9. Finally, let us try to give a short answer to the question: What is Mathematics?

Mathematics is the science of structure.

1.10. The simplicity of this definition is intended. As soon as we include more topics we actually exclude other topics. Structure is a good word, as mathematics is built by algebraic structures, order structures, topological structures, measure theoretical structures or combinations of such structures like for example algebraic structures acting on topological structures. Definitions, theorems and examples form a theoretical structure. The structure of all of mathematics is a structure too and today often described using categorical structures.

1.11. The goal is to illustrate some of these structures from a historical point of view. Each week, an other topic will be covered. Every week therefore is a new start. Our focus is historical. By doing so we also learn a great deal about how mathematics is learned. One of the key insights we have in education is that the difficulties of the pioneers developing some new material is often mirrored today, when we learn the material. Understanding how learning works is not only important for teachers it is also important for students of mathematics. And history is not only fun, it provides key insight on how mathematics works.

1.12. This document has evolved over a decade now. It is also the 10th time, this course is taught here at the Harvard extension school.

A work problem

Every year we try to find a topic that illustrates the **unity of mathematics**. You find older work problems on the websites from 2010 to 2017. Here is the problem from 2021. While doing the problem, also think about which parts of mathematics are getting involved here.

1.13. 1. If you have n people sitting on a circular table who are already acquainted with their neighbors, how many additional connections need to be done? In other words, how many diagonals are there in a polygon with n sides? Lets look at small cases. How many diagonals are there in a pentagon, hexagon or heptagon?
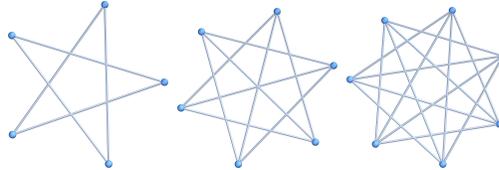
1.14. 2. Can you find a general formula? What would the answer be for a 17-gon?
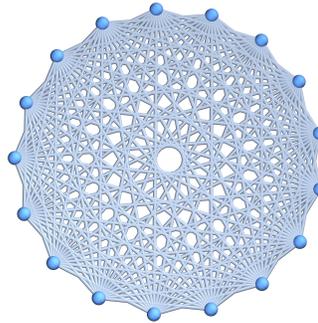
1.15. 3. We can also look at the number of triangles built by diagonals. We require the vertices of the triangle to be points of the polygon. There are 2 triangles in the 6-gon and 7 triangles in the 7 gon. Counting is more difficult. For 17, there are already 442 triangles. Is there a way You filled out already the missing diagonal numbers in problem 1 and 2. Can you find a rule which allows to complete the table?

n-gon	vertices	diagonals	triangles	4-cliques	5-cliques	6-cliques
4	4	2				
5	5	...				
6	6	...	2			
7	7	...	7			
8	8	20	16	2		
9	9	27	30	9		
10	10	35	50	25	2	
11	11	44	77	55	11	
12	12	54	112	105	36	2
13	13					
14	14					
15	15					
16	16					
17	17					