TEACHING MATHEMATICS WITH A HISTORICAL PERSPECTIVE

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Lecture 2: Arithmetic

2.1. The oldest mathematical discipline is **arithmetic**. It is the theory of the construction and manipulation of **number systems**. Humans started doing mathematics by **counting**. **Tally sticks** or **pebbles** were first used as memory devices. The earliest remnants of such devices have been found in **Africa**. The most notable examples are the **Lebombo bone** (44'000 BC) or the **Ishango bone** (20'000 BC). The first steps in building up number systems were initiated independently in various cultures. We have knowledge of early numerical structures erected by **Babylonian**, **Egyptian**, **South American**, **Chinese**, **Indian** or **Greek** thinkers.

FIGURE 1. Left: The four sides of the Ishango Bone. Source: Royal Belgian Institute of National Sciences, “”Have you heard of Ishango”, 2010. Right: Clay tablet from 2041 BCE, from Umma, an ancient city in Sumer. Source: Spurlock Museum, University of Illinois.

2.2. Building the number system starts with the **natural numbers** $1, 2, 3, 4, \dots$. They can be added as well as multiplied. While addition $+$ comes naturally by combining different objects, multiplication $*$ is more subtle: $3 * 4$ means to take 3 copies of 4 and get $4 + 4 + 4 = 12$ while $4 * 3$ means to take 4 copies of 3 to get $3 + 3 + 3 + 3 = 12$. The first factor counts the number of operations while the second factor counts the objects. Spacial insight shows $3 * 4 = 4 * 3$ as one can arrange the 12 objects in a rectangle. The earliest use of multiplication could have been been area computation. This was important as the amount of water to irrigate a field is proportional to its area.

2.3. Rules like commutativity $x + y = y + x, x * y = y * x$, associativity $(x + y) + z = x + (y + z), (x * y) * z = x * (y * z)$ or **distributivity** $x * (y + z) = x * y + x * z$ are guiding principles to extend the number systems. First build negative numbers and fractions and also introduce 0, a number which appeared relatively even later than fractions. Negative numbers could first have become necessary through when owing **dept**. Dividing things up was more natural and lead to

fractions and so the field of **rational numbers**, a preliminary culmination. This build-up from a monoid $\mathbb{N}$ to a group $\mathbb{Z}$ to a ring $\mathbb{Z}$ and eventually to a field of fractions $\mathbb{Q}$ is a prototype also when building algebraic objects.

2.4. Geometry lead naturally to more general numbers. The diagonal of a square was a first example which was not quantified any more by fractions. It has been a puzzling moment for the Pythagoreans to realize that such **real numbers** do exist. Also the area of a circle appeared soon not to be rational. Today, we talk about the **real numbers** and use **limits** to define them. There are two major motivations to **to build new numbers**: we want to be able to **invert operations** and still get a number. Reverting the addition is subtraction for example, reverting multiplication is division, reverting taking the square is taking the square root. The second, related reason is the ability to **solve equations**.

2.5. To find the additive inverse of 3 means solving $x+3=0$. The answer is the negative number $x=-3$. In order to solve solve $x*3=1$ we need a rational number $x=1/3$. To get a solution of $x^2=2$ a real number is needed. It does not stop there, in order to solve $x^2=-2$ we need complex numbers. Finally, in calculus one deals with infinitesimal numbers. This naturally leads to the concept of **surreal numbers**. They can be extended then to **surreal complex numbers**.

Numbers	Operation to complete	Examples of equations to solve
Natural numbers	addition and multiplication	$5+x=9$
Positive fractions	addition and division	$5x=8$
Integers	subtraction	$5+x=3$
Rational numbers	division	$3x=5$
Algebraic numbers	taking positive roots	$x^2=2$, $2x+x^2-x^3=2$
Real numbers	taking limits	$x=1-1/3+1/5-+...$,$\cos(x)=x$
Complex numbers	take any roots	$x^2=-2$
Surreal numbers	transfinite limits	$x^2=\omega$, $1/x=\omega$
Surreal complex	any operation	$x^2+1=-\omega$

2.6. The development and history of arithmetic can be summarized as follows: humans started to count with natural numbers, dealt with positive fractions, reluctantly introduced negative numbers and zero to get the integers. They struggled to "realize" real numbers, were scared to introduce complex numbers, hardly accepted surreal numbers and most do not even know about surreal complex numbers. Ironically, as simple but impossibly difficult questions in number theory show, the modern point of view is the opposite to Kronecker's **"God made the integers; all else is the work of man"**. We would rather say the following: the **surreal complex** numbers are the most **natural** numbers; the **natural** numbers are the most **complex, surreal** numbers. Lets look a bit closer at various number systems.

2.7. Let us first look at **natural numbers** $\mathbb{N}$. Counting can be realized by sticks, bones, quipu knots, pebbles or wampum knots. The **tally stick** concept is still used when playing card games: where bundles of fives are formed, maybe by crossing 4 "sticks" with a fifth. There is a "log counting" method in which graphs are used and vertices and edges count. An old stone age tally stick, the **wolf radius bone** contains 55 notches, with 5 groups of 5. It is probably more than 30'000 years old. The most famous paleolithic tally stick is the **Ishango bone**, the fibula of a baboon. It could be 20'000 - 30'000 years old. It was found in 1962 near Lake Edward in Congo Earlier counting could have been done by assembling **pebbles**, tying **knots** in a string, making **scratches** in dirt or bark but no such traces have survived the thousands of years. We have today still birk bark remnants from 1340 on.

2.8. The **Roman system** improved the tally stick concept by introducing new symbols for larger numbers like $V=5, X=10, L=40, C=100, D=500, M=1000$. in order to avoid bundling too many single sticks. The system is unfit for computations as simple calculations $VIII+VII=XV$ show. **Clay tablets**, some as early as 2000 BC and others from 600 - 300 BC are known. They feature **Akkadian arithmetic** using the base 60. The hexadecimal system with base 60 is convenient because of many factors. It survived: we use 60 minutes per hour. **The Egyptians** used the base 10. The most important source on Egyptian mathematics is the **Rhind Papyrus** of 1650 BC. It was found in 1858 Hieratic numerals were used to write on papyrus from 2500 BC on. **Egyptian numerals** are hieroglyphics. The were found in carvings on tombs and monuments they are 5000 years old, from time of the pyramids.

1	stick
10	heel
100	monkey
1000	flower
10000	finger
1000000	frog
10000000	priest

Remember it as : "the priest points to a frog with his finger and the flower fears a monkey bending a stick."

2.9. The modern way to write numbers like 2021 is the **Hindu-Arab system** which diffused to the West only during the late Middle ages. It replaced the more primitive **Roman system**. Greek arithmetic used a number system with no place values: 9 Greek letters for $1, 2, \dots 9$, nine for $10, 20, \dots, 90$ and nine for $100, 200, \dots, 900$.

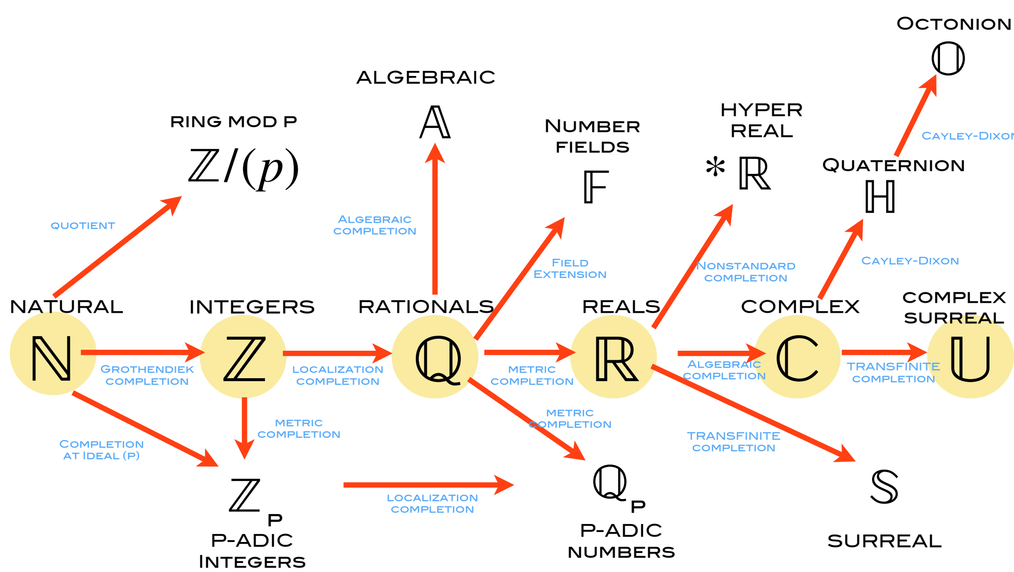


Figure 2. The structure of important number systems. The central bone is $\mathbb{N} \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{R} \to \mathbb{C} \to \mathbb{U}$, which starts with the natural numbers, goes to the integers, fractions, real numbers and finally the complex and surreal complex numbers. Paradoxially, $\mathbb{U}$ is the most natural since we can do anything there. Already solving equations in $\mathbb{N}$ is complex and surreal.

2.10. Now to the **Integers** $\mathbb{Z}$. **Indian Mathematics** morphed the **place-value system** into a modern method of writing numbers. Hindu astronomers used words to represent digits, but the numbers would be written in the opposite order. Independently, also the Mayans developed the **concept of** 0 in a number system using base 20. Sometimes after 500, the Hindus changed to a digital notation which included the symbol 0. Negative numbers were introduced around 100 BC in the **Chinese** text "Nine Chapters on the Mathematica art". Also the **Bakshali manuscript**, written around 300 AD subtracts numbers and carried out additions with negative numbers, where + was used to indicate a negative sign. In Europe, negative numbers were avoided until the 15'th century.

2.11. And now to **fractions** $\mathbb{Q}$ which is a natural **field**, a structure where one can add, subtract, multiply and divide by non-zero elements. Already the **Babylonians** could handle fractions. The **Egyptians** also used fractions, but wrote every fraction a as a sum of fractions with unit numerator and distinct denominators, like $4/5 = 1/2 + 1/4 + 1/20$ or $5/6 = 1/2 + 1/3$. Maybe because of such cumbersome computation techniques, Egyptian mathematics failed to progress beyond a primitive stage. The Egyptians used in the Rhind papyrus other conventions like $2/15 = 1/10 + 1/30$ and not $1/8 + 1/120$. There are still unsolved number theoretical problems involved with Egyptian fractions. And it also leads to puzzles. How do you write $11/17$ for example as a sum of fractions? A general method has been described by Fibonacci in his book "Liber Abaci". The answer is $1/2 + 1/7 + 1/238$. It is unknown for example whether whether Fibonnacci's process finishes after finitely many steps if we insist on odd fractions [**?**]. The example $11/17 = 1/3+1/5+1/9+1/383+1/292995$ shows that things get more complicated with the insistance of having odd fractions. The modern decimal fractions which are used today for numerical calculations were adopted only in 1595 in Europe.

2.12. An interesting transition comes when introducing **real numbers** $\mathbb{R}$ as one has to get a new concept, the concept of **limit** which is a topological or calculus like notion. As noted by the Greeks already, the diagonal of the square of length 1 has a length that is not a fraction. It first produced a crisis. Later, it became clear that "most" numbers are not rational. **Georg Cantor** saw first that the cardinality of all real numbers is much larger than the cardinality of the integers: while one can count all rational numbers but not enumerate all real numbers. One consequence is that most real numbers are **transcendental**: they do not occur as solutions of polynomial equations with integer coefficients. The number π is an example. The concept of real numbers is related to the **concept of limit**. Limits are involved when computing sums like $1 + 1/4 + 1/9 + 1/16 + 1/25 + \dots$. The result is a number which is not rational.

2.13. The **complex numbers** $\mathbb{C}$ were introduced rather late with Gauss. Some polynomials have no real root. To solve $x^2 = -1$ for example, we need new numbers. One idea is to use pairs of numbers (a, b), where $(a, 0) = a$ are the usual numbers and extend addition and multiplication $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$. With this multiplication, the number $(0, 1)$ has the property that $(0, 1) \cdot (0, 1) = (-1, 0) = -1$. It is more convenient to write $a + ib$, where $i = (0, 1)$ satisfies $i^2 = -1$. One can now use the common rules of addition and multiplication. We can interpret the equation $x^2 = -1$ as the search transformation x in the plane which has the property that if one does the transformation twice, one gets the reflection $(a, b) \to -(a, b)$ at the origin. The transformation $(a, b) \to (-b, a)$, a rotation does the job. It satisfies the rule $i^2 = -1$. If we introduce complex numbers as $a+ib$, then multiplying with i gives indeed $i(a+ib) = ia+i^2b = -b+ia$. The algebra of real numbers can now be extended in a natural way to the set of all complex numbers $\{x+iy\}$. For example, $(3+4i)(1-i) = 3-3i+4i+4 = 7+i$. The multiplication with a complex number can be interpreted geometrically as a rotation scaling.

2.14. The **surreal numbers** are a construct which only appeared in the second half of the 20'th century. Similarly as real numbers fill in the gaps between the integers, the surreal numbers fill in the gaps between Cantors ordinal numbers. They are written as $(a, b, c, ...|d, e, f, ...)$ meaning that the "simplest" number is larger than $a, b, c...$ and smaller than $d, e, f, ...$ We have $(|) = 0, (0|) = 1, (1|) = 2$ and $(0|1) = 1/2$ or $(|0) = -1$. Surreals contain already transfinite numbers like $(0, 1, 2, 3...|)$ or infinitesimal numbers like $(0|1/2, 1/3, 1/4, 1/5, ...)$. They were introduced in the 1970'ies by John Conway. The late appearance confirms the pedagogical principle: **late human discovery manifests in increased difficulty to teach it**.
One of the reasons for introducing the surreal numbers is that one compute $\sqrt{\omega}$ for example. Different surreal numbers can mean the same, like $\{1|3\} = 2$. There are some things one has to

be careful about. For example: $x + y$ is not necessarily $y + x$. The number $\omega + 1$ is not the same as $1 + \omega = \omega$.

2.15. Geometry arithmetic: One can also compute with geometric objects like graphs. Our usual arithmetic is based on graphs without edges. Given two graphs, the addition is the disjoint union of the graphs. The multiplication takes the Cartesian product of the vertex sets and connects two if both projections either are vertices or edges. One can now repeat the construction of rational numbers, real numbers and complex numbers in this more general frame work.

2.16. Writing numbers: Geometric representations of numbers are actually already done when writing letters. Instead of carving 5 marks into a bone, one has a symbol 5 for that number. This is a geometric figure. Different cultures have introduced different ways to represent numbers. A big step was the **place value system**. Instead of having a number like 1 Million which the Egyptians wrote as a priest, one would write 1'000'000. But the base 10 system was not not the only one. The Mayan's used base 20, the Sumerians base 60. Modern computers use base 2 (binary), base 8 (octal) or base 16 (hexagesimal) systems.

2.17. Talking knots: Finally, we should mention that also knot representations have appeared in the form of **Khipu**. This also should be seen as a geometric representation of numbers.

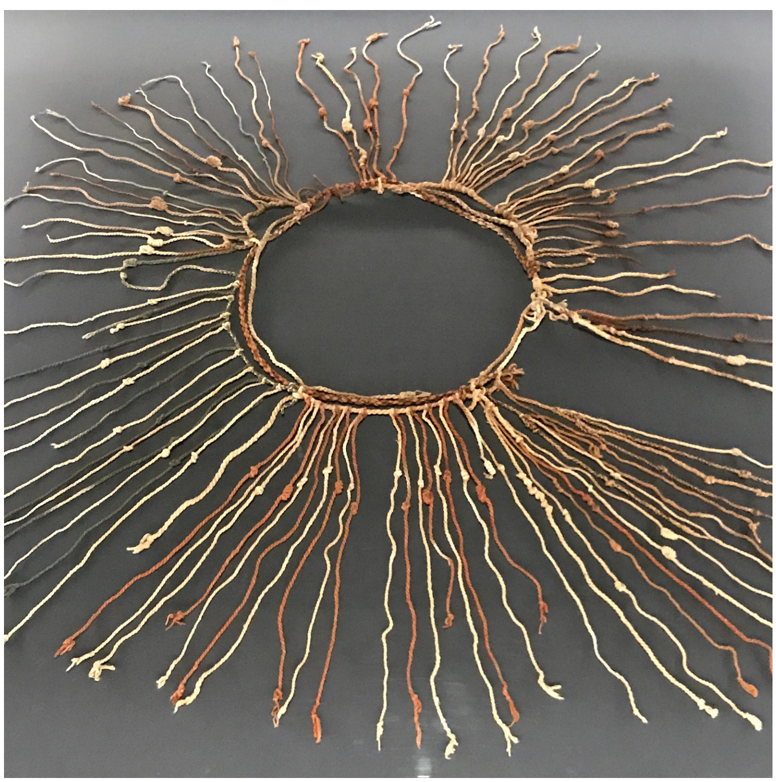
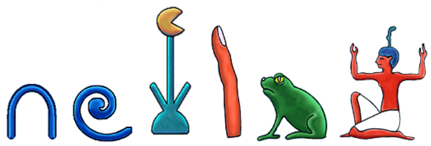


FIGURE 3. A khipu at the Boston museum of fine arts.

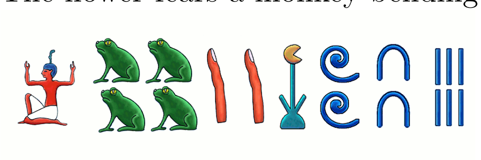
Work problems

1) Here are example proofs: **Theorem:** $\sqrt{3}$ is irrational. **Proof:** $\sqrt{3} = p/q$ implies $3 = p^2/q^2$ or $3q^2 = p^2$. If we make a prime factorization, then on the left hand side contains an odd number of factors 3, while the right hand side contains an even number of factors 3.
Theorem: $\log_{10}(3)$ is irrational. **Proof.** If $\log_{10}(3) = p/q$ then $3 = 10^{p/q}$ or $3^q = 10^p$. This is not possible because the right hand side is not divisible by 3, while the left hand side is.
1a) Show that $\sqrt{17}$ is not rational, 1b) Prove that $\log_{10}(3)$ is irrational.

2) We have learned how to read hieroglyphs. Here are the symbols for 10, 100, 1000, 10000, 100000, 1000000:

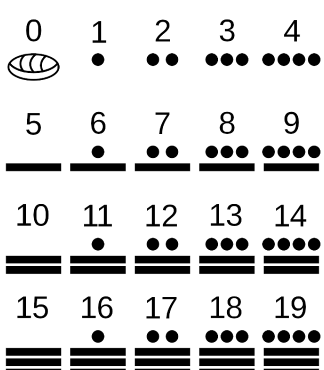


Which integer does this **hieroglyph** represent? Remember: "The priest holds a frog in his finger. The flower fears a monkey bending a stick."



3) In 4000 BC, in the Mesopotamia region, cuneiform were imprinted on a wet clay tablets. An example is "Plimpton 322", a Clay tablet from 1800 BC. The Babylonians already contemplated the square root of 2. We have seen in the presentation the Clay tablet YBC 7289:

How would you write the number 1000 in the hexadecimal system?
4) The Mayan system is based on 20. Here are the first 20 numbers. Note that the Mayans have independently discovered used zero and also had a place-valued system.



How would you write the number 401 in the Mayan system?