TEACHING MATHEMATICS WITH A HISTORICAL PERSPECTIVE

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E-320: Teaching Math with a Historical Perspective **O. Knill, 2010-2021**

Lecture 3: Geometry

3.1. Geometry is a science of **shape, size and symmetry**. It is one of the oldest mathematical disciplines as it appears in the earliest documents of human kind. As shapes have aesthetic value, it also relates to art. While arithmetic focused on numerical structures, geometry builds, relates and described metric structures which appear to our physical world. But it is far from limited to shapes that we can physically realize; geometry also can describe objects of large, fractional or even infinite dimension. For geometry, not even the sky is the limit.

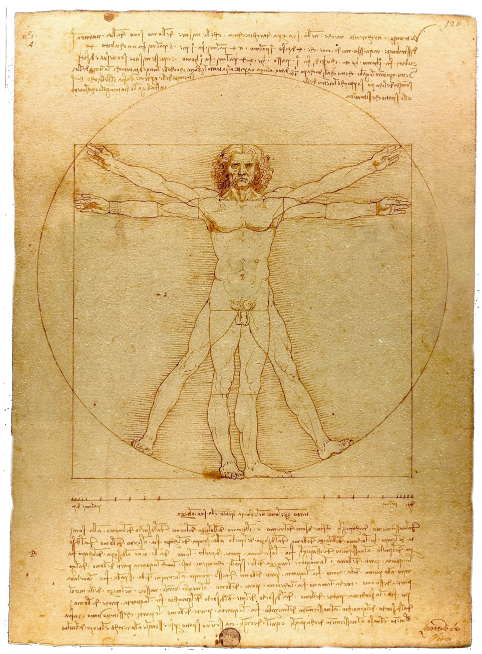


FIGURE 1. The Vitruvian man (1490) by Leonardo da Vinci not only features basic geometric features like angles, proportion, square and circle, it also relates to art. The sextant (Collegium Matopolski Instytut Kultury) is a symbol for measurements on earth done to determine the position on earth.

3.2. Already the first geometric drawings had relations with arithmetic: the multiplication of two numbers as an **area** of a **shape** that is invariant under a physical **symmetry** rotating the rectangle from a $n \cdot m$ to $m \cdot n$ and so justify a naive commutativity assumption for multiplication. Non-commutative or quantum geometries later generalized geometries even further. A guide is Felix Klein's **Erlanger program** used symmetries to distinguish geometries. Symmetries links geometry also with algebra an other core pillar of mathematics.

3.3. One of the earliest connections between number systems and geometry came through identities like the **Pythagorean triples** $3^2 + 4^2 = 5^2$ which were interpreted and drawn geometrically. It is related to the concept of **right angle** which is the most symmetric angle besides 0. Symmetry manifests itself in quantities which are **invariant**. Invariants are most central aspects of geometry and often numerical like Euler characteristic. Generalizing the realm of symmetries leads then to other fields like topology.

3.4. In this lecture, we first look at a few results related to the first discoveries in gometry. We will start with a few smaller miracles happening in triangles as well as a couple of gems: **Pythagoras**, **Thales**,**Hippocrates**, **Feuerbach**, **Pappus** or **Morley**, **Butterfly** which all illustrate the importance of symmetry and also extremely approachable as we can realize and see even build the objects.

3.5. Much of geometry is based on our ability to measure **length**, the **distance** between two points. The etymology of the word geometry comes from measuring distances on the earth. Geometric thinking became more and more important when human extended their horizon. Discovering new worlds requires to travel more efficiently. Also modern global positioning systems are built on simple geometric ideas as differences between distances between various points determine exact positions. Because light travels with a universal speed, distance measurements are also related to clocks measuring time.

3.6. Having a distance $d(A, B)$ between any two points A, B, we can look at the next more complicated object, a **triangle** which is a set A, B, C of 3 points. Given an arbitrary triangle ABC, are there relations between the 3 possible distances $a = d(B, C), b = d(A, C), c = d(A, B)$? If we fix the scale by $c = 1$, then $a + b \geq 1, a + 1 \geq b, b + 1 \geq a$. For any pair of (a, b) in this region, there is a triangle. A triangle also defines angles and knowing angles allows to get from the knowledge of one distance all distances. We humans can estimate the distance to an object by comparing the pictures captured by two eyes. Navigation on earth was initially done by measuring angles using sextants, star and moon positions and clocks.

3.7. The concept of distance leads to **spheres** as the set of points with fixed distance r from a point. In the plane, the sphere is called a **circle**. A natural problem is to find the circumference $L = 2\pi$ of a unit circle, or the **area** $A = \pi$ of a unit disc, the **surface area** $S = 4\pi$ of a unit sphere and the **volume** $V = 4 = \pi/3$ of a unit sphere. Measuring the length of segments on the circle leads to **angle** or **curvature**. Because the circumference of the unit circle in the plane is $L = 2\pi$, angle questions are tied to the number π, which Archimedes already has approximated with rational numbers.

3.8. Volumes were among the first quantities, Mathematicians wanted to measure and compute. A problem on **Moscow papyrus** dating back to 1850 BC explains the formula $h(a^2 + ab + b^2)/3$ for a truncated pyramid with base length a, roof length b and height h. Archimedes achieved to compute the **volume of the sphere** by seeing that it is the volume the complement of the cone inside the cylinder which has at height z a slice of area $\pi - \pi z^2$. The volume of the half sphere therefore is volume of the complement of the cone inside the cylinder which is $\pi - \pi/3 = 2\pi/3$. It later turned out that calculus and analysis are the right language to compute volumes.

3.9. The first geometric steps were done in flat two-dimensional space. Highlights are **Pythagoras theorem**, **Thales theorem**, **Hippocrates theorem**, and **Pappus theorem**. Discoveries in planimetry have been made later on: an example is the Feuerbach 9 point theorem from the 19th century. Ancient Greek Mathematics is closely related to history. It starts with **Thales** goes over Euclid's era at 500 BC and ends with the threefold destruction of Alexandria 47 BC by the Romans, 392 by the Christians and 640 by the Muslims.

3.10. Geometry was also a place, where the **axiomatic method** entered mathematics. More rigorous deductive proofs appeared at the same time also in number theory, especially in the context of prime numbers. The Pythagorean theorem identity leads to the Pythagorean triple Diophantine equation. With axioms and proofs, mathematics became more organized and reliable. The first axioms of geometry were the 5 axioms of Euclid:

1. Any two distinct points A, B determines a line through A and B.
2. A line segment $[A, B]$ can be extended to a straight line containing the segment.
3. A line segment $[A, B]$ determines a circle containing B and center A.
4. All right angles are congruent.
5. If lines L, M intersect with a third so that inner angles add up to $< \pi$, then L, M intersect.

3.11. Euclid wondered whether the fifth postulate can be derived from the first four and called theorems derived from the first four "absolute geometry". Only much later, with **Karl-Friedrich Gauss**, **Janos Bolyai** and **Nicolai Lobachevsky** in the 19'th century realized that for a **hyperbolic space** the 5'th axiom does not hold any more. This was just the beginning for many new geometries. Geometry can be generalized to non-flat, or even much more abstract situations.

3.12. Basic examples of **non-Euclidean geometries** are geometry on a sphere leading to **spherical geometry**. Then there is the geometry on the Poincare disc, an example of a **hyperbolic space**. These geometries are still rather limited. **Riemannian geometry**, which is essential for **general relativity theory** generalizes both concepts to a great extent. An example is the geometry on an arbitrary surface. Curvatures of such spaces can be computed by measuring length alone, which is how long light needs to go from one point to the next. Also Riemannian geometries have been generalized to geometries allowing for rather arbitrary metric spaces.

3.13. An important moment in mathematics was the **merge of geometry with algebra**. This giant step is often attributed to **René Descartes**. Together with algebra, the subject exploded to a today enormous building called algebraic geometry. This geometry can also be tackled with the help of computer algebra systems.

3.14. Here are some examples of geometries which are determined from the amount of symmetry which is allowed:

Euclidean geometry	Properties invariant under a group of rotations and translations
Affine geometry	Properties invariant under a group of affine transformations
Projective geometry	Properties invariant under a group of projective transformations
Spherical geometry	Properties invariant under a group of rotations
Conformal geometry	Properties invariant under angle preserving transformations
Hyperbolic geometry	Properties invariant under a group of Möbius transformations

3.15. We finally show four pictures about the 4 special points in a triangle and with which we will begin the lecture. We will see why in each of these cases, the 3 lines intersect in a common point. It is a manifestation of a **symmetry** present on the space of all triangles. **size** of the distance of intersection points is constant 0 if we move on the space of all triangular **shapes**. It's Geometry!

Work problems

3.16. The existence of the **centroid** (the intersection of medians) B, the **orthocenter** (the intersection of altitudes) O , the **circumcenter** (the center of the circumcircle) C and the **incenter** (the center of the incircle) I are already not completely obvious. We can show their existence algebraically. The points B, O, C are on a line, the **Euler line**.

3.17. The **Pythagorean theorem** does not need any introduction. Like $E = mc^2$, it is a famous quadratic law: for all right angle triangles of side length a, b, c, the quantity $a^2 + b^2 - c^2$ is zero.

3.18. The **3D Pythagoras theorem** tells that areas of the three sides of the pyramid add up to the base area. It is also called the **Faulhaber extension** or **de Gua theorem**. We can verify it by computing the area of the triangle $A = (a, 0, 0), B = (0, b, 0), C = (0, 0, c)$ which is $|\langle a, -b, 0\rangle \times \langle 0, b, -c\rangle|/2 = |\langle bc, ac, ab\rangle| = (bc^2 + ac^2 + ab^2)/4$.

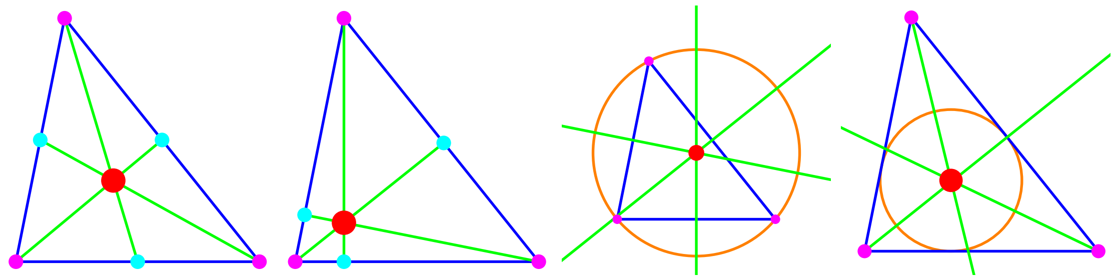

FIGURE 2. The existence of centroid B, orthocenter O, circumcenter C and incenter I are four little miracles for triangles. The points B, O, C are located on a line.

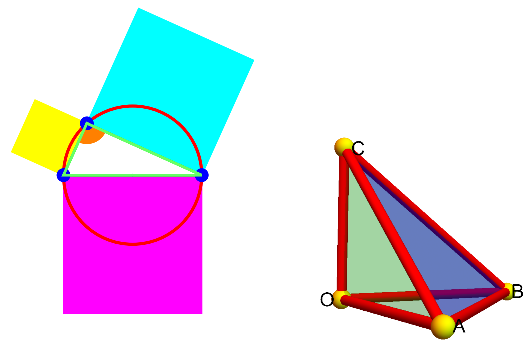

FIGURE 3. Pythagoras theorem.

3.19. Thales of Miletus (625 BC -546 BC) showed that if a triangle inscribed in a fixed circle is deformed by moving one of its points on the circle, then the angle at this point does not change.

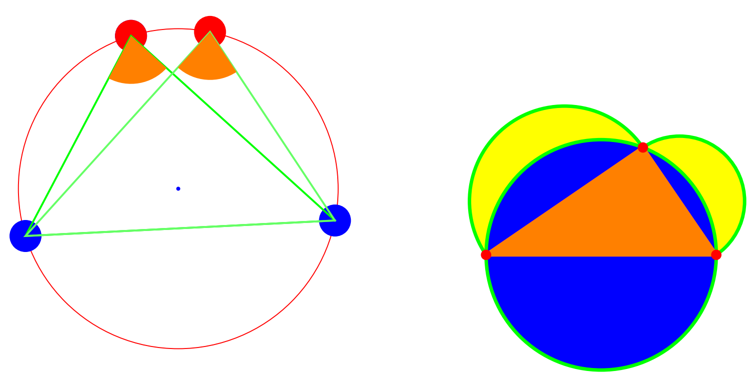

FIGURE 4. Thales theorem assures that the angle does not change when moving the point point on the circle. The Hippocrates theorem equates the areas of the lunes with the triangle area.

3.20. The quadrature of the Lune is due to **Hippocrates of Chios** (470 BC - 400 BC). It is the first rigorous quadrature of a curvilinear area. The sum $L + R$ of the area L of the left moon and the area R of the right moon is equal to the area T of the triangle.

3.21. The proof of Morley's miracle we can decompose the triangle with 7 triangles.

3.22. Given a circle of radius 1 and a point P inside the circle. For any line through P which intersects the circle at points A, B we have $1 - |PO|^2 = |PA||PB|$. Proof with Pythagoras. By

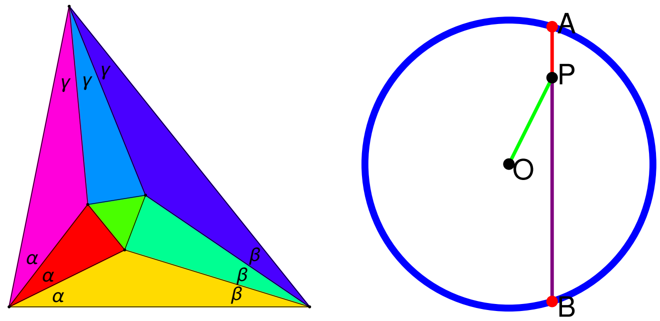

FIGURE 5. Morley's theorem produces an equilateral triangle from angle trisectors of an arbitrary triangle. The Faskreis theorem assures $1 - |PO|^2 = |PA||PB|$.

scaling translation and rotation we can assume the circle is at the origin and that the line through the point $P = (a, b)$ is vertical. The intersection points are then $(a, \pm\sqrt{1 - a^2})$. Now

$$|PA||PB| = (\sqrt{1 - a^2} - b)(\sqrt{1 - a^2} + b) = 1 - a^2 - b^2 = 1 - |PO|^2 .$$

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